TAUTOLOGICAL BUNDLES OVER QUOT SCHEMES

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INTRODUCTION

Let C be a smooth projective curve. Moduli spaces over C such as Jacobians or more generally moduli of vector bundles capture several geometric properties of C. They provide examples of interesting high-dimensional spaces to explore.

Quot schemes parameterize quotients of a fixed vector bundle $E \to C$. They provide a compactification for the space of morphisms from C to Grassmannians. Quot schemes also play an essential role in the study of the moduli space of vector bundles.

Over the past few decades, several geometric aspects of Quot scheme have been studied.

TECHNIQUES

The proof can broadly be divided into three steps:

Universality: Using the arguments similar in spirit to [1], we show that there exists universal series A, B, and C such that

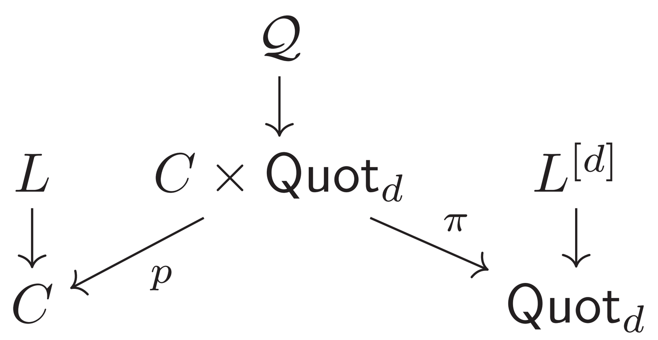
$$\sum_d q^d \chi(\mathsf{Quot}_d, \wedge_y L^{[d]}) = A^{\chi(\mathcal{O}_C)} B^{\deg L} C^{\deg E}.$$

Localization: Universality reduces the calculations to Quot scheme over $\mathbb{P}^1$, where the vector bundle E splits as a direct sum of line bundles. We use equivariant localization (using a torus action) to reduce the calculations to integrals on products of projective spaces.

Combinatorics: We use several combinatorial identities, such as Lagrange-Bürmann formula, to realize the expression as a Schur polynomial evaluated at roots of a polynomial with coefficients involving q and y. We then use Jacobi-Trudi identities to obtain explicit formulas.

PUNCTUAL QUOT SCHEME

Fix a rank N vector bundle E over C. The punctual Quot scheme Quot_d is a smooth scheme that parameterize quotients $[E \twoheadrightarrow Q]$ of zero-dimensional support and length d.



The Quot scheme comes equipped with a universal quotient $\mathcal{Q}$ over $C \times \mathsf{Quot}_d$ and a map $p^*E \twoheadrightarrow \mathcal{Q}$. We use $\mathcal{Q}$ (as a kernel) to obtain tautological bundles over Quot_d. For any line bundle L on C,

$$L^{[d]} := \pi_*(\mathcal{Q} \otimes p^*L).$$

HIGHER RANK QUOTIENTS

Our techniques can be used to obtain the results for the Quot scheme $\mathsf{Quot}_d(E,r)$ parameterizing rank r quotients over $\mathbb{P}^1$.

Theorem 4. *Let E be trivial vector bundle, then*

$$\chi(\mathsf{Quot}_d(E,r), \det L^{[d]}) = u[q^d] s_\lambda(z_1, \ldots, z_N)$$

where u is a sign, s_λ is the Schur polynomials for the rectangular partition $\lambda = (d + \chi(L))^{N-r}$, and $z_1, \ldots, z_N$ are roots of the equation

$$(z-1)^N - z^{r-1} q = 0.$$

We use Jacobi-Trudi identities to obtain corollaries such as

$$\chi(\mathsf{Quot}_d(E,r), \det \mathcal{O}^{[d]}) = \binom{N}{r+d}.$$

Specializing to $d = 0$, we recover the known formulas for Grassmannian $G(N,r)$.

RESULTS

The tautological bundle $L^{[d]}$ is a vector bundle of rank d. For any vector bundle V, let

$$\wedge_y V = \textstyle\sum_k y^k \wedge^k V.$$

Theorem 1. *For any line bundle $L \to C$,*

$$\sum_{d=0}^{\infty} q^d \chi(\mathsf{Quot}_d, \wedge_y L^{[d]}) = \frac{(1+qy)^{\chi(E \otimes L)}}{(1-q)^{\chi(\mathcal{O}_C)}}.$$

We obtain a slightly stronger result involving dual of several tautological inputs.

Theorem 2. *Over Quot_d, the Euler characteristics of $\wedge_y L^{[d]} \otimes_1^r (\wedge_{x_i} M_i^{[d]})^\vee$ equals*

$$[q^d] \frac{(1+qy)^{\chi(E \otimes L)}}{(1-q)^{\chi(\mathcal{O}_C)} \prod (1 - x_i y q)^{\chi(L \otimes M_i^\vee)}},$$

where $M_1, \ldots, M_r$ and L are line bundles and $r < N$.

The nature of the formula provides several interesting vanishing results. For example,

- For any line bundle L and $0 < r < N$: $\chi(\mathsf{Quot}_d, (\det L^{[d]})^{-r}) = 0$.
- When genus $g > 1$ and $d > k + g$: $\chi(\mathsf{Quot}_d, \wedge^k L^{[d]}) = 0$.

We also consider the case of symmetric powers, and obtain a closed-form expression over $\mathbb{P}^1$.

Theorem 3. *For $C = \mathbb{P}^1$ and $d \geq k$,*

$$\chi(\mathsf{Quot}_d, \mathrm{Sym}^k L^{[d]}) = (-1)^k \binom{-\chi(E \otimes L)}{k}.$$

In principle, the universality argument (explained below) gives us an answer for any C. However, we only find a conjectural formula for genus 1.

ANALOGIES WITH HILBERT SCHEME OF POINTS ON A SURFACE

Our results suggest surprising analogy between punctual Quot scheme Quot_d and the Hilbert scheme of points $S^{[d]}$ for a projective surface S.

Exterior powers: For a line bundle $L \to S$, the tautological vector bundle $L^{[d]}$ is defined similarly. The Euler characteristics are given by

$$\sum_{d=0}^{\infty} q^d \chi(S^{[d]}, \wedge_y L^{[d]}) = \frac{(1+qy)^{\chi(L)}}{(1-q)^{\chi(\mathcal{O}_S)}}.$$

Note the striking similarity with the formula for Quot_d. This is proved either via derived category techniques in [4] or via studying Donaldson-Thomas theory of toric Calabi-Yau-3-folds in [3].

Question: Can our formulas for Quot_d be refined to isomorphism of cohomology groups of $\wedge^k L^{[d]}$ and direct sum of cohomology groups of bundles involving $E \otimes L$ and $\mathcal{O}_C$?

Symmetric powers: There are partial results for the symmetric powers of tautological bundles in [3]. The analogy between $\mathsf{Quot}_d(E)$ and $S^{[d]}$ persists! When $\chi(\mathcal{O}_S) = 1$ and $d \geq k$,

$$\chi(S^{[d]}, \mathrm{Sym}^k L^{[d]}) = (-1)^k \binom{-\chi(L)}{k}.$$

Rationality: The generating series involving these tautological bundles over Quot schemes of a surface [2] (or a curve) are rational functions.

*ABOUT OUR WORK

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