Schur bundles over Quot schemes of $\mathbb{P}^1$

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Examples of smooth Quot schemes

Punctual Quot scheme
(Zero dimensional quotient)

Quot scheme over genus 0 curve
(E is a trivial bundle)

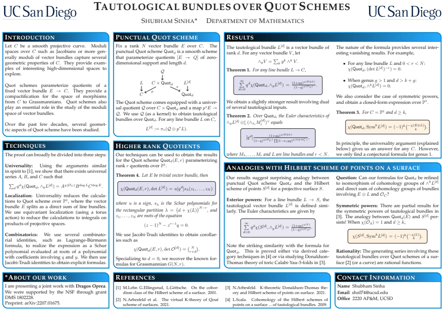
Punctual Quot scheme

Several properties of the Punctual Quot schemes has been studied:

- (Bifet'89, Chen'01): Poincare Polynomial
- (Biswas-Dhillon-Hurtubise'15): Automorphism group
- ⋮
- (Ricolfi'20, Bagnarol-Fantechi-Perroni'20): Motives
- (Oprea'22, Oprea-Pandharipande'18): Positivity and Segre classes of tautological bundles
- (Toda'22) S.O.D of the derived category

Punctual Quot scheme

- (Oprea-**S**'22): Explicit formula for the Euler characteristics of tautological bundles over punctual Quot scheme



UC San Diego **Tautological bundles over Quot Schemes** UC San Diego

Shubham Sinha* Department of Mathematics

Introduction

Let C be a smooth projective curve. Moduli spaces over C such as Jacobians or more generally moduli of vector bundles capture several geometric properties of C. They provide examples of interesting high-dimensional spaces to explore.

Quot schemes parameterize quotients of a fixed vector bundle $E \to C$. They provide a compactification for the space of morphisms from C to Grassmannians. Quot schemes also play an essential role in the study of the moduli space of vector bundles.

Over the past few decades, several geometric aspects of Quot scheme have been studied.

Punctual Quot scheme

Fix a rank N vector bundle E over C. The punctual Quot scheme Quot_d is a smooth scheme that parameterize quotients $[E \twoheadrightarrow Q]$ of zero-dimensional support and length d.

$$\mathcal{Q} \downarrow C \times \mathsf{Quot}_d, \quad p: C \times \mathsf{Quot}_d \to C, \quad \pi: C \times \mathsf{Quot}_d \to \mathsf{Quot}_d, \quad L \to C, \quad L^{[d]} \to \mathsf{Quot}_d$$

The Quot scheme comes equipped with a universal quotient $\mathcal{Q}$ over $C \times \mathsf{Quot}_d$ and a map $p^*E \twoheadrightarrow \mathcal{Q}$. We use $\mathcal{Q}$ (as a kernel) to obtain tautological bundles over Quot_d. For any line bundle L on C,

$$L^{[d]} := \pi_*(\mathcal{Q} \otimes p^*L).$$

Results

The tautological bundle $L^{[d]}$ is a vector bundle of rank d. For any vector bundle V, let

$$\wedge_y V = \sum_k y^k \wedge^k V.$$

Theorem 1. *For any line bundle $L \to C$,*

$$\sum_{d=0}^{\infty} q^d \chi(\mathsf{Quot}_d, \wedge_y L^{[d]}) = \frac{(1+qy)^{\chi(E\otimes L)}}{(1-q)^{\chi(\mathcal{O}_C)}}.$$

We obtain a slightly stronger result involving dual of several tautological inputs.

Theorem 2. *Over Quot_d, the Euler characteristics of $\wedge_y L^{[d]} \otimes_1^r (\wedge_{x_i} M_i^{[d]})^\vee$ equals*

$$[q^d] \frac{(1+qy)^{\chi(E\otimes L)}}{(1-q)^{\chi(\mathcal{O}_C)} \prod (1-x_i yq)^{\chi(L\otimes M_i^\vee)}},$$

where $M_1, \ldots, M_r$ and L are line bundles and $r < N$.

The nature of the formula provides several interesting vanishing results. For example,

- For any line bundle L and $0 < r < N$: $\chi(\mathsf{Quot}_d, (\det L^{[d]})^{-r}) = 0$.
- When genus $g > 1$ and $d > k + g$: $\chi(\mathsf{Quot}_d, \wedge^k L^{[d]}) = 0$.

We also consider the case of symmetric powers, and obtain a closed-form expression over $\mathbb{P}^1$.

Theorem 3. *For $C = \mathbb{P}^1$ and $d \geq k$,*

$$\chi(\mathsf{Quot}_d, \mathrm{Sym}^k L^{[d]}) = (-1)^k \binom{-\chi(E\otimes L)}{k}.$$

In principle, the universality argument (explained below) gives us an answer for any C. However, we only find a conjectural formula for genus 1.

Techniques

The proof can broadly be divided into three steps:

Universality: Using the arguments similar in spirit to [1], we show that there exists universal series A, B, and C such that

$$\sum_d q^d \chi(\mathsf{Quot}_d, \wedge_y L^{[d]}) = A^{\chi(\mathcal{O}_C)} B^{\deg L} C^{\deg E}.$$

Localization: Universality reduces the calculations to Quot scheme over $\mathbb{P}^1$, where the vector bundle E splits as a direct sum of line bundles. We use equivariant localization (using a torus action) to reduce the calculations to integrals on products of projective spaces.

Combinatorics: We use several combinatorial identities, such as Lagrange-Bürmann formula, to realize the expression as a Schur polynomial evaluated at roots of a polynomial with coefficients involving q and y. We then use Jacobi-Trudi identities to obtain explicit formulas.

Higher rank Quotients

Our techniques can be used to obtain the results for the Quot scheme $\mathsf{Quot}_d(E, r)$ parameterizing rank r quotients over $\mathbb{P}^1$.

Theorem 4. *Let E be trivial vector bundle, then*

$$\chi(\mathsf{Quot}_d(E,r), \det L^{[d]}) = u[q^d] s_\lambda(z_1, \ldots, z_N)$$

where u is a sign, s_λ is the Schur polynomials for the rectangular partition $\lambda = (d + \chi(L))^{N-r}$, and $z_1, \ldots, z_N$ are roots of the equation

$$(z-1)^N - z^{r-1} q = 0.$$

We use Jacobi-Trudi identities to obtain corollaries such as

$$\chi(\mathsf{Quot}_d(E,r), \det \mathcal{O}^{[d]}) = \binom{N}{r+d}.$$

Specializing to $d = 0$, we recover the known formulas for Grassmannian $G(N, r)$.

Analogies with Hilbert scheme of points on a surface

Our results suggest surprising analogy between punctual Quot scheme Quot_d and the Hilbert scheme of points $S^{[d]}$ for a projective surface S.

Exterior powers: For a line bundle $L \to S$, the tautological vector bundle $L^{[d]}$ is defined similarly. The Euler characteristics are given by

$$\sum_{d=0}^{\infty} q^d \chi(S^{[d]}, \wedge_y L^{[d]}) = \frac{(1+qy)^{\chi(L)}}{(1-q)^{\chi(\mathcal{O}_S)}}.$$

Note the striking similarity with the formula for Quot_d. This is proved either via derived category techniques in [4] or via studying Donaldson-Thomas theory of toric Calabi-Yau-3-folds in [3].

Question: Can our formulas for Quot_d be refined to isomorphism of cohomology groups of $\wedge^k L^{[d]}$ and direct sum of cohomology groups of bundles involving $E \otimes L$ and $\mathcal{O}_C$?

Symmetric powers: There are partial results for the symmetric powers of tautological bundles in [3]. The analogy between $\mathsf{Quot}_d(E)$ and $S^{[d]}$ persists! When $\chi(\mathcal{O}_S) = 1$ and $d \geq k$,

$$\chi(S^{[d]}, \mathrm{Sym}^k L^{[d]}) = (-1)^k \binom{-\chi(L)}{k}.$$

Rationality: The generating series involving these tautological bundles over Quot schemes of a surface [2] (or a curve) are rational functions.

***About our work**

I am presenting a joint work with **Dragos Oprea**. We were supported by the NSF through grant DMS 1802228.
Preprint: arXiv:2207.01675.

References

[1] M.Lehn G.Ellingsrud, L.Göttsche. On the cobordism class of the Hilbert scheme of a surface. 2001.
[2] N.Arbesfeld et al. The virtual K-theory of Qout scheme of surfaces. 2021.
[3] N.Arbesfeld. K-theoretic Donaldson-Thomas theory and Hilbert scheme of points on surface. 2021.
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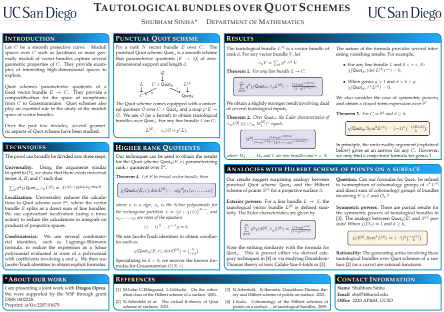
Punctual Quot scheme

- (Oprea-**S**'22): Explicit formula for the Euler characteristics of tautological bundles over punctual Quot scheme

- (Marian-Oprea-Sam'22): Refine the formulas to cohomology in the case of $\mathbb{P}^1$



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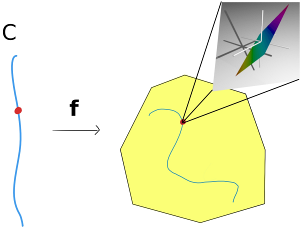
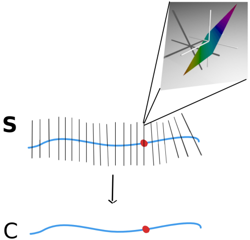
- $\mathbf{Quot}_d(N, r)$ is smooth.
- $\dim \mathbf{Quot}_d(N, r) = Nd + r(N - r)$
- When $d = 0$, then $\mathbf{Quot}_d(N, r) = Gr(N, r)$.

Quot scheme as Morphism Space

$\mathbf{Quot}_d(N, r)$ compactifies $Mor_d(\mathbb{P}^1, Gr(N, r))$!





Maps from C **to** $Gr(N, r)$

Subbundles of $S \subset \mathcal{O}_C^{\oplus N}$

Tautological sequence

Consider the tautological sequence for Quot scheme

$$0 \longrightarrow \mathcal{S} \longrightarrow pr^* \mathcal{O}_{\mathbb{P}^1}^{\oplus N} \longrightarrow \mathcal{Q} \longrightarrow 0$$

$$\begin{array}{ccc} & \downarrow & \\ & \mathbb{P}^1 \times \mathbf{Quot}_d(N, r) & \\ \swarrow^{pr} & & \searrow^{\pi} \\ \mathbb{P}^1 & & \mathbf{Quot}_d(N, r) \end{array}$$

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Remark

The cohomology ring of $\mathbf{Quot}_d(N,r)$ *is generated by Chern classes of*

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$$\mathcal{S}_x^\vee := \mathcal{S}^\vee \mid_{\{x\} \times \mathbf{Quot}_d(N,r)} \quad \text{and} \quad \pi_* \mathcal{S}^\vee .$$

Intersection numbers involving $c_i(\mathcal{S}_x^\vee)$ *is given by Vafa-Intriligator formula.*

History

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- (Buch-Mihalcea '09) Quantum K-theory of Grassmannian

Schur functors

Definition

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an integer partition and $V = \mathbb{C}^n$ (standard representation of $GL_n(\mathbb{C})$). The Schur functor $\mathbb{S}^\lambda$ associates $\mathbb{S}^\lambda(V)$, the unique irreducible representation of $GL_n(\mathbb{C})$ of highest weight λ.

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Example

- For $\lambda = (4)$, we have $\mathbb{S}^\lambda(V) = \mathrm{Sym}^4(V)$
- For $\lambda = (1, 1, 1)$, we have $\mathbb{S}^\lambda(V) = \wedge^3(V)$

Schur bundles on Grassmannian

Let $Gr(N,r)$ be the Grassmannian of rank r subspaces of $\mathbb{C}^N$ with tautological sequence

$$0 \to S \to \mathbb{C}^N \times Gr(N,r) \to Q \to 0.$$

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Proposition

For any partition λ with at most r parts,

$$H^i(Gr(N,r), \mathbb{S}^\lambda(S^\vee)) = \begin{cases} \mathbb{S}^\lambda((\mathbb{C}^N)^\vee) & i = 0 \\ 0 & i > 0. \end{cases}$$

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In particular,

$$\chi(Gr(N, r), \mathbb{S}^\lambda(S^\vee)) = s_\lambda(\underbrace{1, 1, \ldots, 1}_{N \text{ times}}).$$

Result

Recall $\mathcal{S}_x^\vee := \mathcal{S}^\vee|_{\{x\}\times \mathbf{Quot}_d(N,r)}$.

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Theorem (Zhang,S 23)

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where

$$\Lambda = (\lambda_1 + d, \lambda_2 + d, \ldots, \lambda_r + d)$$

and $z_1, z_2, \ldots, z_N$ are roots of

$$(z-1)^N + (-1)^r z^{N-r} q = 0.$$

Vanishing Results

Let $P^{r,N-r}$ be the set of partition contained in $(\underbrace{N-r,\ldots,N-r}_{r \text{ times}})$

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For any partition $\lambda \in P^{r,N-r}$,

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Theorem (Zhang,S 23)

For any partitions $\lambda, \mu \in P^{r,N-r}$ *and* $d > 0$,

$$\chi\big(\mathbf{Quot}_d(N,r), \det \mathcal{S}_x \otimes \mathbb{S}_{\lambda}(\mathcal{S}_x) \otimes \mathbb{S}_{\mu}(\mathcal{S}_x)\big) = 0.$$

Compare with genus 0, 3-pointed quantum K-invariants of Grassmannian:

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,3,d}$$

where $\alpha_1, \alpha_2, \alpha_3 \in K^0(Gr(N, r))$.

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Corollary

For any $F, G \in K^0(Gr(N, r))$ and $d > 0$,

$$\langle [\mathcal{O}_1], F, G \rangle_{0,3,d} = \langle F, G \rangle_{0,2,d}.$$

Thank you!