Algebraic geometry - 9

Last time:

Projective Nullstellensatz:

$\boxed{a}$ For any projective variety $X \subseteq \mathbb{P}^n$, we have

$$Z(I(X)) = X.$$

$\boxed{b}$ For any homogenous ideal $J \subseteq k[x_0, \dots, x_n]$ such that $\sqrt{J} \neq I_0$, we have

$$I(Z(J)) = \sqrt{J}.$$

$$\left\{ \begin{array}{c} \text{Projective alg sets} \\ \text{in } \mathbb{P}^n \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{homogenous radical} \\ \text{ideals } J \subseteq k[x_0, \dots, x_n] \\ \text{with } J = I_0 \end{array} \right\}$$

Proof of $\boxed{b}$. Let $X = Z(J)$, we saw last time that if $X \neq \phi$ (i.e. $J \neq \langle 1 \rangle$ and $\sqrt{J} \neq \langle x_0, \dots, x_n \rangle$), then

$$I(X) = I(C(X)) \subseteq k[x_0, \dots, x_n].$$

Here $C(X) = Z(J) \subseteq \mathbb{A}^{n+1}$.

By Hilbert nullstellensatz for $\mathbb{A}^{n+1}$,

$$I(C(X)) = \sqrt{J} \subseteq k[x_0, \dots, x_n].$$

If $X = \phi$ and $\sqrt{J} \neq \langle x_0, \dots, x_n \rangle$, then $Z(J) = \phi \subseteq \mathbb{A}^{n+1}$.

By Hilbert Nullstellensatz, $J = \langle 1 \rangle$ and hence

$$I(Z(\langle 1 \rangle)) = \langle 1 \rangle = k[x_0, \dots, x_n]$$

Def: Let $X \subseteq \mathbb{P}^n$ be a projective variety, the homogenous coordinate ring of X is

$$S(X) = k[x_0, \dots, x_n] / I(X).$$

Rem: $\boxed{1}$ $k[x_0, \dots, x_n]$ is a graded ring and $I(X)$ is a homogeneous ideal, thus $S(X)$ is also a graded ring. We denote

$$S(X) = S(X)_0 \oplus S(X)_1 \oplus \cdots \oplus S(X)_d \oplus \cdots$$

where $S(X)_d$ is degree d^{th} part of $S(X)$.

$\boxed{2}$ Elements of $S(X)$ does not define a function: Let $d \geq 1$, and Consider $f(x_0, x_1) = x_0^d$. Then f does not define a function on $\mathbb{P}^1$ because

$$f([a_0 : a_1]) = a_0^d \overset{\lambda \neq 1}{\neq} f(\lambda a_0, \lambda a_1) = \lambda^d a_0.$$

When $d = 0$, $f(x_0, x_1) = 1$ is a constant function.

Define: The Zariski topology on a projective variety $X \subseteq \mathbb{P}^n$: The set of closed sets equals

$$\{ Y \subseteq X \subseteq \mathbb{P}^n : Y \text{ is projective alg. set} \}$$
$$= \{ Y = Z(J) \text{ for homogenous ideal } J \subseteq S(X) \}$$

Recall $\mathbb{P}^n$ has affine chart $U_0 = \{ [a_0 : \cdots : a_n] \in \mathbb{P}^n \mid a_0 \neq 0 \} \cong \mathbb{A}^n$.

We will understand how proj varieties in $\mathbb{P}^n$ restricts to U_0.

Def: Let $f \in k[x_0, \dots, x_n]$ be a homogeneous polynomial. The <u>dehomogenization</u> of f (at x_0) is the polynomial

$$\tilde{f} := f(x_0 = 1) \in k[x_1, \dots, x_n].$$

Let $J \subseteq k[x_0, \dots, x_n]$ be homogenous ideal, then $\tilde{J} := \{\tilde{f} : f \in J\} \subseteq k[x_1, \dots, x_n]$ is an ideal.

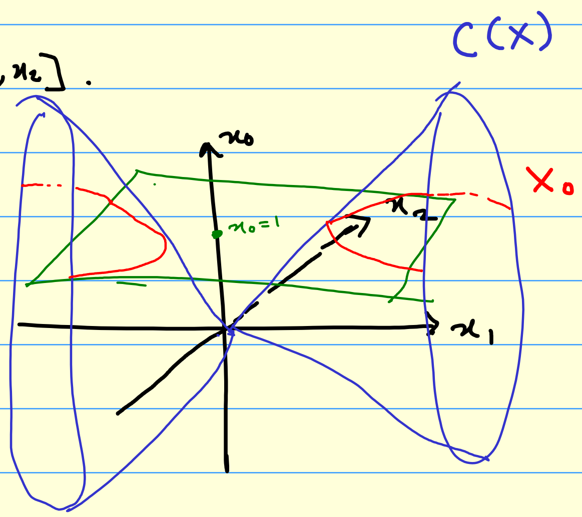
Eg: $f(x_0, x_1) = x_0^2 + x_0 x_1 - 3x_1^2$, $\tilde{f} = 1 + x_1 - 3x_1^2$.

Rem: Let $J \subseteq k[x_0, \dots, x_n]$ be a homogeneous ideal, and $X = Z(J) \subseteq \mathbb{P}^n$. Then

$$Z(\tilde{J}) = X_0 := X \cap U_0 \subseteq \mathbb{A}^n.$$

Eg: $J = \langle x_0^2 - x_1^2 + x_2^2 \rangle \subseteq k[x_0, x_1, x_2]$.

Then $X_0 \cong C(X) \cap Z(x_0 - 1) \subseteq \mathbb{A}^{n+1}$



Def: Let $g \in k[x_1, \ldots, x_n]$ and let $d = \deg(g)$ (max deg of monomials), then we define the homogenization of g by

$$g^h := x_0^d \, g\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in k[x_0, \ldots, x_n].$$

Rem:

$$g = x_1 + x_2^2 \Rightarrow g^h = x_0 x_1 + x_2^2$$
$$f = -x_2^2 \qquad f^h = -x_2^2$$

However,

$$x_0 x_1 = g^h + f^h \neq (g+f)^h = (x_1)^h = x_1 .$$

We define homoginization of ideal as the ideal gen. by all homogeneous ideals,

$$J^h := \langle g^h : g \in J \rangle \subseteq k[x_0, \ldots, x_n].$$

What is $Z(J^h) \subseteq \mathbb{P}^n$?

Proposition: Let $J \subseteq k[x_1, \ldots, x_n]$ be an ideal, and $X = Z(J) \subseteq \mathbb{A}^n$. The Zariski closure of $X \subseteq \mathbb{A}^n = U_0 \subseteq \mathbb{P}^n$ is denoted $\overline{X}$ (we call it projective closure). Then

$$\overline{X} = Z(J^h).$$

Proof: "$\supseteq$" Note that $X^h := Z(J^h)$ is a closed set in $\mathbb{P}^n$, and contains X. Hence $\bar{X} \subseteq X^h$.

"$\subseteq$" Using Projective Nullstellensatz, it is enoug to show that $I(Z(J^h)) \subseteq I(\overline{X})$.

Let $f \in I(Z(J^h)) = \sqrt{J^h}$ be a homogenous element, then

$f^m \in J^h \Rightarrow f^m$ is zero on X

$\Rightarrow f^m$ is zero on $\bar{X}$ ($Z(f^m) \subseteq \mathbb{P}^n$ closed)

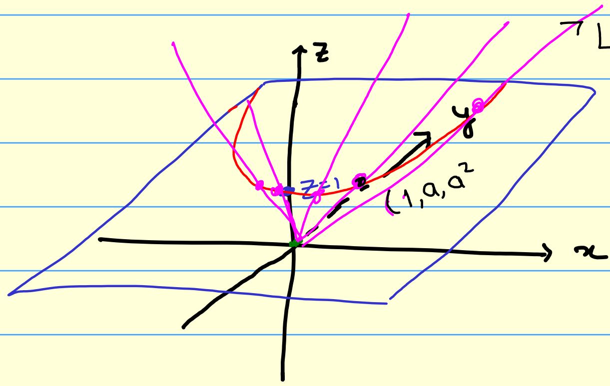
$\Rightarrow f^m \in I(\bar{X})$

$\Rightarrow f \in I(\bar{X})$ (Since $I(\bar{X})$ is radical).

$\square$.

Eg: Projective closure of $X = Z(y - x^2) \subseteq \mathbb{A}^2$ in $\mathbb{P}^2$.

Let $\mathbb{P}^2 = \{[z,x,y]\}$ (note unusual coordinate order!)



$L_a = \{(\lambda, \lambda a, \lambda a^2) : \lambda \in k\}$

$= \{(\frac{\lambda}{a^2}, \frac{\lambda}{a}, \lambda) : \lambda \in k\}$ if $a \neq 0$.

$\lim_{a \to \infty} L_a = \{(0,0,\lambda) : \lambda \in k\}$

$=: L_\infty$

Draw the affine variety $X = Z(y - x^2)$ as $Z(yz - x^2) \cap Z(z-1) \subseteq \mathbb{A}^3$.

Then the cone of $\overline{X}$ consits of all lines passing through points in X. In particular

$$C(\overline{X}) = \bigcup_{a \in k} L_a \cup L_\infty$$

and

$$\overline{X} = \underbrace{\{[1:a:a^2] \mid a \in k\}}_{X \subseteq U_0 \cong \mathbb{A}^2} \cup \{[0:0:1]\} \subseteq \mathbb{P}^2$$