Algebraic Geometry 8

Graded k-algebras and rings

Def. Let R be a ring (or k-alg.). We say R is graded with graded abelian subgroup $R_d \subseteq R$ $\forall$ $d \in \mathbb{N}$ such that
(k-vectorspace)

- $R = \bigoplus_{d \in \mathbb{N}} R_d$. In particular, for every element $f \in R$, $f = \sum_{d \in \mathbb{N}} f^{(d)}$ with finitely non-zero $f^{(d)}$ and $f^{(d)} \in R_d$.

- If $f \in R_d$, $g \in R_e$, then $f \cdot g \in R_{d+e}$.

Eg: $\boxed{\text{I}}$ $R = k[x_0, \dots, x_n]$, $R_d = \bigoplus_{m_0 + \dots + m_n = d} k \cdot x_0^{m_0} \cdot \dots \cdot x_n^{m_n}$

$\dim R_d$ (as k-vectorspace) $=$

$\boxed{\text{II}}$ $R = k[x,y]/\langle xy \rangle$ is graded.

Lemma: Let I, J be ideals in a graded ring R.

$\boxed{a}$ I is homogenous iff I is generated by homogenous elements.

$\boxed{b}$ If I and J are homogenous, then so are $I+J$, $I \cdot J$, $I \cap J$ and $\sqrt{I}$.

$\boxed{c}$ If I is homogenous then R/J is a graded ring with grading given by

$$R/J = \bigoplus_{d \in \mathbb{N}} R_d / R_d \cap I.$$

We will assume R is Noetherian (not required!).

<u>Proof:</u> $\boxed{a}$ Suppose $I = \langle h_1, \ldots, h_m \rangle$, with $h_i \in R^{(d_i)}$ for $i=1,2,\ldots,m$.

Any element $f \in R$ can be expressed as

$$f = \sum_{i=1}^{m} g_i h_i \quad , g_i \in R \quad \text{(not necessarily homog.)}$$

Let $g_i = \sum_{j=0}^{e_j} g_i^{(j)}$, $\quad g_i^{(j)} \in R_j$.

Then $\quad f = \sum_{i=1}^{m} \sum_{j=0}^{e_j} g_i^{(j)} \cdot h_i$.

Note that $\quad f^{(d)} = \sum_{i=1}^{m} g_i^{(d-d_i)} h_i \in I$.

Suppose for every $f \in I$, $f^{(d)} \in I$. Let

$I = \langle h_1, \ldots, h_m \rangle$ (since R is Noetherian)

Let $h_i = h_i^{(0)} + \cdots + h_i^{(d_i)}$.

Then $h_j^{(j)} \in I$ for all $i = 1,2,\ldots,m$ and all $j \geq 0$.

Hence $\quad I = \langle h_1^{(0)}, \ldots, h_1^{(d_1)}, \ldots, h_m^{(0)}, \ldots, h_m^{(d_1)} \rangle$.

$\boxed{b}$ We will only show $\sqrt{I}$ is homogenous. We shall use induction on degree d, where

$$f = f^{(0)} + \cdots + f^{(d)}.$$

Assume $f \in \sqrt{I}$, then there exists $n \in \mathbb{N}$ such that

$$f^n = (f^{(0)} + \cdots + f^{(d)})^n \in I$$

$$= (f^{(d)})^n + \underbrace{\sum_{i=1}^{n} \binom{n}{i} (f^{(d)})^{n-i} (f^{(0)} + \cdots + f^{(d-1)})^i}_{\text{lower degree terms!}}$$

Thus, $(f^{(d)})^n \in I \Rightarrow f^{(d)} \in \sqrt{I}$.

$$\Rightarrow f - f^{(d)} \in \sqrt{I}$$

Note $f - f^{(d)} = \underbrace{f^{(0)} + \cdots + f^{(d-1)}} \in \sqrt{I}$, we use induction hypothesis to argue $f^{(j)} \in \sqrt{I}$ $\forall$ $i = 0, \ldots, d-1$.

$\square$

$\boxed{c}$ Note that $R_d/_{R_d \cap I} \to R/_I$ is an injective group homomorphism.

$$\bar{f} \longrightarrow \bar{f}$$

Thus $R_d/_{R_d \cap I} \subseteq R/_I$ is an abelian group / k-vector space.

Let $\bar{f} \in R/_I$ and $f \in R$ be the representative of $\bar{f}$.
Consider the decomposition

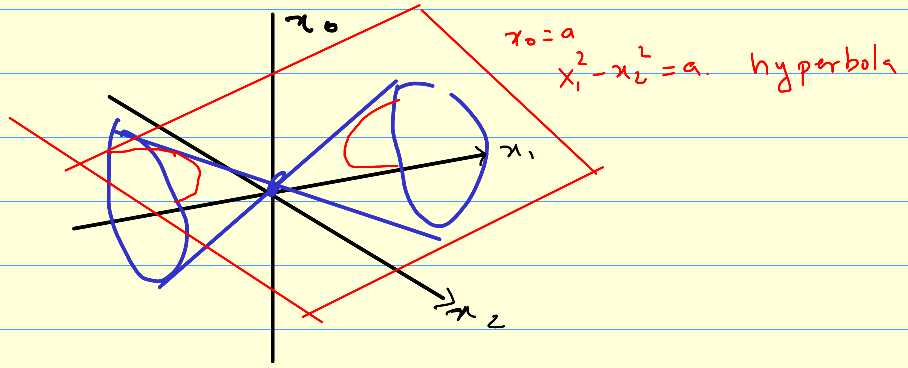
$$f = f^{(0)} + \cdots + f^{(d)}, \qquad f^{(j)} \in R_j.$$

Note that $\bar{f} = \overline{f^{(0)}} + \cdots + \overline{f^{(d)}} \in R/_I$. and $\overline{f^{(j)}} \in R_d/_{R_d \cap I}$.

The decomposition above is unique: Suppose $\bar{f} = \sum_{j=0}^{d} \overline{g^{(j)}}$ and

$g^{(d)} \in R$ be representative. Then $f^{(j)} - g^{(j)} \in I \Rightarrow \overline{f^{(j)}} = \overline{g^{(j)}}$.

Eg: Let $f = x_1^2 - x_2^2 - x_0^2 \in k[x_0, x_1, x_2]$. What is the affine algebraic set $Z(f) \subseteq \mathbb{A}^3$?



Note that if $p = (a_0, a_1, a_2) \in Z(f) \Rightarrow \lambda \cdot p = (\lambda a_0, \lambda a_1, \lambda a_2) \in Z(f)$.
for all $\lambda \in k$.

How are $Z(f) \subseteq \mathbb{A}^3$ and $Z(f) \subseteq \mathbb{P}^2$ related?
Cone!

Def: Let $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the projection.
$(x_0, \dots, x_n) \to [x_0 : \dots : x_n]$

$\boxed{a}$ An affine variety $X \subseteq \mathbb{A}^{n+1}$ is called a cone if $0 \in X$ and for every point $p \in X$, $\lambda \cdot p \in X$.

$\boxed{b}$ For a cone X, we call

$$\mathbb{P}(X) = \pi(X \setminus \{0\}) = \{[x_0 : \dots : x_n] \in \mathbb{P}^n \mid (x_0, \dots, x_n) \in X \setminus \{0\}\} \subseteq \mathbb{P}^n$$

the projectivization of X.

$\boxed{c}$ For a projective variety $X \subseteq \mathbb{P}^n$, the cone of X is

$$C(X) = \{0\} \cup \pi^{-1}(X) = \{0\} \cup \{(x_0, \dots, x_n) \mid [x_0 : \dots : x_n] \in X\} \subseteq \mathbb{A}^{n+1}.$$

Rem: $\boxed{a}$ Let $S \subseteq k[x_0, \dots, x_n]$ be a set of homogeneous polynomials, then $Z(S) \subseteq \mathbb{A}^{n+1}$ is a cone: If $p \in Z(S)$, then $\lambda \cdot p \in Z(S)$

Since $f(\lambda \cdot p) = \lambda^{\deg f} f(p) = 0$ for any $f \in S$.

$\boxed{b}$ Let $X \subseteq \mathbb{A}^{n+1}$ be a cone, then $I(X)$ is a homogeneous ideal.

Let $f \in I(X)$ and let $f = f^{(0)} + \dots + f^{(d)}$ be homogeneous decomposition.

For all $p \in X$ and $\lambda \in k$, we have $\lambda \cdot p \in X$ (X is cone). Thus

$$0 = f(\lambda \cdot p) = \sum_{j=0}^{d} \lambda^j f^{(j)}(p). \quad \forall \lambda \in k.$$

This implies $f^{(j)}(p) = 0$ for all $j = 0, \dots, d$.

Hence $f^{(j)} \in I(X)$.

Def: Let $X \subseteq \mathbb{P}^n$ be a subset. The ideal of X is defined to be

$$I(X) := I(C(X)) = \{ f \in k[x_0,..,x_n] : f(p) = 0 \ \forall p \in C(X) \} \subseteq k[x_0,..,x_n].$$

By remark above, $I(X)$ is a homogenous ideal.

Lemma: There is a bijection

$$\{\text{Cones in } \mathbb{A}^{n+1}\} \longleftrightarrow \{\text{Projective alg. sets in } \mathbb{P}^n\}$$

$$\hat{X} \longrightarrow \mathbb{P}(\hat{X})$$

$$C(X) \longleftarrow X$$

Proof: Check that for any $S \subseteq k[x_0,..,x_n]$ of non-constant homogenous polynomials,

$$C(\mathbb{P}(\hat{X})) = \hat{X} \quad \text{and} \quad \mathbb{P}(C(X)) = X .$$

Question: Is there a bijection between homogenous ideals and projective varieties?

(Irrelevant ideal)

Consider $I_0 = \langle x_0, .., x_n \rangle \subseteq k[x_0,..,x_n]$. Note that

$$Z(I_0) = \{0\} \subseteq \mathbb{A}^{n+1} \quad \text{and} \quad Z(I_0) = \phi \subseteq \mathbb{P}^n$$

However, $Z(1) = \phi \subseteq \mathbb{P}^n$.

Projective Nullstellensatz:

a) For any projective variety $X \subseteq \mathbb{P}^n$, we have

$$Z(I(X)) = X.$$

b) For any homogenous ideal $J \subseteq k[x_0, \dots, x_n]$ such that $\sqrt{J} \neq I_0$, we have

$$I(Z(J)) = \sqrt{J}.$$

$$\left\{ \begin{array}{l} \text{Projective alg sets} \\ \text{in } \mathbb{P}^n \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{homogenous radical} \\ \text{ideals } J \subseteq k[x_0, \dots, x_n] \\ \text{with } J = I_0 \end{array} \right\}$$