Algebraic Geometry 7

Projective space: Lines in Affine space $\mathbb{A}^{n+1}$ passing through origin.

<u>Def</u>: We define an equivalence relation on $k^{n+1}\setminus\{0\}$ by

$$(a_0,\dots,a_n) \sim (\lambda a_0,\dots,\lambda a_n), \text{ for all } \lambda \in k^* = k\setminus\{0\}.$$

The projective space as a set

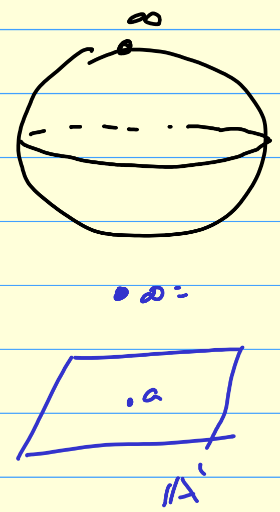
$$\mathbb{P}^n := \left(k^{n+1}\setminus\{0\}\right)/\sim$$

with the quotient topology. We denote the representatives of points in $\mathbb{P}^n$ by

$$[a_0:a_1:\dots:a_n] = \text{equivalence class of } \{(\lambda a_0,\dots,\lambda a_n) : \lambda \in k^*\}.$$

<u>Eg</u>: $k=\mathbb{C}$, $\mathbb{P}^1 = \underbrace{\{[1:a], a\in k\}}_{\cong\, \mathbb{A}^1} \cup \underbrace{\{[0:1]\}}_{=\,\infty}$

$$i : \mathbb{A}^1 \to \mathbb{P}^1$$
$$a \to [1:a]$$



<u>Affine charts</u>: Let $U_i := \{[a_0:\dots:a_n] \in \mathbb{P}^n \mid a_i \neq 0\}$ for $i=0,1,\dots,n$.

We have a bijection

$$\varphi_i : U_i \to \mathbb{A}^n$$
$$[a_0:\dots:a_n] \to \left(\frac{a_0}{a_i},\dots,\boxed{\frac{a_i}{a_i}},\dots,\frac{a_n}{a_i}\right)$$

remove this element! $\frac{a_i}{a_i} = 1$

Inverse is given by

$$\varphi_i^{-1} : \mathbb{A}^n \longrightarrow U_i$$

$$\underbrace{(b_0, \dots, b_{i-1}, b_{i+1}, \dots, b_n)}_{n \text{ coordinates}} \longrightarrow [b_0 : \dots : b_{i-1} : 1 : b_i : \dots : b_n]$$

We call $\left(\frac{a_0}{a_i}, \dots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \dots, \frac{a_n}{a_i}\right)$ the affine coordinates

of $p = [a_0 : \dots : a_n] \in U_i$ with respect to U_i.

Show that $\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \dots \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^0 = \{pt\}$!

<u>Algebraic set in $\mathbb{P}^n$</u> :

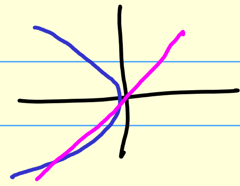
<u>Def</u>: A homogeneous polymial is an element $f \in k[x_0, \dots, x_n]$ such that f can be expressed as a sum of monomials of equal degrees: that is,

$$f = \sum_{m_0 + \dots + m_n = d} a_{m_0, \dots, m_n} x_0^{m_0} x_1^{m_1} \cdots x_n^{m_n},$$

and d is called the homogeneous degree of f.

<u>Rem</u>: If f is a homogeneous poly. of degree d, then

$$f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n) \qquad \forall \lambda \in k.$$

Eg: $f(x,y) = x + y^2$ is not homogenous. 

$f(-1,-1) = 0$ but $f(1,1) \neq 0$.

Def: Let $f \in k[x_0, \dots, x_n]$ be a homogenous poly. of degree d,

$$Z(f) = \{ p \in \mathbb{P}^n \mid f(p) = 0 \}.$$

(deg d hypersurface in $\mathbb{P}^n$)

Eg: $Z(0) = \mathbb{P}^n$ and $Z(1) = \phi$.

Def: Let S be a set of homogenous polynomials,

$$Z(S) = \{ p \in \mathbb{P}^n \mid f(p) = 0 \quad \forall f \in S \}.$$

We call $Z(S)$ Projective algebraic set.

Eg: Projective alg. sets in $\mathbb{P}^1$ = $\{ \phi, \mathbb{P}^1, \text{finite subsets} \}$

Note that it is enoug to show that for any non-constant polynomial $f \in k[x,y]$, $Z(f)$ is a finite set.

Let $f(x,y) = a_d y^d + a_{d-1} y^{d-1} x + \dots + x^d \in k[x,y]$

Case 1: Suppose $a_d \neq 0$. WLOG $a_d = 1$.

$$\begin{aligned} f(x,y) &= y^d + a_{d-1} y^{d-1} x + \dots + a_0 x^d \\ &= x^d \left((y/x)^d + a_{d-1} (y/x)^{d-1} + \dots + a_0 \right) \\ &= x^d \prod_{i=1}^{d} (y/x - \alpha_i) \qquad (k \text{ is alg. closed}) \\ &= \prod_{i=1}^{d} (y - \alpha_i x) \end{aligned}$$

$$\Rightarrow Z(f) = \bigcup_{i=1}^{d} Z(y - \alpha_i x) = \{ [1:\alpha_1], \dots, [1:\alpha_d] \} \subseteq \mathbb{P}^1.$$

Case 2: $a_d = 0$. We may write

$$f(x,y) = x^m \left(b_{d-m} y^{d-m} + \cdots + b_0 x^{d-m} \right), \quad m \geq 1.$$

$$\begin{aligned} Z(f) &= Z(x^m) \cup Z(b_{d-m} y^{d-m} + \cdots + b_0 x^{d-m}) \\ &= \{\underset{\infty}{\underset{\shortparallel}{[0:1]}}\} \cup \{\text{finite set by case 1}\} \subseteq \mathbb{P}^1. \end{aligned}$$

Question: Are $\mathbb{A}^1$ and $\mathbb{P}^1$ (Zariski) homeomorphic?

Def: Any polynomial $f \in k[x_0, \ldots, x_n]$ can be written in a unique way

$$f = f^{(0)} + f^{(1)} + \cdots + f^{(d)}$$

where $f^{(m)}$ is homogenous polynomial of degree m.

Def: An ideal $I \subseteq k[x_0, \ldots, x_n]$ is called a homogenous ideal if for every $f \in I$, all homogenous polynomials $f^{(i)} \in I \quad \forall\, i = 0, 1, \ldots, d$.

Prop: An ideal $I \in k[x_0, \ldots, x_n]$ is homogenous if and only if $I = \langle f_1, \ldots, f_m \rangle$ where f_j is a homogenous polynomial.

Proof: HW

Eg: $I = \langle x, y \rangle \subseteq k[x,y]$ what is $Z(I) \subseteq \mathbb{P}^1$?

We would like to formulate Hilbert Nullstellensatz for projective space.