Algebraic Geometry 6

Theorem: For any two affine varieties X and Y there is a bijection

$$\{\text{morphisms } X \to Y\} \overset{1:1}{\longleftrightarrow} \{k\text{-algebra homomorphisms } A(Y) \to A(X)\}$$

$$\varphi \longrightarrow \varphi^*$$

Proof: If $\varphi : X \to Y$, then $\varphi^* : \mathcal{O}_Y(Y) = A(Y) \longrightarrow \mathcal{O}_X(X) = A(X)$ is a k-algebra homomorphism.

Let $\Phi : A(Y) \longrightarrow A(X)$ be a k-alg. homomorphism. We will construct a morphism $\varphi : X \to Y$ such that $\varphi^* = \Phi$.

$Y \subseteq \mathbb{A}^n$ is closed. Let $y_1, \dots, y_n \in A(Y)$ restriction of coordinate element in $A(\mathbb{A}^n)$. Define

$$f_i = \Phi(y_i) \in A(X)$$

Then consider the map

$$\varphi := (f_1, \dots, f_n) : X \longrightarrow \mathbb{A}^n .$$

- $\varphi(X) \subseteq Y$. Indeed, let $h \in I(Y)$ then

$$h \circ \varphi = h(f_1, \dots, f_n) = \Phi(h(y_1, \dots, y_n)) = 0$$

since $h(y_1, \dots, y_n) = 0$ in $A(Y)$. Thus

$$\varphi(X) \subseteq Z(I(Y)) = Y .$$

- φ is a morphism. Next lemma:

Lemma: Let X and Y be affine varieties, and $U \subseteq X$ open subset. Then $\varphi: U \to Y$ is a morphism if and only if

$$\varphi = (f_1, .., f_n) : U \longrightarrow Y \subseteq \mathbb{A}^n$$

where $f_1, .., f_n \in \mathcal{O}_X(U)$.

Proof: Assume φ is a morphism. Let $f_1, .., f_n \in \mathcal{O}_X(U)$ as $\varphi^* y_1, .., \varphi^* y_n$, where $y_1, .., y_n$ are coordinate elements in $A(Y) \subseteq A(\mathbb{A}^n)$.

Then, $\varphi = (f_1, .., f_n)$ by definition of φ^*: For any $p \in U$, $\varphi(p) = (b_1, .., b_n) \in Y \subseteq \mathbb{A}^n$, then $b_i = y_i(\varphi(p)) = f_i(p)$.

Assume $\varphi = (f_1, .., f_n)$, with $f_i \in \mathcal{O}_X(U)$.

- φ is continuous: Let $W \subseteq Y$ closed subset, hence

$$W = Z(g_1, .., g_m) \text{ for some } g_1, .., g_m \in A(Y)$$

Note that

$$f^{-1}(W) = \{x \in U : g_i(f_1(x), .., f_n(x)) = 0 \ \forall\ i = 1, 2, .., m\}$$

Each function $x \to g_i(f_1(x), .., f_n(x))$ is regular on U, since $g_i \in A(Y)$ is expressed as poly, and $\mathcal{O}_X(U)$ is a k-algebra.

Thus,

$$f^{-1}(W) = \bigcap_{i=1}^{m} \{x \in U : g_i(f_1(x), \dots, f_n(x))\} \text{ is closed.}$$

- Let $V \subseteq Y$ open and $h \in \mathcal{O}_Y(V)$, then $\varphi^* h \in \mathcal{O}_X(\varphi^{-1}(V))$: Note that

$$\varphi^* h = h \circ \varphi : \varphi^{-1}(V) \longrightarrow k$$
$$x \longrightarrow h(f_1(x), .., f_n(x))$$

If h is locally a quotient $h = \frac{P}{Q}$, (on some open set $\tilde{V} \subseteq V$) then so does

$$\varphi^* h = \frac{P(f_1, .., f_n)}{Q(f_1, .., f_n)} \quad \text{on } \varphi^{-1}(\tilde{V}).$$

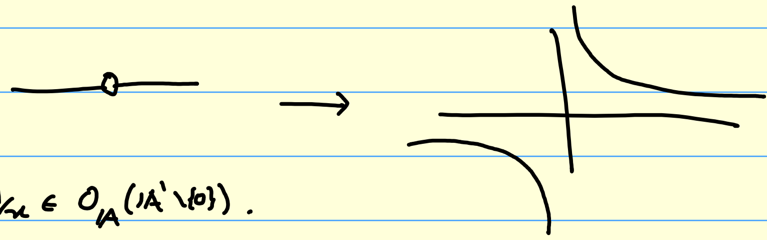
(Check $Q(f_1, .., f_n)$ has no zeros on $\varphi^{-1}(\tilde{V})$)

$\square$.

Def: An affine variety is a ringed space $X, \mathcal{O}_X$ that is isomorphic to an affine variety (closed subsets in $\mathbb{A}^n$).

Eg: $\mathbb{A}^1 \setminus \{0\} \cong Z(x \cdot y - 1) \subseteq \mathbb{A}^2$.

$$(x) \xrightarrow{\varphi} (x, \tfrac{1}{x})$$



$\boxed{\mathrm{I}}$ φ is a morphism, since $x, 1/x \in \mathcal{O}_{\mathbb{A}}(\mathbb{A}^1 \setminus \{0\})$.

$\varphi^{-1}: Z(xy-1) \to \mathbb{A}^1 \setminus \{0\}$ is also a morphism of ringed spaces.
$(x, y) \to (x)$

$\Rightarrow$ φ is an isomorphism.

Def: Let R be a finitely generated k-algebra. We say R is reduced if R has no nil-potent elements.

Prop: R is reduced fin. gen. k-algebra if and only if $R \cong A(X)$ for an affine variety X.

Proof: $\Rightarrow$ Let $x_1, \ldots, x_n$ be gen of R as k-algebra. Then, we have a surjective homomorphism

$$g : k[x_1, \ldots, x_n] \twoheadrightarrow R.$$

Let $I = \ker(g)$. Note that if $f^r \in I$ for some $f \in k[x_1, \ldots, x_n]$, then $\tilde{f}^r = 0$ in $k[x_1, \ldots, x_r]/I \cong R$. ($\tilde{f}$ represents f in the quotient ring).

Since R is reduced,

$$\tilde{f} = 0 \text{ in } R \Leftrightarrow f \in I.$$

Hence I is radical and hence for $X = Z(I)$,

$$R = A(X).$$

($\Leftarrow$) Try to do it yourselves!

$$\{\text{affine varieties}\}/_{\text{isomorphism}} \xleftrightarrow{1:1} \{\text{fin. gen. reduced } k\text{-algebra}\}/_{\text{isomorphism}}$$

<u>Prop</u>: Dinstinguished open sets $D(f) \subseteq X$, for any $f \in A(X)$, are affine varieties themselves.

<u>Proof</u>: $X \subseteq \mathbb{A}^n$, and consider $A(\mathbb{A}^n) = k[x_1, \dots, x_n]$, Let x_{n+1} be new variable. Consider

$$Z := Z\left(I(X) + \langle f x_{n+1} - 1 \rangle\right) \subseteq \mathbb{A}^{n+1}.$$

We want to show that the $Z \subseteq \mathbb{A}^{n+1}$ is affine variety isomorphic to $D(f)$ as ringed spaces.

- Let
$$\varphi : D(f) \longrightarrow Z \subseteq \mathbb{A}^{n+1}$$
$$(x_1, \dots, x_n) \longrightarrow \left(x_1, \dots, x_n, \tfrac{1}{f}\right)$$

  f is non-zero on $D(f)$, and $x_1, \dots, x_n, \frac{1}{f} \in \mathcal{O}_X(D(f))$. By Lemma, φ is a morphism.

- φ is bijective, with inverse given by $(x_1, \dots, x_n, x_{n+1}) \xrightarrow{\varphi^{-1}} (x_1, \dots, x_n)$.

- Note that φ^{-1} is indeed a morphism!

<u>HW</u>: Show that $A(D(f)) = A(X)_f$.

Eg: $\mathbb{A}^2 \setminus \{0\}$ is not an affine variety.

Suppose $\mathbb{A}^2 \setminus \{0\}$ is an affine variety. Since $\mathcal{O}_{\mathbb{A}^2\setminus\{0\}}(\mathbb{A}^2\setminus\{0\}) \overset{HW2}{\cong} k[s,t]$.

Consider the injection

$$\varphi : \mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1.$$

Note that $\varphi^* : A(\mathbb{A}^1) \to A(\mathbb{A}^1 \setminus \{0\})$ is an isomorphism.
By applying Theorem (pg 1 of this file), if $\mathbb{A}^1 \setminus \{0\}$ is an affine variety,

$\varphi : \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ is an isomorphism.

(but clearly it is not surjective).