Algebraic Geometry 5

Recall: $D(f) \subseteq X$ distinguished open sets.

$\boxed{D(f) = X \setminus Z(f)}$.

Lemma: Regular functions on $D(f)$ are global quotients, i.e.,

$$\mathcal{O}_X(D(f)) \cong \left\{ \frac{g}{f^r} \;:\; g \in A(X),\; r \in \mathbb{N} \right\}.$$

In particular, $\mathcal{O}_X(X) \cong A(X)$.

Proof: "$\supseteq$": Every global quotient $\frac{g}{f^r}$ is regular on $D(f)$.

"$\subseteq$" Let $\varphi : D(f) \to k$ be a regular function.

For any $a \in D(f)$, there exists an open set U_a and $f_a, g_a \in A(X)$ such that

$$\varphi\big|_{U_a} = \frac{g_a}{f_a} \quad \text{on } U_a.$$

After shrinking U_a, we may assume $U_a = D(h_a)$ a distinguished open.

By replacing $\frac{g_a}{f_a} = \frac{g_a \cdot h_a}{f_a \cdot h_a}$. $\boxed{\text{new } f_a, g_a}$, we may assume

$(*)$ $\boxed{f_a, g_a \in \langle h_a \rangle}$.

Since both h_a, f_a vanish on $Z(h_a)$ and not on $D(h_a)$,

we may assume $\boxed{h_a = f_a}$ and $U_a = D(f_a)$.
$(**)$

For any two points $a, b \in D(f)$,

$$g_a \cdot f_b = g_b \cdot f_a \quad \text{on } D(f_a) \cap D(f_b).$$

$\left(\varphi = \frac{g_a}{f_a} = \frac{g_b}{f_b} \text{ when restricted to } D(f_a) \cap D(f_b) \right)$

By $(*)$ and $(**)$ $g_a f_b = g_b f_a = 0$ on $Z(f_a) \cup Z(f_b)$

Hence,

$(\dagger)$ $\boxed{g_a f_b = g_b f_a \text{ on } D(f)} = D(f_a) \cap D(f_b) \cup Z(f_a) \cup Z(f_b).$

Now, $D(f) = \bigcup_{a \in D(f)} D(f_a)$ implies

$$Z(f) = \bigcap_{a \in D(f)} Z(f_a) = Z(\{f_a : a \in D(f)\}).$$

$$= Z(\langle f_1, \dots, f_\ell \rangle)$$

(By Noetherian property only finitely many points $\{a_1, \dots, a_\ell\} \in D(f)$.)

$\boxed{\begin{array}{l} f_i = f_{a_i} \\ g_i = g_{a_i} \end{array}}$

By relative Hilbert Nullstellensatz,

$$f \in I(Z(f)) = \sqrt{\langle f_1, \dots, f_\ell \rangle}.$$

This means $f^r = \sum_{i=1}^{\ell} b_i \cdot f_i$ where $b_i \in A(X)$

Set $g := \sum_{i=1}^{\ell} b_i g_i$.

Then we claim $\varphi = \frac{g}{f^r}$ on $D(f)$:

For all $a \in D(f)$, we have

$\varphi = \frac{g_a}{f_a}$ on $D(f_a)$, and

$$g f_a = \sum_{i=1}^{\ell} b_i g_i f_a \overset{(\dagger)}{=} \sum_{i=1}^{\ell} b_i g_a \cdot f_i = g_a f^r$$

$\Rightarrow \varphi = \frac{g_a}{f_a} = \frac{g}{f^r}$ on $D(f_a)$.

$\square$

<u>Prop</u>: (HW) $\mathcal{O}_X(D(f)) \cong A(X)_f$.

Morphism of Affine Varieties

What should be a "morphism" from affine variety X to Y?

Def: (Polynomial maps) Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties. A map $\varphi : X \to Y$ is a polynomial map if
$$\varphi(p) = (f_1(p), \dots, f_m(p))$$
where $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ or equivalently $f_1, \dots, f_m \in A(X)$.

Eg: Let $C = Z(y - x^2) \subseteq \mathbb{A}^2$. Consider the polynomial maps
$$\varphi : \mathbb{A}^1 \to C \qquad \text{and} \qquad \psi : C \to \mathbb{A}^1 .$$
$$t \to (t, t^2) \qquad\qquad (x, y) \to (x)$$
Observe $\varphi \circ \psi = id_C$ and $\psi \circ \varphi = id_{\mathbb{A}^1}$. (Isomorphism!)

Polynomial maps are compatible with polynomial functions:

Def: Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties and φ be a polynomial map. The pullback of $h \in A(Y)$ is
$$\varphi^*(h) := h \circ \varphi \in A(X).$$

Claim: $\varphi^*(h) \in A(X)$

Proof: If $h \in A(Y)$, let $h = \tilde{h}|_Y$ for $\tilde{h} \in k[y_1, \dots, y_m]$.
$$\varphi^*(h) = \tilde{h}(f_1, f_2, \dots, f_m)\big|_X \quad \text{where} \quad \varphi = (f_1, \dots, f_m).$$
Hence $\varphi^*(h) \in A(X)$.

Rem:
- $\varphi^*(f+g) = \varphi^*(f) + \varphi^*(g)$
- $\varphi^*(f \cdot g) = \varphi^*(f) \cdot \varphi^*(g)$.

$\Rightarrow$ $\varphi^* : A(Y) \to A(X)$ is a ring homomorphism.

Morphisms of affine varieties (works more generally!)

Def: A ringed space is a topological space X with a sheaf of ring $\mathcal{O}_X$ on X.

Def: Let X, Y be $\boxed{\text{affine varieties}}$ or ringed space. and $\mathcal{O}_X, \mathcal{O}_Y$ are sheaf of regular functions. A map

$$\varphi : X \to Y$$

is called a morphism of affine varieties if:

$\boxed{a}$ φ is continuous (Zariski).

$\boxed{b}$ φ preserves regular functions:

For any open $U \subseteq Y$ and regular function $f \in \mathcal{O}_Y(U)$,

$$\varphi^* f := f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(U))$$

is regular function on open set $\varphi^{-1}(U)$.

Note that

(i) $\varphi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$ is a ring homomorphism.

(ii) $id_X : X \to X$ is a morphism, and $id_X^* : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ is identity.

(iii) If $\varphi : X \to Y$ and $\psi : Y \to Z$ be morphisms of affine varieties. Then $\psi \circ \varphi : X \to Z$ is a morphism. Indeed, for any open $U \subseteq Z$ and $f \in \mathcal{O}_Z(U)$

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^* (f) \in \mathcal{O}_X((\psi \circ \varphi)^{-1}(U))$$

(iv) A morphism $\varphi : X \to Y$ is an isomorphism if there exists an inverse $\varphi^{-1} : Y \to X$ such that

$$\varphi \circ \varphi^{-1} = id_Y \quad \text{and} \quad \varphi^{-1} \circ \varphi = id_X .$$

Show that it is enough to assume $\varphi^{-1} : Y \to X$ is a morphism.

(v) Let $\varphi : X \to Y$ be a map. Assume there is an open cover $(U_i)_{i\in I}$ of X such that

$$\varphi|_{U_i} : U_i \to Y$$

is a morphism. Then φ is a morphism:

For any $W \subseteq Y$ open, $\varphi^{-1}(W) = \bigcup_{i\in I} \varphi_i^{-1}(W) \cap U_i$ is open and if $f \in \mathcal{O}_Y(W)$ is regular then the regular functions $f\circ\varphi|_{U_i} \in \mathcal{O}_{U_i}(\varphi_i^{-1}(W))$ glue to unique $f\circ\varphi \in \mathcal{O}_X(\varphi^{-1}(W))$.

(vi) Let $S \subseteq X$ and $T \subseteq Y$ be subvarieties and $\varphi: X \to Y$ a morphism such that $\varphi(S) \subseteq T$. Then $\varphi|_S : S \to T$ is a morphism.
(HW)

Theorem: For any two affine varieties X and Y there is a bijection

$$\{\text{morphisms } X \to Y\} \overset{1:1}{\longleftrightarrow} \{k\text{-algebra homomorphisms } A(Y) \to A(X)\}$$

$$\varphi \longrightarrow \varphi^*$$

Proof: Next class!

Eg: (1) $C = Z(x_2 - x_1^2,\ x_3 - x_1^3) \subseteq \mathbb{A}^3$.

Let $\varphi : \mathbb{A}^1 \to C$, $(t) \to (t, t^2, t^3)$. This is an isomorphism

with $\varphi^{-1} : C \to \mathbb{A}^1$, $(x,y,z) \to (x)$. In HW2, you showed $A(C) \cong A(\mathbb{A}^1)$.

(ii) $C = Z(x^2 - y^3) \subseteq \mathbb{A}^2$. Let $\varphi : \mathbb{A}^1 \to C$
$(t) \mapsto (t^2, t^3)$.

Note that φ is bijection as sets.
But φ is not an isomorphism (Inverse $\psi : C \to \mathbb{A}^1$, $(x,y) \to (y/x)$ is not a morphism)
since $A(C) \not\cong A(\mathbb{A}^1)$.

(iii) $C = Z(y^2 - x^3 + x) \subseteq \mathbb{A}^2$

As in (ii), $C \not\cong \mathbb{A}^1$ since $A(C) \not\cong \mathbb{A}^1$.

Is there any non-constant morphism $\varphi : \mathbb{A}^1 \to C$?
No!
If $\varphi : \mathbb{A}^1 \to C$ is a morphism, then
$\varphi^* : A(C) \to A(\mathbb{A}^1) = k[t]$ is a k-algebra homomorphism.

This induces a morphism of fields $k(x)[y]/\langle y^2 - x^3 + x \rangle \to k[t]$.

<u>Lemma</u>: There is no non-trivial field homomorphism
$$\phi : k(x)[y]/\langle y^2 - x^3 + x \rangle \to k(t)$$

<u>Proof</u>: Suppose there exists ϕ non-trivial, then
$\phi(x), \phi(y) \in k(x)$ are non-zero elements.
Let $\phi(x) = \dfrac{P(t)}{Q(t)}$ and $\phi(y) = \dfrac{f(t)}{g(t)}$, where $P, Q, R, S \in k[t]$.

We will assume $\gcd(P,Q) = \langle 1 \rangle$ and $\gcd(f,g) = \langle 1 \rangle$ (no common factors)

Since $y^2 - x^3 + x = 0$, $\phi(y)^2 - \phi(x)^3 + \phi(x) = 0$
$$\Rightarrow \frac{P(t)^2}{Q(t)^2} - \frac{f(t)\left(f(t) - g(t)\right)\left(f(t) + g(t)\right)}{g(t)^3} = 0$$
$$\Rightarrow P(t)^2 \cdot S(t)^3 = Q(t)^2 f(t)\left(f(t) - g(t)\right)\left(f(t) + g(t)\right)$$

Using the fact that $k[t]$ is a UFD (see HW2), and the gcd condition,

$$P^2 = f(f-g)(f+g) \quad \text{and} \quad Q^2 = g^2$$

(in $k[t]$ up to constant multiplication

Since f and g has no common factor,

$$\gcd(f, f-g) = 1, \quad \gcd(f, f+g) = 1 \quad \text{and} \quad \gcd(f-g, f+g) = 1.$$

Hence $f, g, f+g, f-g$ are perfect squares.

<u>Claim:</u> There are no perfect square polynomials $f, g, f+g, f-g \in k[t]$

<u>Proof:</u> Suppose there exists such a pair f, g with minimum possible value of $\max\{\deg f, \deg g\}$.

Let $u, v \in k[t]$ such that $f = u^2$ and $g = v^2$.

Then

$$\left.\begin{aligned} (f-g) &= (u^2 - v^2) = (u-v)(u+v) \\ \text{and } (f+g) &= (u^2 + v^2) \end{aligned}\right\} \text{ perfect squares.}$$

Since u and v has no common factor,

$u-v$ and $u+v$ are perfect squares in $k[t]$.

Let $\tilde{f} := \dfrac{u-v}{\sqrt{2}}$ and $\tilde{g} := \dfrac{i(u+v)}{\sqrt{2}}$, where $i^2 = -1$, $(\sqrt{2})^2 = 2$ $\in k$.

Note that $\tilde{f}, \tilde{g} \in k[t]$ are perfect squares.

and $\gcd(\tilde{f}, \tilde{g}) = 1$.

$$(\tilde{f} - \tilde{g})(\tilde{f} + \tilde{g}) = \tilde{f}^2 - \tilde{g}^2 = u^2 + v^2 \text{ is perfect square.}$$

Again by UFD property we have $(\tilde{f} - \tilde{g})$ and $(\tilde{f} + \tilde{g})$ are perfect squares.

In conclusion, we started with (f, g) and produced another pair $(\tilde{f}, \tilde{g})$ such that $\tilde{f}, \tilde{g}, \tilde{f} - \tilde{g}, \tilde{f} + \tilde{g}$ are perfect squares and

$$\max(\deg \tilde{f}, \deg \tilde{g}) \leq \frac{1}{2} \max(\deg f, \deg g). \qquad \square$$