Algebraic Geometry 4

Sheaf

Central object of study in geometry and topology!
It captures the structure of collections of
- "functions" on top. space X
- "vector fields" on a manifold X. etc.

Def: A sheaf (of rings) on a topological space X is the collection of the data:

(i) For every open $U \subseteq X$, there is a ring $\mathcal{F}(U)$
(ii) For every inclusion of open $U \subseteq V \subseteq X$,
(Restriction) $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ ring homomorphism.

Such that the following conditions are satisfied:

Presheaf!
(i) $\mathcal{F}(\phi) = 0$
(ii) $\rho_{U,U} = \text{identity} \quad \forall\, U \subseteq X$
(iii) For inclusions $U \subseteq V \subseteq W \subseteq X$, the restriction maps are compatible

$$\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}.$$

(iv) Gluing Property holds: If $U \subseteq X$ open has an open cover $\{U_i : i \in I\}$.
If there is collection $\{\varphi_i \in \mathcal{F}(U_i) : i \in I\}$ which are compatible, that is,

$$\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_j \cap U_i}$$

then there exists a unique $\varphi \in \mathcal{F}(U)$ such that

$$\varphi_i = \rho_{U,U_i}(\varphi) \quad \forall\, i \in I.$$

Eg: 1) $X = \mathbb{R}^n$ and for open $U \subseteq X$,

$\mathcal{F}(U) = \{ \varphi : U \to \mathbb{R}$ continous functions$\}$

(note that this is a ring).

2) $X = \mathbb{C}$ and for any open $U \subseteq \mathbb{C}$,

$\mathcal{F}(U) = \{ \varphi : U \to \mathbb{C}$ holomorphic functions$\}$

$\boxed{3}$ $X \subseteq \mathbb{A}^n$ affine variety and for open $U \subseteq X$,

$\mathcal{O}_X(U) = \{ \varphi : U \to k : \varphi$ is regular function$\}$

For $U \subseteq V$ opens, we have restriction maps

$$\rho_{V,U} : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$$
$$\varphi \to \varphi|_U$$

- $\mathcal{O}_X(\phi) = 0$ • $\mathcal{O}_X(U) \xrightarrow{id} \mathcal{O}_X(U)$
- $U \subseteq V \subseteq W$, functions are compatible under restriction.

$$\begin{tikzcd} \mathcal{O}_X(W) \arrow[r] \arrow[rr, bend right] & \mathcal{O}_X(V) \arrow[r] & \mathcal{O}_X(U) \end{tikzcd}$$

- (Gluing) Built in definition of regular functions:

Recall: $\varphi \in \mathcal{O}_X(U) \iff \varphi$ is locally a quotient of polynomial functions.

Let $\{U_i : i \in I\}$ be an open cover and $\{\varphi_i \in \mathcal{O}_X(U_i) : i \in I\}$

For every point $a \in U$ and $i \in I$,

The sheaf of regular function is $\mathcal{O}_X$!

<u>Observe</u>: Let $X \subseteq \mathbb{A}^n$ be an irreducible affine variety.

Each regular function define $\varphi \in \mathcal{O}_X(U)$ define an element in $K(X)$: Write

$$\varphi = \frac{f}{g} \quad \text{for some open } V \subsetneq U.$$

Then $\frac{f}{g} \in k(X)$ is independent of the cloice of f, g, V.

Suppose $\varphi = \frac{f'}{g'}$ for $V' \subseteq V$, then

$$\frac{f}{g} = \frac{f'}{g'} \quad \text{on } V' \cap V.$$

$$\Rightarrow \quad fg' - f'g = 0 \quad \text{on } V' \cap V \overset{\text{open}}{\subseteq} X$$

$$\Rightarrow \quad fg' - f'g = 0 \quad \text{on closure } \overline{V' \cap V} = X$$

$$\Rightarrow \quad fg' - f'g = 0 \quad \text{on } A(X) \qquad \square$$

<u>Show</u>: $\mathcal{O}_X(U) \hookrightarrow k(X)$ inclusion!

<u>Lemma</u>: Let $U \subseteq X$ and $\varphi \in \mathcal{O}_X(U)$. Then

$$Z(\varphi) = \{ p \in U : \varphi(p) = 0 \}$$

is closed in U.

<u>Proof</u>: For any point $a \in U$, $\exists (f, g, U_a)$ so that $\varphi = \frac{f}{g}$ on $U_a \subseteq U$.

$$U_a \setminus Z(\varphi) = \{ p \in U_a : \varphi(p) \neq 0 \} = U_a \setminus Z(g_a) \text{ is open.}$$

Thus, $U \setminus Z(\varphi) = \bigcup_{a \in U} U_a \setminus Z(g_a)$ is open. $\square$

Lemma: (Identity principal) Let $X \subseteq \mathbb{A}^n$ irreducible affine variety.
Let $V \subseteq X$ be open, and $\varphi_1, \varphi_2 \in \mathcal{O}_X(V)$ such that

$$\varphi_1|_U = \varphi_2|_U \in \mathcal{O}_X(U) \text{ for some open } \emptyset \neq U \subseteq V$$

Then $\varphi_1 = \varphi_2$.

Proof: Consider $Z(\varphi_1 - \varphi_2) \subseteq V$. By previous lemma,

$Z(\varphi_1 - \varphi_2)$ is closed in V.

$\Rightarrow$ $Z(\varphi_1 - \varphi_2)$ contains $\overline{U} = V$.

Distinguishe open sets: Let X be an affine variety.

Let $0 \neq f \in A(X)$, we call

$$D(f) = X \setminus Z(f) = \{ p \in X : f(p) \neq 0 \}.$$

Distinguished open subset of f in X.

Rem: $\{ D(f) : f \in A(X) \}$ form a basis for Zariski topology:
(Check def!)

$U \subseteq X$ open $\Leftrightarrow$ $X \setminus U = Z(f_1, \ldots, f_m)$ for some $f_1, \ldots, f_m \in A(X)$

$\Leftrightarrow$ $X \setminus U = Z(f_1) \cap \cdots \cap Z(f_m)$

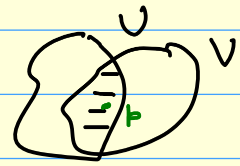
$\Leftrightarrow$ $U = D(f_1) \cup \cdots \cup D(f_m)$.

Note that every open set $U \subseteq X$ are union of finitely many open sets. This will be quite useful!

<u>Definition</u>: Let X be an affine variety and $p \in X$. We define the stalk of the sheaf of regular function $\mathcal{O}_X$ at p as

$$\mathcal{O}_{X,p} := \left\{ (\varphi, U) : p \in U \overset{\text{open}}{\subseteq} X \text{ and } \varphi \in \mathcal{O}_X(U) \right\} \big/ \sim$$

Here "$\sim$" is the equivalence relation



$$(\varphi, U) \sim (\psi, V)$$

if $\quad \varphi|_{U \cap V} = \psi|_{U \cap V}$.

- Note that $\mathcal{O}_{X,p}$ is a ring:
  - Adding $\quad [(\varphi, U)] + [(\psi, V)] = \left[\left(\varphi|_{U\cap V} + \psi|_{U \cap V}, U \cap V\right)\right]$.
  - multiplication is similar.

- $\mathcal{O}_{X,p}$ is a local ring (has a unique maximal ideal).

<u>HW</u>: Show that $\mathcal{O}_{\mathbb{A}^1, p} = \left\{ \frac{f}{g} \in k(x) : f, g \in k[x],\ g(p) \neq 0 \right\}$.

<u>Lemma</u>: Regular functions on $D(f)$ are global quotients:

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^r},\ g \in A(X) \right\}$$

In particular $\quad \mathcal{O}_X(X) = A(X)$.

<u>Proof</u>: Next lecture!