Algeraic Geometry 3

Last time: - $X \subseteq \mathbb{A}^n$ affine variety
- Zariski topology on X
- X is irreducible $\Leftrightarrow$ $A(X) = k[x_1,\dots,x_n]/I(X)$ is integral domain.

Recall: Let $I \subseteq J \subseteq k[x_1,\dots,x_n]$ be ideals. Let $\tilde{J} = J/I \subseteq k[x_1,\dots,x_n]/I$. Then

$$k[x_1,\dots,x_n]/J \cong \left(k[x_1,\dots,x_n]/I\right)/\tilde{J}. \quad (*)$$

Rem: Let $Y \subseteq X$ be a subvariety. Then Y is irreducible subvariety of X if $I(Y) \subseteq A(X)$ is a prime ideal, that is,

$$A(Y) \overset{(*)}{\cong} A(X)/I_X(Y) \text{ is an integral domain.}$$

Results (last time) (i) Affine variety $X \subseteq \mathbb{A}^n$ is Noetherian.

(ii) There exists a unique decomoposition of X into irreducibles

$$X = \bigcup_{i=1}^{m} X_i \quad \text{such that} \quad X_i \nsubseteq X_j \text{ for } i \neq j.$$

Rem: This means that any radical ideal $I \subseteq k[x_1,\dots,x_n]$ can be written as finit intersection of prime ideals

$$I = \bigcap_{i=1}^{m} \mathfrak{p}_i \quad \text{when} \quad \mathfrak{p}_i = I(X_i).$$

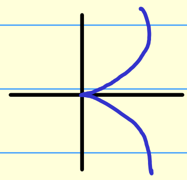
Dimension theory

$\mathbb{A}^1 \rightsquigarrow \dim 1$

$C = Z(y^2 - x^3)$

What is $\dim(C)$?

One, it is a curve!



Def: Let X be an irreducible affine variety. The <u>dimension</u> of X, denoted $\dim(X)$, is the largest integer $n \in \mathbb{N}$ such that there exists an ascending chain

$$\phi = X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_n = X$$

of irreducible subvarieties of X.

Eg: [I] points have dimension 0.

[II] $\mathbb{A}^1$ has dimension 1: Any irreducible subvariety $Y \subseteq \mathbb{A}^1$ is $Y = \{a\}$.
(Prime ideal in $k[x]$ is of the form $\langle (x-a) \rangle$)

[III] Fact: $\mathbb{A}^n$ has dimension n. ($\dim(\mathbb{A}^n) \geq n$ is easy! Consider $\phi \subseteq \mathbb{A}^1 \subseteq \cdots \subseteq \mathbb{A}^n$.)

Def: Let $Y \subseteq X$ be irreducible subvariety, then $\operatorname{codim}_Y X$ is the largest integer n such that there exists

$$Y = Y_0 \subsetneq Y_1 \subsetneq \cdots \subseteq Y_n \subseteq X \quad , \quad Y_i\text{'s are irreducible.}$$

Codimension 1 subvarieties are called Hypersurface.

Eg: (Twisted Affine cubic) Consider $C = \{(t, t^2, t^3) : t \in k\} \subseteq \mathbb{A}^3$.

- C is an affine variety $Z(xz-y^2, x^3-z) \subseteq \mathbb{A}^3$.
- $\dim(C) = 1$
- $\operatorname{codim}_C \mathbb{A}^3 = 2$

$\left.\begin{matrix} \dim(C)=1 \\ \operatorname{codim}_C \mathbb{A}^3 = 2 \end{matrix}\right\}$ HW!

Regular functions (analogue to holomorphic function in CA)

Let X be an irreducible affine variety.

$A(X)$ = coordinate ring which is an integral domain.

Def: The quotient field $K(V)$ of $A(V)$ is called the field of rational functions on X and elements of $k(X)$ are called rational functions.

Eg: Let $U = \mathbb{C}\setminus\{0\} \subseteq \mathbb{C}$ be open set. We would like to define "regular functions" on U.

In this case, "regular functions" are holomorphic rational functions on U $= \left\{ \dfrac{P(z)}{z^m} : P \in k[z],\ m \in \mathbb{N} \right\}$

"Wrong def": Let $X \subseteq \mathbb{A}^n$ be an affine variety, and let $U \subseteq X$ be an open set (Zariski). A regular function
$$\varphi : U \to k$$
is an element of the form $\frac{f}{g}$ when $f, g \in A(X)$ and $g(p) \neq 0 \ \forall p \in U$.

Why do we need better definition?

1) $A(X)$ may not be integral domain.

2) We would like to have a local definition. (as in complex holomorphic function). Consider the next example.

Eg: Let $X = Z(x_1x_4 - x_2x_3)$ and $U = X \setminus Z(\langle x_2, x_4\rangle)$.
Consider the function

$$\varphi : U \longrightarrow k$$
$$(x_1,x_2,x_3,x_4) \to \begin{cases} \frac{x_1}{x_2} & \text{if } x_2 \neq 0 \\ \frac{x_3}{x_4} & \text{if } x_4 \neq 0 \end{cases}$$

well defined! [Note that for points $(x_1,x_2,x_3,x_4) \in U$ with $x_2 \neq 0$ and $x_4 \neq 0$,
$x_1x_4 - x_2x_3 = 0 \Rightarrow \frac{x_1}{x_2} = \frac{x_3}{x_4}$.

HW: There is no elements $f, g \in A(X)$ such that
$\varphi(p) = \frac{f}{g}(p) \quad \forall p \in U$.

Correct definition: Let X be an affine variety and $U \subseteq X$ is open.
A regular function on U is a map
$$\varphi : U \to k$$
such that: For every $p \in U$, there exists an open set $U_p \subseteq U$ and polynomial functions $f, g \in A(X)$ with

$$\varphi(q) = \frac{f(q)}{g(q)} \quad \text{for all } q \in U_p.$$

Def: Let $\mathcal{O}_X(U)$ = set of all regular functions on U.

Rem: $\mathcal{O}_X(U)$ is a k-algebra:
1) $\varphi, \psi \in \mathcal{O}_X(U) \Rightarrow \varphi + \psi \in \mathcal{O}_X(U)$
2) $\varphi, \psi \in \mathcal{O}_X(U) \Rightarrow \varphi \cdot \psi \in \mathcal{O}_X(U)$
3) $k \subseteq \mathcal{O}_X(U)$ subalgebra of constant functions.

Eg: $X = \mathbb{A}^1$, $U = \mathbb{A}^1 \setminus \{p_1, \dots, p_r\}$.

$$\mathcal{O}_X(U) \cong \left\{ \frac{f(z)}{(z-p_1)^{m_1} \cdots (z-p_r)^{m_r}} : f \in k[z],\ m_1, \dots, m_r \in \mathbb{N} \right\}$$

(Suppose $\varphi \in \mathcal{O}_X(U)$, for any point $a \in U$ let $U_a = U \setminus \{z_1, \dots, z_s\}$ such that there exists $f_a, g_a \in k[x]$

$$\varphi(q) = \frac{f_a(q)}{g_a(q)} \quad \forall\ q \in U_b$$

Moreover, assume that f_a and g_a do not have a common factor and $g_a(z)$ is monic (call it reduced form).

Then for any $a, b \in U$,

$$\frac{f_a}{g_a} = \frac{f_b}{g_b} \quad \text{on} \quad U_a \cap U_b$$

implies $f_a \cdot g_b(x) = f_b \cdot g_a(x)$ on infinitely many values $x \in k$

Hence $f_a \cdot g_b \equiv f_b \cdot g_a \in k[z]$ as polynomials. Since $k[z]$ is a unique factorization domain, and f_a/g_a and f_b/g_b are in reduced form,

$$f_a \equiv f_b \text{ and } g_a \equiv g_b \text{ as polynomial.}$$

Since the above holds for all $a, b \in U$, we have globally defined polynomials $f, g \in k[z]$ such that

$$\varphi(p) = \frac{f(p)}{g(p)} \quad \forall\ p \in U.$$

Since $g(z) \neq 0$ $\forall\ z \in \mathbb{A}^1 \setminus \{p_1, \dots, p_r\}$, $g(z) = \prod_{i=1}^{r} (z-p_i)^{m_i}$.)

Eg: Note that for $U = \mathbb{A}^1 \setminus \{0\}$,

$$\mathcal{O}_{\mathbb{A}^1}(U) \neq \mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1) \cong k[z].$$

Because $\varphi(z) = \frac{1}{z} \in \mathcal{O}_{\mathbb{A}^1}(U)$ does not have removable singularity at $z = 0$.

Hartog's removable singularity theorem: Let $\boxed{n \geq 2}$ and

Let $U = \mathbb{A}^n \setminus \{(0,0,\dots,0)\} \subseteq \mathbb{A}^n$ be the open set. Then

$$\mathcal{O}_{\mathbb{A}^n}(U) \cong \mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n) \cong k[x_1,\dots,x_n].$$

Equivalently, every regular function φ on $U = \mathbb{A}^n \setminus \{(0,\dots,0)\}$

has a "removable sigularity" (φ extends to regular function on $\mathbb{A}^n$) at the origin.

Rem: By previous example, the above theorem fails for $n = 1$.