Algebraic Geometry 2

Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set (or affine variety)

Def: A <u>polynomial function</u> on X is a map

$f: X \to k$ where f is the restriction of a polynomial function $P \in k[x_1, \ldots, x_n]$
$\left(f(p) = P|_X(x) \quad \forall p \in X \right)$

Note that if $P, Q \in k[x_1, \ldots, x_n]$ such that $P - Q \in I(X)$, then $P|_X = Q|_X$ (since $(P-Q)(p) = 0 \quad \forall p \in X$).

Def: The <u>coordinate ring</u> of X is the quotient ring

$$A(X) := k[x_1, \ldots, x_n] / I(X)$$

Def: (a) For any set $S \subseteq A(X)$, denote

affine subvariety

$$Z(S) = Z_X(S) := \{ p \in X : f(p) = 0 \ \forall f \in S \} \subseteq X.$$

(b) For any subset $Y \subseteq X$,

$$I(Y) = I_X(Y) := \{ f \in A(X) : f(p) = 0 \ \forall p \in Y \} \subseteq A(X).$$

Relative Hilbert Nullstellensatz: (Ex. 1.23 Gathmann)

$$\left\{ \begin{array}{c} \text{affine subvarieties} \\ \text{in } X \end{array} \right\} \stackrel{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{radical ideals} \\ \text{in } A(X). \end{array} \right\}$$

Zariski on affine variety

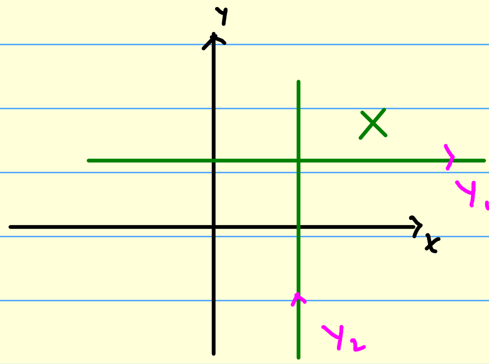
Def: Let $X \subseteq \mathbb{A}^n$ be an affine variety. The Zariski topology on X is:

(I) Induced topology on X from $\mathbb{A}^n$,

or equivalently,

(II) Closed sets are exactly affine subvarieties of X.

Eg: $X = Z((x-a)\cdot(y-b)) \subseteq \mathbb{A}^2$



$Y_1 = Z(y-b)$

$Y_2 = Z(x-a)$

What is Zariski topology on X?

closed sets consists of
$$\begin{cases} \phi,\ X,\ \text{finit sets}, \\ Y_1 \cup \{\text{finite sets}\}, \\ Y_2 \cup \{\text{finite sets}\} \end{cases}$$

Observe: X is connected in Zariski topology

(There is no subset $V \subseteq X$ such that V and $X \setminus V$ are both closed excep $V = \phi$ and $V = X$.)

What are Y_1 and Y_2?

Irreducible components!

Def: Let X be a topological space.

X is <u>reducible</u> if there exists closed sets $Y_1, Y_2 \subsetneq X$ such that

$$X = Y_1 \cup Y_2 .$$

If X is not reducible, we call it irreducible.

Eg: $X = Z(\langle (x-a)\cdot(y-b)\rangle)$, then coordinate ring

$$A(X) = k[x,y] \big/ \langle (x-a)\cdot(y-b)\rangle .$$

In the quotient ring, $(x-a) \neq 0 \in A(X)$

$(y-b) \neq 0 \in A(X)$

But $(x-a)\cdot(y-b) = 0$ in $A(X)$. (zero divisors!)

Hence, $A(X)$ is not an integral domain!

Proposition: A non-empty affine variety $X \subseteq \mathbb{A}^n$ is irreducible if and only if $A(X)$ is an integral domain.

Proof: Since X is non-empty $A(X) \neq 0$.

$\Rightarrow$ Suppose $A(X)$ is not integral domain. Let $f, g \in A(X) \setminus \{0\}$ such that $f \cdot g = 0$ in $A(X)$.

Let $X_f = Z(f) \subseteq X$ and $X_g = Z(g) \subseteq X$ be closed sets.

- $X_f \neq X$ and $X_g \neq X$ (since f and g are not zero)
- $X_f \cup X_g = Z(f) \cup Z(g) = Z(f \cdot g) = 0$

This contradicts that X is irreducible.

$\Leftarrow$ Suppose X is reducible, i.e., $\boxed{X = Y_1 \cup Y_2}$.

for some $Y_1, Y_2 \subsetneq X$ closed sets.

By relative Hilb. Nullstellensatz,

$I(Y_i) \neq 0 \subseteq A(X)$ for $i=1,2$.

Let $f_1 \in I(Y_1)$, $f_2 \in I(Y_2)$,

then $f_1 \cdot f_2(p) = 0 \ \forall p \in X$.

$\Rightarrow$ $f_1 \cdot f_2 = 0$ in $A(X)$.

- Recall an ideal $I \subseteq R$ is prime if and only if R/I is an integral domain.
- If $I \subseteq R$ is a prime ideal, $I = \sqrt{I}$.

Consequence: We have a bijection

$$\left\{ \text{prime ideals in } k[x_1, \dots, x_n] \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{Irreducible subvarieties} \\ \text{in } \mathbb{A}^n \end{array} \right\}$$

Eg: 1) $\mathbb{A}^n$ is irreducible. (Since $A(\mathbb{A}^n) = k[x_1, \dots, x_n]$ is integral domain)

2) If $f \in k[x_1, \dots, x_n]$ is an irreducible polynomial, then $Z(f)$ is irreducible
(Since $k[x_1, \dots, x_n]$ is unique factorization domain, $\langle f \rangle$ is a prime ideal.)

Def: A topological space X is <u>Noetherian</u> if every descending chain

$$X \supseteq X_1 \supseteq X_2 \supseteq \cdots$$

becomes stationary.

Lemma: Any affine variety is a Noetherian space (Zariski top.).

Proof: Let $X \subseteq \mathbb{A}^n$ be an affine variety. Note that

$$A(X) = k[x_1, \dots, x_n] / I(X) \quad \text{is Noetherian ring}$$

(by Hilb. Basis theorem and HW 1).

A descending chain $X \supseteq X_1 \supseteq X_2 \supseteq \cdots$ of subvariety corresponds to an ascending chain of ideals

$$I(X) \subseteq I(X_1) \subseteq I(X_2) \subseteq \cdots \subseteq A(X).$$

By ACC criterion, the above chain becomes stationary, that is, there exists $N \in \mathbb{N}$ such that

$$I(X_N) = I(X_{N+1}) = \cdots$$

By rel Hilb. Nullstellensatz, $X_N = X_{N+1} = \cdots$

$\square$

Proposition 2.14 (Gathmann) : Every Noetherian topological space can be written as a finite union $X = X_1 \cup X_2 \cup \cdots \cup X_m$ of irreducible closed subsets $X_i \subseteq X$ $\forall$ $1 \le i \le m$.

Furthermore, if $X_i \nsubseteq X_j$ for all $1 \le i, j \le m$, then the irreducible components $X_1, \ldots, X_m$ are unique.

Proof: Suppose X does not have finite irreducible decomposition. Then X is reducible and

$$X = X_1 \cup Y_1 \quad \text{for some closed sets } X_1, Y_1 \subsetneq X$$

Either X_1 or Y_1 does not have finite decomposition. (Say X_1) Then X_1 is reducible and

$$X_1 = X_2 \cup Y_2 \quad \text{for} \quad \text{closed sets } X_2, Y_2 \subsetneq X_1.$$

Repeat the process, we obtain a strictly descending chain

$$X \supsetneq X_1 \supsetneq X_2 \supseteq \cdots,$$ which contradicts X is Noetherian.

For uniqueness, assume there are two such decomposition.

$$X = X_1 \cup \cdots \cup X_m = Y_1 \cup \cdots$$

Then

$$X_i = \bigcup_j (Y_j \cap X_j),$$ so by irreducibility of X_i, we get $X_i \subseteq Y_j$ for some j.

Similarly $Y_j \subseteq X_k$ for some k. Thus $X_j \subseteq \underbrace{Y_j \subseteq}_{} X_k \Rightarrow k = j$

and $X_i = Y_j$.