Algebraic Geometry 17

non-constant!

Prop: Let $f \in k[x_1,\dots,x_n]$ and $X = Z(f)$ is the hypersurface. A point $p \in X$ is singular if and only if

$$\frac{\partial f}{\partial x_i}(p) = 0 \qquad \forall\ i \in \{1,2,\dots,n\}.$$

Proof: Assume $p=0$. Then the tangent cone

$$C_p(X) = Z(f_{in}) \subseteq \mathbb{A}^n. \qquad \text{(initial homogenous term)}$$

On the othe hand, the tangent space

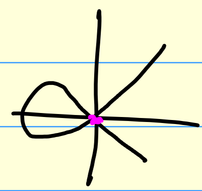
$$T_p(X) = Z(f_1) \subseteq \mathbb{A}^n. \qquad \text{(Linear terms).}$$

We have $C_p(X) \subseteq T_p(X)$, and the equality holds if and only if $f_1 \neq 0$.

Suppose $f_1 = \sum_{i=1}^{n} a_i x_i$. Suppose $a_i \neq 0$ for some $i \in \{1,2,\dots,n\}$, then

$$\frac{\partial f}{\partial x_i}(0) = a_i \neq 0.$$

Eg: $X = Z(y^2-x^2-x^3) \subseteq \mathbb{A}^2$ has singular at $p=0$. X is not singular anywhere else.



Prop: Let $X \subseteq \mathbb{A}^n$ be a hypersurface. Then

$$X_{reg} := \{p \in X \mid p \text{ is a smooth point in } X\} \subseteq X$$

is an open subset.

Proof: Let $X = Z(f)$. The set of singular points

$$X_{sing} = X \setminus X_{reg} = Z\left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

is a closed subset.

<u>Rem:</u> X_{reg} is not empty. In particular, X irred $\Rightarrow$ $X_{reg} \subseteq X$ dense.

<u>Proof:</u> We may assume $X = Z(f)$ such that $\sqrt{\langle f \rangle} = \langle f \rangle$.

(note that radical of principle ideal is principle in UFD!)

$X_{sing} = X$ implies

$$\frac{\partial f}{\partial x_i} \in \langle f \rangle \quad \text{for all } i \in \{1,2,..,n\}.$$

But $\deg\left(\frac{\partial f}{\partial x_i}\right) < \deg f$, hence $\frac{\partial f}{\partial x_i} = 0 \ \forall\, i \in \{1,2,\dots,n\}$

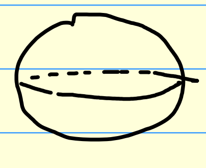
This implies f is a constant function.

<u>HW:</u> Read Jacobian description for tangent space $T_p(X)$ for $X = Z(f_1,..,f_r) \subseteq \mathbb{A}^n$ and $X = Z(f_1,..,f_r) \subseteq \mathbb{P}^n$.

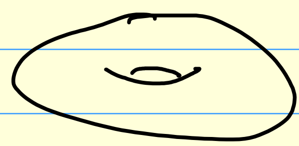
Non-Singular Curves

$C \subseteq \mathbb{P}^n$ is non-singular if every point $p \in C$ is singular.

<u>Eg:</u> $C = \mathbb{P}^1$



$C = Z(y^2z - x^3 - xz^2) \subseteq \mathbb{P}^2$



- When $k = \mathbb{C}$, a smooth projective curve is a Riemann surface. (1-dim complex manifold).
- Riemann-Roch Theorem relates degree of C to genus of C.

We want to understand $\mathcal{O}_{C,p}$ more concretely.

Lemma: (Nakayama) Let A be a local ring with maximal ideal m. Let M be a finitely gen. module.

$$M = mM \Rightarrow M = 0.$$

Def: A local ring A is called a Discrete Valuation Ring (DVR) if:

- A is an integral domain
- The maximal ideal $m \subseteq A$ is principal. Write $m = \langle t \rangle$, then t is called uniformizer parameter.
- Every element $a \in A$ can be written as
  $$a = u \cdot t^n \quad \text{where} \quad u \in A^* \ (u \text{ is a unit}),\ n \in \mathbb{N}.$$

Theorem: Let p be a smooth point on a curve C. Then the local ring $\mathcal{O}_{C,p}$ is a DVR.

Proof: Let $K(C)$ be rational field, then

$$\mathcal{O}_{C,p} \subseteq K(C) \text{ is a subring}$$

Let $m_p \subseteq \mathcal{O}_{C,p}$ be the maximal ideal.
From last time, the tangent space

$$T_p C \cong \left(m_p / m_p^2\right)^\vee \cong k \quad (\text{since } p \text{ is a smooth point}).$$

Let $t \in m_p$ such that its class $\bar{t} \in m_p / m_p^2$ is non-zero.
Claim: $m_p = \langle t \rangle$

Proof: Let $A = \mathcal{O}_{C,p}$. Consider m_p, m_p^2 and $\langle t \rangle$ as A-modules.
(A is a Noetherian ring, check yourself)

We are given that $\bar{t} \in m_p/m_p^2 \cong k$ (i.e $\bar{t}$ spans m_p/m_p^2)

This implies $m_p = \langle t \rangle + m_p^2$ and

$$m_p \cdot \left(m_p/\langle t \rangle\right) = \left(m_p^2 + m_p\langle t \rangle\right)/\langle t \rangle = \left(m_p/\langle t \rangle\right)$$

By Nakayama's Lemma, $m_p/\langle t \rangle = 0 \Leftrightarrow m_p = \langle t \rangle$.

$\square$

Let $M = \bigcap_{n>0} m_p^n$. By definition,

$$m_p \cdot M = M \Rightarrow M = 0.$$

This implies if $a \in \mathcal{O}_{C,p}$, $a \in \langle t^n \rangle \setminus \langle t^{n+1} \rangle$ for some $n \in \mathbb{N}$. Thus we may write

$$a = u t^n.$$

$\square$

Def: For any point $p \in C$ (smooth point), there is a valuation map

$$v_p : \mathcal{O}_{C,p} \setminus \{0\} \longrightarrow \mathbb{N}$$

$$v_p(a) = n \quad \text{if} \quad a = ut^n, \ u \text{ unit}, \ t \text{ uniformizer}.$$

It satisfies

- $v_p(a \cdot b) = v_p(a) + v_p(b)$
- $v_p(f+g) \geq \min(v_p(f), v_p(g))$
- $v_p(a) = 0 \Leftrightarrow a$ is a unit.

Def: Let $k(C)$ be the rational field. Note that $\mathcal{O}_{C,p} \subseteq k(C)$ and $k(C)$ is the quotient field. The valuation map is defined on $k(C)$:

$$v_p : k(C) \longrightarrow \mathbb{Z}$$
$$v_p\left(\frac{f}{g}\right) = v_p(f) - v_p(g).$$

For any rational function $h = f/g \in k(C)$, we say

- h has zero of order n at p if $v_p(h) = n$
- h has pole of order n at p if $v_p(h) = -n$,

Theorem: There is one-one correspondes

$$\left\{ \begin{array}{c} p \in C \\ \text{for a smooth} \\ \text{projective curve} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{DVR } R \subseteq k(C) \text{ such that } k \subseteq R \subseteq k(C) \\ \text{and } k(C) \text{ is fraction field of } R \end{array} \right\}.$$

We will not prove it!

Note that this suggests the following:

Theorem: Let C and C' be smooth projective birational curves, then

$$C \cong C'.$$

Proof: Let $U \subseteq C$ and $U' \subseteq C'$ be open dense with isomorphism

$$\varphi_0 : U \xrightarrow{\sim} U', \quad \psi_0 = \varphi_0^{-1} : U' \xrightarrow{\sim} U.$$

By Prop below, we have morphisms extending φ_0 and ψ_0

$$\varphi : C \to C' \quad \text{and} \quad \psi : C' \to C.$$

Here $\psi \circ \varphi : C \to C$ is an identity on an open dense set, hence $\psi \circ \varphi = id : C \to C$. Similarly, $\varphi \circ \psi = id : C' \to C'$, thus $C \cong C'$.

$\square$

Prop: Let C be a curve and $Y \subseteq \mathbb{P}^n$ be a projective variety. Any rational map

$$\varphi_0 : C \dashrightarrow Y$$

extends to a morphism $\varphi : C \longrightarrow Y$.

Proof: (See page 92 Gottsce's notes).

Rem: Y being projective is required: Let $f_0 : \mathbb{A}^1 \setminus \{0\} \longrightarrow \mathbb{A}^1$, $z \longmapsto 1/z$

then there is no $f : \mathbb{A}^1 \to \mathbb{A}^1$ such that $f|_{\mathbb{A}^1 \setminus \{0\}} = f_0$.