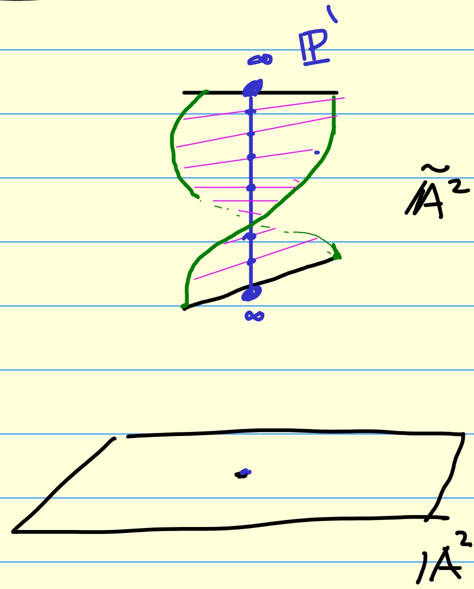
Algebraic Geometry 15



Blow up a point in $\mathbb{A}^n$:

Let $p = (0,0,\dots,0) \in \mathbb{A}^n$.

__Def:__ The blowup of $\mathbb{A}^n$ at p is defined to be

$$\tilde{\mathbb{A}}^n = \{ (x = (x_1,\dots,x_n) \in \mathbb{A}^n,\ y = [y_1,\dots,y_n] \in \mathbb{P}^{n-1} \mid x_i y_j = x_j y_i \ \forall 1 \le i,j \le n \} \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}.$$

__Prop:__ $\tilde{\mathbb{A}}^n$ is an irreducible variety of dimension n and there is a morphism

$$\pi : \tilde{\mathbb{A}}^n \longrightarrow \mathbb{A}^n$$
$$\pi(x,y) = x$$

such that

(a) $\pi^{-1}(p = (0,\dots,0)) \cong \mathbb{P}^{n-1}$ called exceptional set (divisor).

(b) The restriction of π to $\tilde{\mathbb{A}}^n \setminus \pi^{-1}(p)$ is an isomorphism
$$\pi : \tilde{\mathbb{A}}^n \setminus \pi^{-1}(p) \xrightarrow{\sim} \mathbb{A}^n \setminus \{p\}.$$

__Proof:__ Let $U = \mathbb{A}^n \setminus \{p\}$, then

$$f : U \to \mathbb{P}^{n-1}$$
$$(x_1,\dots,x_n) \to [x_1,\dots,x_n]$$

(U is the cone of $\mathbb{P}^{n-1}$)

is a morphism. Thus, the graph of f

$$\Gamma_f = \{ (x,y) \in U \times \mathbb{P}^{n-1} \mid f(x) = \mathbb{P}^{n-1} \} \subseteq U \times \mathbb{P}^{n-1}$$

is closed in $U \times \mathbb{P}^{n-1}$. Note that $y = [y_1, \dots, y_n] = f(x)$ is equivalent to the condition

$$x_i y_j - x_j y_i = 0 \quad \forall \ i, j \in \{1, 2, \dots, n\}.$$

Therefore, $\tilde{\mathbb{A}}^n = \overline{\Gamma_f}$ is the closure in $\mathbb{A}^n \times \mathbb{P}^{n-1}$.

Note that Γ_f is irreducible (since U is irreducible), thus $\tilde{\mathbb{A}}^n = \overline{\Gamma_f}$ is irreducible. (and $\dim U = n = \dim \Gamma_f = \dim \tilde{A}_n$).

The map π is the composition

$$\overline{\Gamma_f} \hookrightarrow \mathbb{A}^n \times \mathbb{P}^{n-1} \xrightarrow{\pi_{\mathbb{A}^n}} \mathbb{A}^n .$$

Since $p = (0, \dots, 0)$

$$\pi^{-1}(p) = \{(\{p\}, y) : y_j \cdot 0 - y_i \cdot 0 = 0\} \subseteq \{p\} \times \mathbb{P}^{n-1}$$
$$\cong \mathbb{P}^{n-1}.$$

On the open set $U = \mathbb{A}^n \setminus \{p\}$,

$$\pi^{-1}(U) \cong \Gamma_f \subseteq U \times \mathbb{P}^{n-1}$$

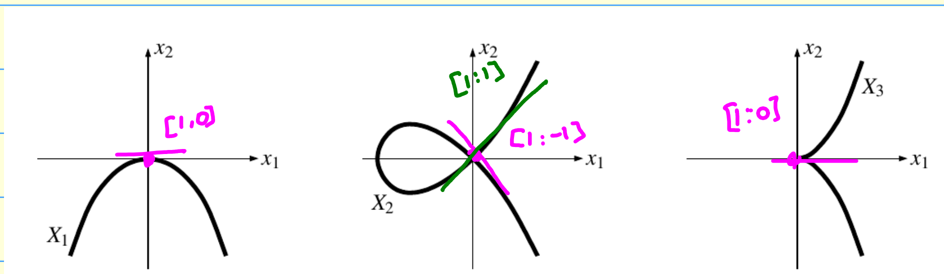
and hence

$$\pi : \Gamma_f \to U \quad \text{is an isomorphism}$$

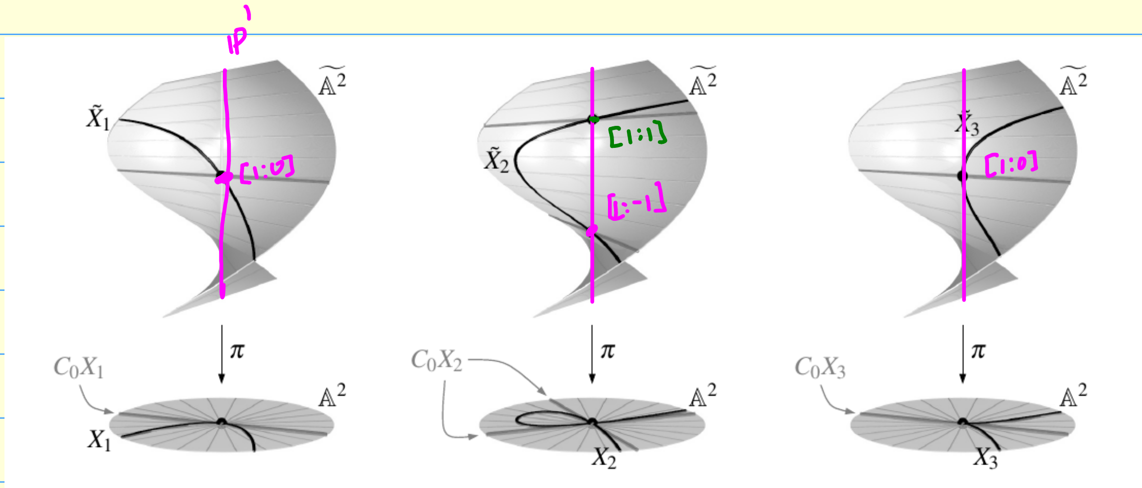
with inverse $(x, f) : U \to \Gamma_f \subseteq U \times \mathbb{P}^{n-1}$.

$\square$

$\tilde{A}_n$ captures all the tangent directions at point p!



$X_1 = V(x_2 + x_1^2)$ $\quad$ $X_2 = V(x_2^2 - x_1^2 - x_1^3)$ $\quad$ $X_3 = V(x_2^2 - x_1^3)$



Eg: (a) Let $C = Z(x_2 + x_1^2) \subseteq \mathbb{A}^2$.

The lift of C to $\tilde{\mathbb{A}}^2$ is the closure $\tilde{C} = \overline{\pi^{-1}(C \setminus \{(0,0)\})} \subseteq \tilde{\mathbb{A}}^2$.

Explicitly, $\tilde{C} = Z(x_1y_2 - x_2y_1,\ x_2^2 + x_1,\ y_2 + x_1y_1) \subseteq \mathbb{A}^2 \times \mathbb{P}^1$

$\tilde{C} \cap \pi^{-1}((0,0)) = \{[1:0]\} \subseteq \mathbb{P}^1$

(b) Let $C = Z(x_2^2 - x_1^2 - x_1^3) \subseteq \mathbb{A}^2$. Then

$$\tilde{C} = Z(x_1y_2 - x_2y_1,\ x_2^2 - x_1^2 - x_1^3,\ y_2^2 - y_1^2 - x_1y_1^2) \subseteq \mathbb{A}^2 \times \mathbb{P}^1.$$

$\tilde{C} \cap \pi^{-1}((0,0)) = \{[1:-1], [1:1]\}. \subseteq \mathbb{P}^1$

(c) Let $C = Z(x_2^2 - x_1^3) \subseteq \mathbb{A}^2$. Then

$$\tilde{C} = Z(x_1y_2 - x_2y_1,\ x_2^2 - x_1^3,\ y_2^2 - x_1y_1^2) \subseteq \mathbb{A}^2 \times \mathbb{P}^1.$$

Note that $\tilde{C} \cap \pi^{-1}((0,0)) = \{[1:0]\} \subseteq \mathbb{P}^1$.

Comes with "multiplicity" two!

Blow up of X at a point

Def: Let $X \subseteq \mathbb{A}^n$ be an affine v and $p \in X$ be a point (assume $p = (0, .., 0)$ is the point in X). Let $U = X \setminus \{p\}$, then there is morphism

$$f : U \longrightarrow \mathbb{P}^{n-1}$$
$$x \longrightarrow [x_1, \dots, x_n]$$

where $x_1, x_2, \dots, x_n$ are coordinate functions. The blow up of X at point p is

$$\tilde{X} = \overline{\Gamma_f} = \{(x, y) \in X \times \mathbb{P}^{n-1} \mid x_i y_j - x_j y_i = 0 \quad \forall \ i, j \in \{1, 2, .., n\}\}$$

is the closure of the graph of f in $X \times \mathbb{P}^{n-1}$.

We let $\pi : \tilde{X} \to X$ be the birational map given by the

$$\begin{tikzcd} \tilde{X} \arrow[r, hook] \arrow[rr, bend right, "\pi"'] & X \times \mathbb{P}^n \arrow[r] & X \end{tikzcd}.$$

HW: Show that the definition of $\tilde{X}$ is independent of the choice of coordinate functions $x_1, \dots, x_n \in A(X)$.

Def: Let X be a variety and $p \in X$ be a point. Let $U \subseteq X$ be an open affine neighbourhood containing p. Let $\tilde{U}$ be blow up of U at point p, and define

$\tilde{X}$ = blow up of X at point p

:= "Gluing" varieties $X \setminus \{p\}$ and $\tilde{U}$ along $X \setminus \{p\} \cap \tilde{U} \cong U$.

In particular, we can define blow up of $\mathbb{P}^2$ at a point.

Rem: There is a morphism $\pi : \tilde{X} \to X$ (given by $\pi : \tilde{U} \to U$ as above) is birational and $E := \pi^{-1}(\{p\})$ is called exceptional set.
We have isomorphism
$$\pi : \tilde{X} \setminus E \xrightarrow{\sim} X \setminus \{p\}$$

Rem: The exception set E is a projective variety. Indeed,
$$E \subseteq \overline{\Gamma_f} \cap \{p\} \times \mathbb{P}^{n-1}$$
(Let $U \subseteq \mathbb{A}^n$ affine as chosen above).
where $\overline{\Gamma_f}$ is closed in $U \times \mathbb{P}^{n-1}$.

Def: (Tangent cone) Let X be a variety and $\{p\} \in X$ be a point.
Let $\tilde{X}$ be the blowup of X at p. The tangent cone
$$C_p X = \text{Cone of } E \subseteq \{p\} \times \mathbb{P}^{n-1}.$$

Eg: $\boxed{\text{Blow up of } \mathbb{P}^1 \times \mathbb{P}^1 \text{ at one point}} \cong \boxed{\text{Blow up of } \mathbb{P}^2 \text{ at two distinct points (succesively).}}$

$$X = \mathbb{P}^1 \times \mathbb{P}^1 = Z(z_0 z_3 - z_1 z_2) \subseteq \mathbb{P}^3$$

Let $a = [0:0:0:1] \in X$, then the blowup of X at the point a can be calculated by considering the affine chart
$$X \cap U_3 \subseteq \mathbb{A}^3 \quad \text{where } U_3 = \mathbb{P}^3 \setminus Z(z_3) \cong \mathbb{A}^3$$
and blowing up the point $a = [0:0:0:1]$ that restricts to $(0,0,0)$.

The blow up of $a \in X$ is explicitly describle by blowing up $(0,0,0) \in \underset{X \cap U_3}{\underset{\shortparallel}{Z(z_0 - z_1 z_2)}} \subseteq \mathbb{A}^3$.

$$\tilde{X} \quad \left\{ (z,y) \in \mathbb{P}^3 \times \mathbb{P}^2 \;\middle|\; z_0 z_3 = z_1 z_2 \;;\; z_0 y_1 = z_1 y_0 \;;\; z_0 y_2 = z_2 y_0 \;;\; z_1 y_2 = z_2 y_1 \right\} \subseteq \mathbb{P}^3 \times \mathbb{P}^2$$

Segre

blow up! $[z_0 : z_1 : z_2] = [y_0 : y_1 : y_2]$

Let $b = [0:1:0]$, $c = [0:0:1] \in \mathbb{P}^2$ (distinct points).

We can blow up b in the open affine charts $U_1 = \mathbb{P}^2 \setminus Z(x_1) \cong \mathbb{A}^2$
and blow up c in the open affine charts $U_2 = \mathbb{P}^2 \setminus Z(x_2) \cong \mathbb{A}^2$.

Explicitly, it is given by

$$\tilde{\tilde{\mathbb{P}}}^2 := Z\left((x,s,t) \in \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \;\middle|\; \begin{array}{c} x_0 s_1 = x_2 s_0 \\ x_0 t_1 = x_1 t_0 \end{array} \right) \subseteq \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$$

Now use the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$

$$[s_0 : s_1], [t_0 : t_1] \longrightarrow [\underbrace{s_0 t_0}_{w_0} : \underbrace{s_0 t_1}_{w_1} : \underbrace{s_1 t_0}_{w_2} : \underbrace{s_1 t_1}_{w_3}]$$

$$\tilde{\tilde{\mathbb{P}}}^2 = Z\left(x, w \in \mathbb{P}^2 \times \mathbb{P}^3 \;\middle|\; \begin{array}{l} x_0 w_2 = x_2 w_0 \;;\; x_0 w_1 = x_1 w_0 \;; \quad w_0 w_2 = w_1 w_3 \\ x_0 w_1 = x_2 w_3 \;;\; x_0 w_2 = x_1 w_3 \;; \end{array} \right)$$

C<u>laim</u>: The isomorphism $f : \mathbb{P}^3 \times \mathbb{P}^2 \xrightarrow{\sim} \mathbb{P}^2 \times \mathbb{P}^3$ induces
$(z, y) \longrightarrow (x = y, w = z)$
isomorphism of subvarieties $f : \tilde{X} \xrightarrow{\sim} \tilde{\tilde{\mathbb{P}}}^2$.

<u>HW:</u> Check!