Algebraic Geometry 14.

<u>Theorem:</u> $G(r,n)$ is an irreducible variety of dimension $r(n-r)$.

<u>Proof:</u> Consider the Plucker embedding

$$f : G(r,n) \hookrightarrow \mathbb{P}^{\binom{n}{r}-1}.$$

Let $(z_I)_{\substack{I \subseteq \{1,2,\dots,n\} \\ |I| = r}}$ be the coordinates of $\mathbb{P}^{\binom{n}{r}-1}$.

We will show that for any $I \in \{1,2,\dots,n\}$, $|I| = r$,

$$f(G(r,n)) \cap \{z_I = 0\} \cong \mathbb{A}^{r(n-r)}.$$

Recall an element $V = \mathrm{Lin}(v_1, \dots, v_r) \in G(r,n)$ $\left(\text{here } v_1, \dots, v_r \in k^n\right)$ can be represented by a matrix of rank r

$$A = \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{r1} & \cdots & a_{rn} \end{bmatrix} \longmapsto \left[\det A_I\right] \in \mathbb{P}^{\binom{n}{r}}$$

where for $I = \{i_1, \dots, i_r\}$, $A_I = \begin{bmatrix} a_{1i_1} & \cdots & a_{1i_r} \\ \vdots & & \\ a_{ri_1} & \cdots & a_{ri_r} \end{bmatrix}$

Note that if $B = TA$ for $T \in GL_r(k)$, then $f(B) = f(A) \in \mathbb{P}^{\binom{n}{r}-1}$.

This means we may perform row-operations on A keeping the $\mathrm{Lin}(v_1, \dots, v_r) \in G(r,n)$ same.

For now assume $I = \{1,2,..,r\}$ and note that

$$f^{-1}(\{z_I \neq 0\}) = \{ A \mid A_I \text{ is invertible matrix} \}.$$

By performing row operations,

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 & a_{1r+1} & \cdots & a_{1n} \\ 0 & 1 & & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & 0 & & & \\ 0 & \cdots & 0 & 1 & a_{rr+1} & & a_{rn} \end{bmatrix}.$$

<u>HW</u>: Show that $f^{-1}(\{z_I \neq 0\}) \cong Mat_{r \times n-r} = \{ \text{set of } r \times n-r \text{ matrices} \}$

Note that $\mathbb{A}^{r(n-r)} \cong M_{r \times n-r}$. Hence $G(r,n)$ admits an affine cover

$$G(r,n) \cong \bigcup_{\substack{I \subseteq \{1,2,..,n\} \\ |I| = r}} \mathbb{A}^{r(n-r)}.$$

This implies $G(r,n)$ has dimension $r(n-r)$.

<u>HW</u>: • Show that for $I, J \subseteq \{1,2,..,n\}$, $|I| = |J| = r$,

$$f^{-1}(\{z_I \neq 0\}) \cap f^{-1}(\{z_J \neq 0\}) \subseteq \mathbb{A}^{r(n-r)} = f^{-1}(\{z_I \neq 0\})$$

is an open subset.

- Conclude irreducibility of $G(r,n)$.

<u>Rem</u>: Note that $G(2,4) \not\cong \mathbb{P}^2 \times \mathbb{P}^2 \not\cong \mathbb{P}^4$ but they are all covered by $\mathbb{A}^4$.

Rational maps

Def: Let X and Y be irreducible varieties. A rational map
$$f: X \dashrightarrow Y$$
is a morphism $f: U \to Y$ for an open subset (non-empty) $U \subseteq X$.

Let $f_1, f_2 : X \dashrightarrow Y$ be rational maps (with $U_1, U_2 \subseteq X$ open)
We say $f_1 = f_2$ if
$$f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}.$$

Eg: $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ is a rational.
$x \longrightarrow 1/x$

Def: A rational map $f: X \dashrightarrow Y$ is called dominant if $f(U) \subseteq Y$ contain a non-empty open subset. Here $U \subseteq X$ is any open subset with morphism $f: U \to Y$.

Let $f: X \dashrightarrow Y$ and $g: Y \dashrightarrow Z$ be rational maps.
Assume f is dominant. Then we define the composition
$$g \circ f : X \dashrightarrow Z$$
which is a rational map:

Let $U \subseteq X$ open with $f: U \to Y$. and $V \subseteq Y$ with $g: V \to Z$ be morphisms. Since f is dominant, there exists an open subset $V_0 \subseteq Y$ with $V_0 \subseteq f(U)$. Then consider

$$U_0 = f^{-1}(V_0) \cap f^{-1}(V), \text{ and define}$$

$$g \circ f|_{V_0} : U_0 \longrightarrow Z.$$

Def: A rational map $f: X \dashrightarrow Y$ is <u>birational</u> if f is dominant and there is a rational map $g: Y \dashrightarrow X$ such that
$$f \circ g = id_Y \quad \text{and} \quad g \circ f = id_X.$$

In this case, we call X and Y birational.

<u>HW:</u> X is birational to Y if and only if there are open subsets $U \subseteq X$ and $V \subseteq Y$ and an isomorphism
$$f: U \xrightarrow{\sim} V.$$

Function field

Recall if $X \subseteq \mathbb{A}^n$ irreducible affine variety, then $A(X)$ is an integral domain, and
$$K(X) := \text{field of fractions of } A(X).$$

We may view
$$K(X) = \{\text{rational maps } f: X \dashrightarrow \mathbb{A}^1\}. \quad (\text{check it yourself!})$$

Def: Let X be an irreducible variety. A rational map $f: X \dashrightarrow \mathbb{A}^1$ is called a <u>rational function</u>. We define

Function field of X $= K(X) = \{\text{rational functions } f: X \to \mathbb{A}^1\}$
$$= \{ f \in \mathcal{O}_X(U) \mid U \subseteq X \text{ open} \} / \sim$$

where $(f_1 \in \mathcal{O}_X(U_1)) \sim (f_2 \in \mathcal{O}_X(U_2))$ if
$$f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2} \in \mathcal{O}_X(U_1 \cap U_2).$$

Rem: (a) $K(X)$ is a field: If $f_1 \in \mathcal{O}_X(U_1)$ and $f_2 \in \mathcal{O}_X(U_2)$, then $f_1 \cdot f_2$ and $f_1 + f_2$ defined on $U_1 \cap U_2$.

If $0 \neq f \in \mathcal{O}_X(U)$, then

$$f^{-1} \in \mathcal{O}_X(\underbrace{U \setminus Z(f)}_{\text{open}})$$

and hence $f^{-1} \in K(X)$.

(b) If $\varphi : X \dashrightarrow Y$ is a rational dominant map, then

$$\varphi^* : K(Y) \longrightarrow K(X) \quad \text{is a } k\text{-alg. morphism.}$$
$$f \longrightarrow f \circ \varphi$$

Prop: Let X be irreducible and $U \subseteq X$ open. Then

$$K(U) \cong K(X).$$

Proof: The isomorphism of function field is given by

$$\varphi : K(U) \longrightarrow K(X),$$
$$f \in \mathcal{O}_U(V) \rightarrow f \in \mathcal{O}_X(V)$$

with inverse given by

$$\varphi^{-1} : K(X) \longrightarrow K(U)$$
$$f \in \mathcal{O}_X(V) \longrightarrow f|_{U \cap V} \in \mathcal{O}_X(U \cap V).$$

Theorem: Let X and Y be irreducible varieties. TFAE

(a) X and Y are birational

(b) $\exists$ open subsets $U \subseteq X$ and $V \subseteq Y$ with $U \cong V$.

(c) $K(X) \cong K(Y)$ as k-algebra.

Proof: We already established (a) $\Leftrightarrow$ (b) and (b) $\Rightarrow$ (c) $\left(\begin{array}{c} K(X) \cong K(U) \\ \wr\wr \\ K(Y) \cong K(V) \end{array}\right)$.

(c) $\Rightarrow$ (a) : Let $\theta : K(Y) \xrightarrow{\sim} K(X)$ be an isomorphism.

We will construct a birational map

$$f : X \dashrightarrow Y.$$

It is enough to assume X and Y are affine (By prop above!)

Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties.

Let $y_1, \dots, y_m$ be coordinat functions on Y, and consider

$$\theta(y_1), \dots, \theta(y_m) \in K(X).$$

Let $U \subseteq X$ open subset where $\theta(y_1), \dots, \theta(y_m)$ are regular.

Thus, we have

$$\theta : A(Y) \to \mathcal{O}_X(U) \text{ is injective (since } \theta \text{ is isomorphism)}$$

These, we obtain a morphism

$$f : U \longrightarrow Y$$

such that $f^* = \theta$.

<u>HW</u>: f^* is injective implies $f(U)$ is dense in Y.

Hence, we obtain a rational dominant map.

$$f : X \dashrightarrow Y$$

with $f^* = \theta : K(Y) \xrightarrow{\sim} K(X)$.

Similarly, define rational dominant

$$g : Y \dashrightarrow X$$

with $g^* = \theta^{-1} : K(X) \xrightarrow{\sim} K(Y)$.

Then, $f \circ g : X \dashrightarrow X$ such that

$$(f \circ g)^* = \mathrm{id}_X : K(X) \to K(X).$$