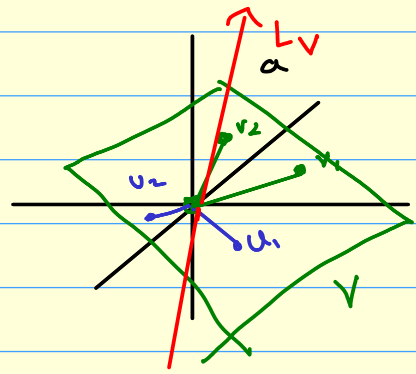
Algebraic Geometry 13

Grassmannian

$G(r,n) = \{ V \subseteq k^n \mid \dim_k V = r \}.$

Eg: $Gr(2,3)$



Let $v_1, v_2 \in k^3$, then Let

$Lin(v_1,v_2) = k\cdot v_1 + k v_2$ be

the linear span of v_1, v_2.

When does $Lin(v_1,v_2) = Lin(u_1,u_2)$

where $v_1 \neq \lambda v_2$ and $u_1 \neq \lambda u_2$?

(Cross product!) Note that $V \subseteq k^2$ dim 2 vec. subspace is uniquely determined by Line $L_V = \{ a = (a_0,a_1,a_2) \in k^3 \mid a_0 v_0 + a_1 v_1 + a_2 v_2 = 0 \ \forall \ v = (v_0,v_1,v_2) \in k^3 \}$

Hence, $G(2,3) = \{ L_V \subseteq k^3 \mid \dim L_V = 1 \} = \mathbb{P}^2$.

Def: Let V, W be a k-vec. space. A multilinear map

$$f : V^m \to W$$

is called alternating if $f(v_1, .., v_n) = 0$ for all $v_1, .., v_n$ with $v_i = v_j$ for some $i \neq j$.

Eg: Let $V = k^3$ and $m = 2$. The cross product

$$f : V^2 \to V$$

$$(\vec{a}, \vec{b}) \to \vec{a} \times \vec{b} := (a_2 b_3 - a_3 b_2, \ a_3 b_1 - a_1 b_3, \ a_1 b_2 - a_2 b_1)$$

is an alternating bilinear map.

Def: An exterior/alternating product $T = \Lambda^m V$ is a vector space and an m-fold multilinear map $\tau : V^m \to T$ that satisfy the following universal property:

For any m-fold alternating multilinear map

$$f : V^m \to W$$

there is a unique linear map $g : T \to W$ such that

$$f = g \circ \tau .$$

$$\begin{tikzcd} V^m \arrow[r, "f"] \arrow[d, "\tau"'] & W \\ T \arrow[ur, dashed, "\exists ! g"'] & \end{tikzcd}$$

Existance: Let $V^{\otimes m} = V \otimes V \otimes \cdots \otimes V$ be m-fold tensor product.

Consider $K = \text{Span}\{ v_1 \otimes \cdots \otimes v_m \in V^{\otimes m} \mid v_i = v_j \text{ for some } i \neq j \}$.

Then

$$\Lambda^m V \equiv T := V^{\otimes m}/K \quad \text{is the unique alternating prod}$$

with the τ given by the composition

$$\begin{tikzcd} \tau : V^m \arrow[r] \arrow[rr, bend right] & V^{\otimes m} \arrow[r] & V^{\otimes m}/K . \end{tikzcd}$$

Here $v_1 \wedge v_2 \wedge \cdots \wedge v_m := \tau(v_1, \ldots, v_m)$.

(use universal property to prove uniqueness).

Eg: $V = k^2$, then

$$\Lambda^0 V \cong k$$
$$\Lambda^1 V = V$$
$$\Lambda^2 V = k$$

$V = k^3$, then

$$\Lambda^0 V \cong k$$
$$\Lambda^1 V \cong V$$
$$\Lambda^2 V \cong V$$
$$\Lambda^3 V \cong k .$$

Basis: Let $V = k e_1 \oplus \cdots \oplus k e_n$. Then

$$V^{\otimes m} \cong \bigoplus_{i_1, \ldots, i_m \in \{1, 2, \ldots, n\}} k \, e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_m}$$

$$\Lambda^m V \cong \bigoplus_{1 \le i_1 < i_2 < \cdots < i_m \le n} k \, e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m} .$$

Note that $\dim(\Lambda^m V) = \binom{n}{m}$.

Eg: $V = k^3$ $\quad \Lambda^2 V = k\, e_1 \wedge e_2 \oplus k\, e_2 \wedge e_3 \oplus k\, e_1 \wedge e_3$.

here $e_i \wedge e_j + e_j \wedge e_i = (e_i + e_j) \wedge (e_i + e_j) = 0$ $\left(\text{in } V^{\otimes 3}/K\right)$.

Lemma 1: Let $v_1, v_2, \ldots, v_m \in k^n$ for $m \leq n$. Then

$$v_1 \wedge v_2 \wedge \cdots \wedge v_m = 0 \in \Lambda^m k^n \iff v_1, v_2, \ldots, v_m \text{ are linearly independent.}$$

Proof: Let $\{e_1, \ldots, e_n\}$ be a basis for k^n, and

$$v_1 = a_{11} e_1 + \cdots + a_{1n} e_n$$
$$v_2 = a_{21} e_1 + \cdots + a_{2n} e_n$$
$$\vdots$$
$$v_m = a_{m1} e_1 + \cdots + a_{mn} e_n .$$

Note that

$\{v_1, v_2, \ldots, v_m\}$ is linearly independent $\iff A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ has rank m

$\iff$ There exists $I = \{i_1, \ldots, i_m\} \subseteq \{1, 2, \ldots, n\}$, the $m \times m$ minor such that $A_I = \begin{bmatrix} a_{1 i_1} & \cdots & a_{1 i_m} \\ \vdots & & \vdots \\ a_{m i_1} & \cdots & a_{m i_m} \end{bmatrix}$ is invertible.

On the other hand,

$$v_1 \wedge v_2 \wedge \cdots \wedge v_m = \sum_{\{i_1, \ldots, i_m\} \subseteq \{1, 2, \ldots, n\}} \left(\sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) \cdot a_{1 i_{\sigma(1)}} \cdots a_{m i_{\sigma(m)}} \right) e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}$$

$$= \sum_{I = \{i_1, \ldots, i_m\} \subseteq \{1, 2, \ldots, n\}} \det A_I \; e_{i_1} \wedge \cdots \wedge e_{i_m}$$

is non-zero if and only if there exist $I = \{i_1, \ldots, i_m\} \subseteq \{1, 2, \ldots, n\}$ such that A_I is invertible. $\square$

Lemma 2: Let $v_1, \dots, v_m \in k^n$ and $w_1, \dots, w_m \in k^n$ are both linearly independent. Then dim-m vector spaces $Lin(v_1, \dots, v_m) = Lin(w_1, \dots, w_m) \subseteq k^n$ if and only if $w_1 \wedge w_2 \wedge \dots \wedge w_m = \lambda(v_1 \wedge v_2 \wedge \dots \wedge v_m)$ for $\lambda \in k \setminus \{0\}$.

Proof: Suppose $Lin(v_1, \dots, v_m) = Lin(w_1, \dots, w_m)$, then consider change of basis matrix $T = (t_{ij})_{i,j=1}^m$

$$w_i = \sum t_{ij} v_j .$$

Then

$$w_1 \wedge w_2 \wedge \dots \wedge w_m = \sum_{\sigma \in S_m} sgn(\sigma)\, t_{1\sigma(1)} \cdots t_{m\sigma(m)}\, v_1 \wedge v_2 \wedge \dots \wedge v_m$$

$$= \det T \; v_1 \wedge v_2 \wedge \dots \wedge v_m .$$

Note that T is invertible, so $\det T \neq 0$.

Suppose $w_1 \wedge w_2 \wedge \dots \wedge w_m = \lambda(v_1 \wedge v_2 \wedge \dots \wedge v_m)$. We will show that $w_i \in Lin(v_1, \dots, v_m)$ for all $1 \le i \le m$. Indeed,

$$w_i \wedge (v_1 \wedge v_2 \wedge \dots \wedge v_m) = \lambda^{-1} w_i \wedge w_1 \wedge \dots \wedge w_m = 0 . \quad \text{for all } 1 \le i \le m.$$

By Lemma 1, $\{w_i, v_1, \dots, v_m\}$ is linearly dependant. Hence, $w_i \in Lin(v_1, \dots, v_m)$.

$\square$

Remark: For any $v \in \Lambda^m k^n$, define the linear map

$$\begin{aligned} \phi_v : \; k^n &\longrightarrow \Lambda^{m+1} k^n . \\ w &\longrightarrow w \wedge v \end{aligned}$$

Set $v = v_1 \wedge v_2 \wedge \dots \wedge v_m$. Then the kernel of ϕ_v is given by

$$\begin{aligned} Ker(\phi_v) &= \{ w \in k^n \mid w \wedge v_1 \wedge \dots \wedge v_m = 0 \} \\ &= Lin(v_1, \dots, v_m) . \end{aligned}$$

<u>Plucker embedding</u>: Consider the map

$$f: G(r,n) \longrightarrow \mathbb{P}^{\binom{n}{r}-1}$$

$$\left\{\begin{array}{l} V \subseteq k^n, \dim V = r \\ \text{for any basis} \\ \{v_1,\dots,v_r\} \text{ of } V \end{array}\right\} \longrightarrow \{v_1 \wedge \cdots \wedge v_r \in \Lambda^r k^n\} \subseteq \Lambda^r k^n \cong k^{\binom{n}{r}}$$

f is well-defined

For any vector subspace $V \subseteq k^n$ of dim r, and linear basis $\{v_1,\dots,v_r\}$ and $\{w_1,\dots,w_r\}$ of V.

$0 \neq v_1 \wedge v_2 \cdots \wedge v_r \in \Lambda^r k$ by Lemma 1, and $v_1 \wedge \cdots \wedge v_r = \lambda\, w_1 \wedge \cdots \wedge w_r$ for some $\lambda \in k \setminus \{0\}$ by Lemma 2.

<u>Theorem</u>: $Gr(r,n)$ is a projective subvariety of $\mathbb{P}^{\binom{n}{r}-1}$ given by Plucker embedding.

<u>Lemma</u>: For a fixed non-zero element $w \in \Lambda^r k^n$ with $r < n$, consider the linear map

(HW)

$$\phi: k^n \longrightarrow \Lambda^{r+1} k^n$$
$$v \longrightarrow v \wedge w$$

Then $\operatorname{rank}(\phi) \geq n-r$, with equality holding if and only if $w = v_1 \wedge v_2 \wedge \cdots \wedge v_r$ for some $v_1,\dots,v_r \in k^n$. (Pure exterior power).

<u>Proof of Theorem</u>: $G(n,n) \subseteq \mathbb{P}^{\binom{n}{n}-1} = \mathbb{P}^0$ is a point, hence a projective variety.

Assume $r < n$.

By construction $[w] \in \mathbb{P}^{\binom{n}{r}-1}$ lie in $f(G(r,n))$ if and only if w is a pure tensor $w = v_1 \wedge v_2 \wedge \cdots \wedge v_r$.

By Lemma, w is a pure tensor $\iff$ $\phi : k^n \to \wedge^{r+1} k^n$, $v \mapsto v \wedge w$ has rank $n-r$.

We know that $\operatorname{rank}(\phi) \geq n-r$ in general,

$\operatorname{rank} \phi = n-r \iff$ all $(n-r+1)\times(n-r+1)$ minors vanish.

Note that the matrix defining ϕ has coefficients in coordinates of $w = [v_{I_1} : \cdots : v_{I_{\binom{n}{r}}}] \in \mathbb{P}^{\binom{n}{r}-1}$, when

$I_1, \dots, I_{\binom{n}{r}}$ are all r-element subsets of $\{1,2,\dots,n\}$.

Hence image $f(G(r,n)) \subseteq \mathbb{P}^{\binom{n}{r}-1}$ is a closed set.

Eg: Fix basis $\{e_1, e_2, e_3, e_4\}$ of k^4. Then the Plucke embedding of $G(2,4)$ in coordinates is given by $f : G(2,4) \to \mathbb{P}^5$

$$\left\{ \begin{array}{c} v_1 \\ v_2 \end{array} \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}}_{A} \right\} \xrightarrow{f} \Big[\overset{\omega_{\{1,2\}}}{a_{11}a_{22} - a_{21}a_{12}} : \overset{\omega_{\{1,3\}}}{a_{11}a_{23} - a_{13}a_{21}} : \overset{\omega_{\{1,4\}}}{a_{11}a_{24} - a_{21}a_{14}} : \underset{\omega_{\{2,3\}}}{a_{12}a_{23} - a_{22}a_{13}} : \underset{\omega_{\{2,4\}}}{a_{12}a_{24} - a_{22}a_{14}} : \underset{\omega_{\{3,4\}}}{a_{13}a_{24} - a_{23}a_{14}} \Big] \in \mathbb{P}^5$$

Check that for any for any 2×2 invertible matrix T,

$f(T \times A) = f(A)$ (i.e the map is well defined).

HW: Find explicitly the image $f(G(2,4)) \subseteq \mathbb{P}^5$.