Algebraic Geometry 11-12

Segre Embedding

For any $n, m \geq 0$, Let $N = (n+1)(m+1) - 1$. The segre morphism is

$$f : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N$$

$$([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]) \longrightarrow \begin{aligned}[t] [\, & x_0 y_0 : x_0 y_1 : \cdots : x_0 y_m : \\ & x_1 y_0 : x_1 y_1 : \cdots : x_1 y_m : \\ & \quad \vdots \\ & x_n y_0 : x_n y_1 : \cdots : x_n y_m \,] \end{aligned}$$

Let $[z_{ij} : 0 \leq i \leq n, 0 \leq j \leq m]$ denote the coordinates of $\mathbb{P}^N$.

Prop: $\boxed{\text{I}}$ The image $X = f(\mathbb{P}^n \times \mathbb{P}^m)$ is a projective variety given by

$$X = Z(z_{ij} z_{kl} - z_{il} z_{kj} : 0 \leq i, k \leq n,\ 0 \leq j, l \leq m).$$

$\boxed{\text{II}}$ f is a bijection, both f and f^{-1} are given by polynomial maps (f defines isomorphism).

Proof: We will prove here for $n = m = 1$.

$$f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

$\boxed{\text{I}}$ Note $f(\mathbb{P}^1 \times \mathbb{P}^1)$ is contained in X. conversely, let $z = [z_{00} : z_{01} : z_{10} : z_{11}] \in X$. We will show $z \in \mathrm{Im}(f)$.
Assume $z_{00} \neq 0$ (other case can be dealt similarly).
Set $z_{00} = 1$, and $x_0 = y_0 = 1$, $x_1 = z_{10}$, $y_1 = z_{01}$.
Using $\boxed{z_{11} = z_{10} \cdot z_{01}}$,
$f([x_0 : x_1], [y_0 : y_1]) = [1 : y_1 : x_1 : x_1 y_1] = z$.

$\boxed{\text{II}}$ Continue assuming $z_{00} = 1$ and consider affine variety

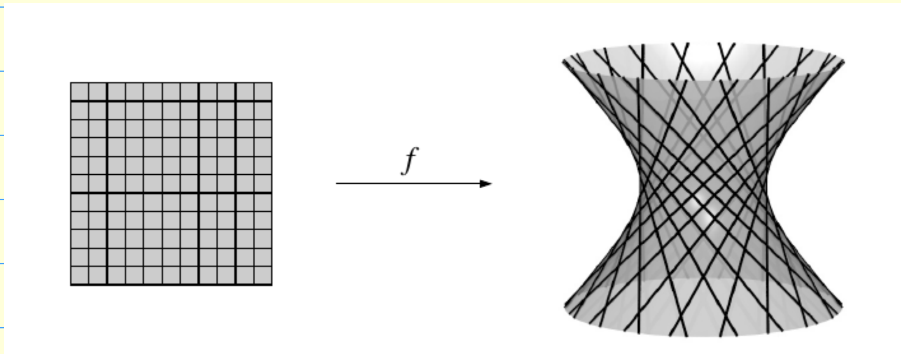
$$U_0 = X \cap \{z_{00} \neq 0\} \subseteq \mathbb{A}^3.$$
$$= \{(z_{01}, z_{10}, z_{11}) : z_{11} = z_{01} \cdot z_{10}\} \subseteq \mathbb{A}^3$$

On the other hand,

$$V_0 = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \{x_0 \neq 0,\ y_0 \neq 0\} \cong \mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$$
$$= \{(x_1, y_1) \in \mathbb{A}^2\}$$

$$\left.\begin{array}{lll} f : V_0 \longrightarrow U_0 & & \text{morphism} \\ (x_1, y_1) \longrightarrow (y_1, x_1, x_1 y_1) & & \\ f^{-1} : U_0 \longrightarrow V_0 & & \\ (z_{01}, z_{10}, z_{11}) \to (z_{01}, z_{10}) & & \text{morphism} \end{array}\right\} \Rightarrow f \text{ is isomorphism.}$$

Similarly, f induces isomorphism for each affine chart and thus $f : \mathbb{P}^1 \times \mathbb{P}^1 \to X$ is isomorphism of Variety.



$\mathbb{P}^1 \times \mathbb{P}^1$ $\qquad$ $X \subset \mathbb{P}^3$

Corollary: The diagonal $\Delta_{\mathbb{P}^n} = \{(x, y) : x = y \in \mathbb{P}^n\} \subseteq \mathbb{P}^n \times \mathbb{P}^n$ is closed.

Proof: Observe that

$$\Delta_{\mathbb{P}^n} = \{[x_0 : \cdots : x_n], [y_0 : \cdots : y_n] \mid x_i y_j = x_j y_i \ \forall\ 0 \le i, j \le n\}.$$

In Segre embedding,

$$\Delta_{\mathbb{P}^n} \hookrightarrow \mathbb{P}^n \times \mathbb{P}^n \xrightarrow[\sim]{f} X \subseteq \mathbb{P}^N \qquad (N = (n+1)^2 - 1)$$

Segre relations

$$f(\Delta_{\mathbb{P}^n}) = \left\{ z = [z_{00} : \cdots : z_{0n} : \cdots : z_{nn}] \;\middle|\; \begin{array}{l} z_{ij} z_{kl} = z_{il} z_{kj} \\ \text{and } \; z_{ij} = z_{ji} \end{array} \right\}$$

Diagonal relation

So, $f(\Delta_{\mathbb{P}^n}) \subseteq \mathbb{P}^N$ is closed, hence $\Delta_{\mathbb{P}^n} \subseteq \mathbb{P}^n \times \mathbb{P}^n$ is closed.

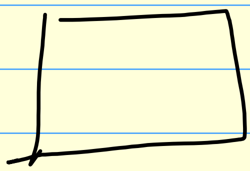
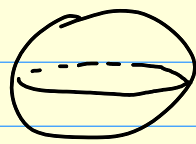
<u>Corollary</u>: Let $X \subseteq \mathbb{P}^n$ be a proj. variety, then $\Delta_X = \{(x,x) \mid x \in \mathbb{P}^n\} \subseteq X \times X$ is closed.

<u>Proof</u>: Note that $\Delta_X = \underbrace{\Delta_{\mathbb{P}^n}}_{\text{closed}} \cap X \times X \subseteq \mathbb{P}^n \times \mathbb{P}^n$.

It is enough to show $X \times X \subseteq \mathbb{P}^n \times \mathbb{P}^n$ is closed.

HW: Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be closed subsets. Then $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is a closed subset and hence a projective variety using Plucker embedding.

<u>Compactness</u>: In Eucledean topology $k = \mathbb{C}$,

$\mathbb{A}^1$ =  $\mathbb{P}^1$ = 

not compact compact

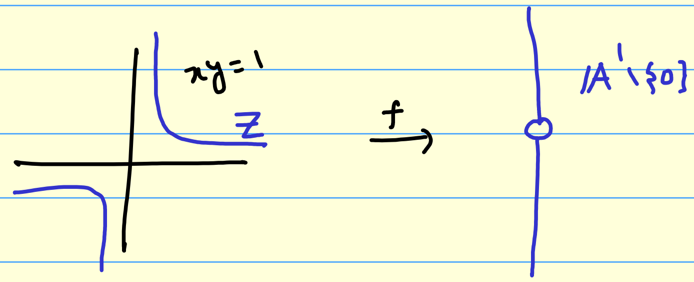
But in Zariski topology, $\mathbb{A}^1$ and $\mathbb{P}^1$ is both compact!

Def: A map $f: X \to Y$ is called closed if for every $A \subseteq X$ closed, $f(A) \subseteq Y$ is closed.

Eg: ① If X, Y are Hausdorff, X compact, then continous map $f: X \to Y$ is closed.

② $f: \mathbb{A}^2 \to \mathbb{A}^1$ $\qquad$ $f(Z(xy-1)) = \mathbb{A}^1 \setminus \{0\}$ not closed!
$(x,y) \to (y)$

$\left\{ \mathbb{A}^1, \mathbb{A}^2 \text{ are compact but not Hausdorff} \right\}$



Def: A ringed topological space $(X, \mathcal{O}_X)$ is called a Variety, if
(i) There is a finite open cover $X = \bigcup_{i=1}^{n} U_i$, such that U_i is an affine variety with sheaf of regular function $\mathcal{O}_{U_i} \cong \mathcal{O}_X|_{U_i}$; and
(ii) $\Delta_X \subseteq X \times X$ is closed.

Def: We say a variety X is complete if for every variety Y, the projection
$$\pi: X \times Y \longrightarrow Y$$
is a closed map.

Theorem: The projective space $\mathbb{P}^n$ is a complete variety.

We will first prove a technical proposition!

Proposition: The projection map $\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is closed.

Proof: Let $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be a closed subset. Then

$Z = Z(f_1, \dots, f_r)$, where $f_i(x, y)$ = bihomogenous polynomial in $(x_0, \dots, x_n ; y_0, \dots, y_m)$.

Why bihomogenous?
In Segree coordinates $[z_{00} : \cdots z_{nm}]$, f_i should be homogenous $\Rightarrow$ since $z_{e,k} = x_e \cdot y_k$, degree d poly in $z_{e,k}$'s means degree d bihomogenous polynomial in $(x_0, \dots, x_n)$ and $(y_1, \dots, y_n)$.
Here, $f(x_0, x_1 ; y_0, y_1) = x_0^2 y_1 y_0 + x_0 x_1 y_0^2$ is deg 2 bihomog. poly.

We may assume $f_1, \dots, f_r$ has bidegree d.
Consider a fixed point $a \in \mathbb{P}^m$;

$$g_i(x_0, \dots, x_n) := f_i(x_0, \dots, x_n, a_0, \dots, a_m) \in k[x_0, \dots, x_n]$$

Note that
$a \notin \pi(Z) \subseteq \mathbb{P}^m$. $\Leftrightarrow$ there is no $x \in \mathbb{P}^n$ such that $(x, a) \in Z$.

$$\Leftrightarrow \quad Z(g_1, \dots, g_r) = \phi$$

$$\Leftrightarrow \quad \sqrt{\langle g_1, \dots, g_r \rangle} \supseteq \langle x_0, \dots, x_n \rangle$$

$$\Leftrightarrow \quad \exists \; k_0, \dots, k_n \text{ such that } x_i^{k_i} \in \langle g_1, \dots, g_r \rangle$$

$$\Leftrightarrow \quad k[x_0, \dots, x_n]_k \subseteq \langle g_1, \dots, g_r \rangle \text{ for } k = k_1 + \dots + k_r.$$

Note that $k \geq d$ and $k[x_0,\dots,x_n]_k = \langle g_1,\dots,g_r\rangle_k$.

Define map

$$\Phi_k : \left(k[x_0,\dots,x_n]_{k-d}\right)^r \longrightarrow k[x_0,\dots,x_n]_k$$

$$(h_1,\dots,h_r) \longrightarrow \sum_{i=1}^{r} h_i g_i$$

$$\Phi_k \text{ is surj} \quad \Longleftrightarrow \quad r\binom{n+k-d}{k-d}\overbrace{\begin{bmatrix} \underbrace{\begin{matrix} \vdots & - & - & \vdots \\ \vdots & & & \vdots \\ \vdots & - & - & \end{matrix}}_{I} & \quad \end{bmatrix}}^{\binom{n+k}{k}}$$

There is $\binom{n+k}{k}$ - minor that is non zero.

$\det(M_I(a)) \neq 0$. Entries of M_I are polynomial in a.

$(\det M_I(a) \neq 0)$ is open in $\mathbb{P}^n$.

Hence, $a \in \mathbb{P}^n$ such that $a \notin \pi(Z)$ is open in $\mathbb{P}^n$.

$\Rightarrow \quad \pi(Z)$ is closed.

<u>Cor</u>: For any variety Y, $\pi : \mathbb{P}^n \times Y \to Y$ is closed.

<u>Proof.</u> Let $Y \subseteq \mathbb{A}^m$ be an affine variety. Let $Z \subseteq \mathbb{P}^n \times Y$ be closed and let $\bar{Z} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be closure of Z.

$\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is closed $\Rightarrow$ $\pi(\bar{Z}) \subseteq \mathbb{P}^m$ is closed.

Note $\pi(Z) = \pi(\bar{Z} \cap \mathbb{P}^n \times Y) = \pi(\bar{Z}) \cap Y$ is closed in Y.

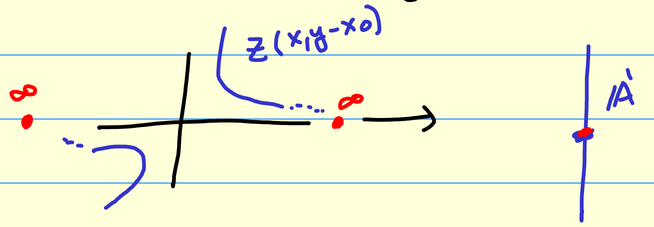
(If Y is any variety, Y is covered by affines) ✓

Conclusion: $\mathbb{P}^n$ is complete.

HW: Show that a closed subvariety Y of a complete subvariety is complete. In particular, all projective varieties $X \subseteq \mathbb{P}^n$ are complete.

Eg: $\mathbb{P}^1 \times \mathbb{A}^1 \xrightarrow{f} \mathbb{A}^1$ is a closed map.
$([x_0 : x_1], y) \to y$

$$f(Z(x_1 y - x_0)) = \mathbb{A}^1.$$



Prop: Let X be a complete variety. Let $f: X \to Y$ be a morphism, then $f(X) \subseteq Y$ is a closed subvariety.

Proof: Consider the graph of f, $\Gamma_f := \{(x, y) \in X \times Y \mid y = f(x)\}$.

$$\begin{tikzcd} X \arrow[r, "(id{,}f)"] & X \times Y \arrow[d, "\pi \leftarrow \text{closed}"] \\ & Y \end{tikzcd}$$

We will show that Γ_f is closed in $X \times Y$: Consider the morphism $X \times Y \xrightarrow{(f, id)} Y \times Y$. (use universal property!)

Then $\Gamma_f = (f, id)^{-1} \Delta_Y$. Recall Δ_Y is closed for any variety, hence Γ_f is closed.

Cor: For any complete connected variety X, $\mathcal{O}_X(X) \cong k$.

Proof: A regular function $f \in \mathcal{O}_X(X)$ defines a morphism

$f: X \to \mathbb{A}^1$. Compose it with $i: \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$,

$i \circ f: X \to \mathbb{P}^1$ and $(i \circ f)(X) \subseteq \mathbb{A}^1$.

By Prop., $(i \circ f)(X) \subseteq \mathbb{P}^1$ is closed and connected. (and not $\mathbb{P}^1$),

$$\Rightarrow (i \circ f)(X) = \{a\} \subseteq \mathbb{A}^1 \subseteq \mathbb{P}^1.$$

$\Rightarrow$ f is constant function. $\Rightarrow$ $\mathcal{O}_X(X) \cong k$.

<u>Grassmannian</u>:

<u>Def:</u> Let r and n be positive integers and $r < n$. Grassmannian
$G(r,n) = \{ V \subseteq k^n : V \text{ is } r\text{-dim vector subspace} \}$.

<u>Eg</u>: $\mathbb{P}^r = G(1, r+1) = \{ L \subseteq k^{r+1} \mid L \text{ is dim-1 vector subspaces} \}$

$$= \left(k^{r+1} \setminus \{0\}\right) / \text{scaling}.$$

We will show that $G(r,n)$ is a projective variety.