Algebraic Geometry 10

Recall: Let $X \subset \mathbb{P}^n$ be projective variety. Then

$$S(X) = \bigoplus_{d=0}^{\infty} S(X)_d \quad \left(= k[x_0, \dots, x_n]/I(X)\right)$$

is a the homogeneous coordinate ring.

Def: Let $U \subseteq X \subseteq \mathbb{P}^n$ be an open subset of X. We say a function $\varphi : U \to k$ is regular if:

For any point $a \in U$, there exists an open set $U_a \subseteq U$ containing a, such that

$$\varphi(x) = \frac{g(x)}{h(x)}, \text{ where } g(x), h(x) \in S(X)_d \text{ for some } d \in \mathbb{N}.$$

Def: The sheaf of regular function on X is denoted by $\mathcal{O}_X$: for any open $U \subseteq X$,

$$\mathcal{O}_X(U) := k\text{-algebra of regular function on } U.$$

For any subset $V \subseteq X$, let $\mathcal{O}_V := \mathcal{O}_X|_V$. Then $V, \mathcal{O}_V$ is a ringed topological space (not necessarily an affine or projective variety).

Proposition: Let $X \subseteq \mathbb{P}^n$ be a proj. variety. Then

$$U_i = \{ [x_0 : \dots : x_n] \in X \mid x_i \neq 0 \}$$

is an affine variety.

Proof: Let $X = Z(J) \subseteq \mathbb{P}^n$, where $J \subseteq k[x_0, \dots, x_n]$ is a homogeneous ideal. We will assume $i = 0$ (the proof works for all i, after appropriate modification).

Let $\tilde{J} \subseteq k[x_1, \dots, x_n]$ be dehomogenization of J (set $x_0 = 1$), and define the affine variety

$$Y := Z(\tilde{J}).$$

We will describe an isomorphism (of ringed topological spaces)

$$F: Y \longrightarrow U_0$$
$$(x_1, \dots, x_n) \longrightarrow [1 : x_1 : \dots : x_n]$$

- F is a bijection (check that $F^{-1}([x_0 : \dots : x_n]) = (\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$ defines an inverse of F).

- F is continuous: Any closed set $V \subseteq X \subseteq \mathbb{P}^n$ given by $I = I(V)$. Then $F^{-1}(V \cap U_0) = Z(\tilde{I})$ is closed in Y, where $\tilde{I} \subseteq k[x_1, \dots, x_n]$ is dehomogenization of I.

F^{-1} is continous: For any closed set $W \subseteq Y \subseteq \mathbb{A}^n$, Let $I = I(W)$, then

$$F(W) = Z(I^h)$$

where $I^h \subseteq k[x_0, \dots, x_n]$ is homogenization of I

- F and F^{-1} pull back regular function to regular function: Let φ be a regular function on Y, then "locally"

$$\varphi = \frac{g(x_1, \dots, x_n)}{f(x_1, \dots, x_n)} \text{ on open subsets } W \subseteq Y$$

Thus "locally", we have

$$(F^{-1})^* \varphi = \frac{g(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})}{f(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})} = \frac{x_0^d \, g(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})}{x_0^d \, f(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})}.$$

Let $d = \max(\deg g, \deg f)$, then the denominator and numerator are both of same degree. Hence $(F^{-1})^* \varphi$ is regular.

Similarly, one can show that $F^* \varphi$ is regular: Locally,

$$F^*(\varphi) = \left. \frac{g(x_0, \dots, x_n)}{f(x_0, \dots, x_n)} \right|_{x_0 = 1} = \frac{g(1, x_1, \dots, x_n)}{f(1, x_1, \dots, x_n)}.$$

Recall lemma from Lecture 6:

Lemma: Let X and Y be affine varieties, and $U \subseteq X$ open subset. Then $\varphi : U \to Y$ is a morphism if and only if
$$\varphi = (f_1, \dots, f_n) : U \longrightarrow Y \subseteq \mathbb{A}^n$$
where $f_1, \dots, f_n \in \mathcal{O}_X(U)$.

Lemma: Let $X \subseteq \mathbb{P}^n$ be an proj. variety and let $f_0, \dots, f_m \in S(X)$ be homogeous polynomials of same degree d.
Then, on the open set $U = X \setminus Z(f_0, \dots, f_m)$,
$$f : U \longrightarrow \mathbb{P}^m$$
$$x \to [f_0(x) : \cdots : f_m(x)] \quad \text{is a morphism.}$$

Proof: f is well-defined since for any $x \in U$, $(f_0(x), \dots, f_m(x)) \neq (0, \dots, 0)$ and for any $\lambda \in k^*$,
$$\left(f_0(\lambda x_0, \dots, \lambda x_n), \dots, f_m(\lambda x_0, \dots, \lambda x_n) \right) = \left(\lambda^d f_0(x_0, \dots, x_n), \dots, \lambda^d f_m(x_0, \dots, x_n) \right).$$

f is a morphism: Let $V_i = \mathbb{P}^m \setminus Z(y_i) = \{ [y_0 : \cdots : y_m] \mid y_i \neq 0 \}$ and
Let $U_i := f^{-1}(V_i) = \{ x \in U \mid f_i(x) \neq 0 \}$.
Using gluing property, it is enough to show that
$$f\big|_{U_i} : U_i \to V_i \cong \mathbb{A}^n$$
is a morphism of affine varieties. Note that in affine coordinates of V_i,
$$f\big|_{U_i}(x) = \left(\frac{f_0(x)}{f_i(x)}, \dots, \frac{f_{i-1}(x)}{f_i(x)}, \frac{f_{i+1}(x)}{f_i(x)}, \dots, \frac{f_m(x)}{f_i(x)} \right)$$
where $\frac{f_j}{f_i}$ is quotient of polynomial functions on U_i.

Using Lemma $f|_{U_i}$ is a morphism for each $0 \le i \le m$. $\square$

Eg: $\boxed{1}$ Let $Gl_{n+1}(k) := \left\{ A = \begin{bmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & & \\ a_{n0} & \cdots & a_{nn} \end{bmatrix} : \det A \neq 0 \right\}$.

For any $A \in Gl_{n+1}(k)$, we have an isomorphism

$$\begin{aligned} f: \mathbb{P}^n &\to \mathbb{P}^n \\ x &\to Ax \end{aligned}.$$

Check: f is an isomorphism.

$\boxed{11}$ Projection from a point to a hyperplane.

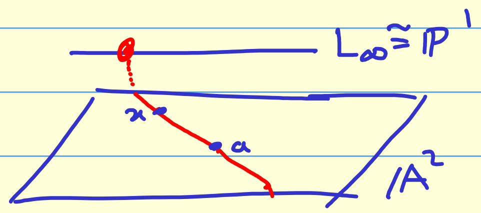
Let $a \in \mathbb{P}^n$ and $H = Z(c_0x_0 + \cdots + c_nx_n) \cong \mathbb{P}^{n-1} \subseteq \mathbb{P}^n$.
Assume $a \notin H$. Then

$$\begin{aligned} f: \mathbb{P}^n \setminus \{a\} &\longrightarrow H \cong \mathbb{P}^{n-1} \\ x &\longrightarrow H \cap \underbrace{\{[sx_0 + ta_0 : \cdots : sx_n + ta_n] \mid [s:t] \in \mathbb{P}^1\}}_{\text{line passing connecting } a \text{ and } x.} \end{aligned}$$

is a morphism.

Eg: $a = [1:0:0] \in \mathbb{A}^2 \subseteq \mathbb{P}^2$

$$\begin{aligned} f: \mathbb{P}^2 \setminus \{a\} &\to \mathbb{P}^1 \\ [x_0 : x_1 : x_2] &\to [x_1 : x_2] \end{aligned}$$



f is a morphism. (use previous lemma!)

Product: Recall $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$ is affine variety, while $\mathbb{A}^1 \times \mathbb{A}^1$ does not have the product topology.

Def: Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties, then $X \times Y \subseteq \mathbb{A}^{n+m}$ is an affine variety with

$$I(X \times Y) = \langle I(X), I(Y) \rangle \subseteq k[x_1, \ldots, x_n, y_1, \ldots, y_m].$$

Universal property of product: Let X, Y be affine variety. Then for any affine variety Z and morphism
$$f_X : Z \to X \quad \text{and} \quad f_Y : Z \to Y$$
there exists a unique morphism $f : Z \to X \times Y$ such that
$$f_X = \pi_X \circ f \quad \text{and} \quad f_Y = \pi_Y \circ f .$$
where π_X and π_Y are projection maps $X \times Y$ to X and Y respectively.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "f_Y"] \arrow[ddr, bend right, "f_X"'] & & \\
& X \times Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \\
& X &
\end{tikzcd}$$

Proof: HW.

How to define product $\mathbb{P}^1 \times \mathbb{P}^1$? $\mathbb{P}^1 \times \mathbb{P}^1 \neq \mathbb{P}^2$ (why?)

Construction: $f : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow Z(x_0x_3 - x_1x_2) \subseteq \mathbb{P}^3$
$$([s:t],[u:v]) \to [su : sv : tu : tv]$$

Show that f is a bijection.

Note that
$$f^{-1}(x_0 : x_1 : x_2 : x_3) = \begin{cases} ([x_0 : x_2], [x_0 : x_1]) & x_0 \neq 0 \\ ([x_1 : x_3], [x_0 : x_1]) & x_1 \neq 0 \\ ([x_0 : x_2], [x_2 : x_3]) & x_2 \neq 0 \\ ([x_1 : x_3], [x_2 : x_3]) & x_3 \neq 0 \end{cases}$$

Show that this is well-defined!

Segre Embedding

For any $n, m \geq 0$, Let $N = (n+1)(m+1) - 1$. The segre morphism is

$$f: \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N$$

$$([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]) \longrightarrow [x_0 y_0 : x_0 y_1 : \cdots : x_0 y_m : x_1 y_0 : x_1 y_1 : \cdots : x_1 y_m : \cdots : x_n y_0 : x_n y_1 : \cdots : x_n y_m]$$

Let $[z_{ij} : 0 \leq i \leq n, 0 \leq j \leq m]$ denote the coordinates of $\mathbb{P}^N$.

Prop: ① The image $X = f(\mathbb{P}^n \times \mathbb{P}^m)$ is a projective variety given by

$$X = Z(z_{ij} z_{kl} - z_{il} z_{kj} : 0 \leq i,k \leq n, 0 \leq j,l \leq m).$$

② f is a bijection, both f and f^{-1} are given by polynomial maps (f defines isomorphism).