Algebraic Geometry-1

Lemma: (i) If $S, T \subseteq k[x_1, \dots, x_n]$, then
$$Z(S) \cup Z(T) = Z(ST).$$
finite union of algebraic sets

(ii) If $\{S_i\}_{i \in I}$ is a family of subsets of $k[x_1, \dots, x_n]$, then
$$\bigcap_{i \in I} Z(S_i) = Z\left(\bigcup_{i \in I} S_i\right).$$
arbitrary intersections of algebric sets

Proof: (i) "$\subseteq$" Let $p \in Z(S) \cup Z(T)$. Let $f \in S$ and $g \in T$, then $f(p) = 0$ or $g(p) = 0$. This impies
$$f \cdot g(p) = 0 \Rightarrow p \in Z(S \cdot T)$$
"$\supseteq$" Let $p \in Z(S \cdot T)$ such that $p \notin Z(S)$. Then there exists $f \in S$ such that $f(p) \neq 0$.
For any $g \in T$, $(f \cdot g)(p) = 0 \Rightarrow g(p) = 0$. Hence $p \in Z(T)$.

(ii) $\quad p \in Z(S_i) \ \forall i \in I \iff f(p) = 0 \ \forall f \in \bigcup_{i \in I} S_i$
$\iff p \in Z\left(\bigcup_{i \in I} S_i\right).$

The above suggests us to define a topology for $\mathbb{A}^n$.

Recall: A topology on a set X is a collection of open sets such that

(i) ϕ and X are open

(ii) A finite intersetion of open is open

(iii) An arbitrary union of open is open.

$B \subseteq X$ is open $\Leftrightarrow$ $X \setminus B \subseteq X$ is closed

Topology on X is a collection of closed sets such that

(i) X and ϕ are closed

(ii) A finite union of closed sets is closed

(iii) A arbitrary intersection of closed sets is closed.

Def: The Zariski topology on $\mathbb{A}^n$ is the topology whose closed sets are the affine algebraic sets.

Eg: (1) Closed subsets in $\mathbb{A}^1$: All finite subsets of $\mathbb{A}^1$ + $\mathbb{A}^1$ + ϕ.

Proof: Any non-constant polynomial $f \in k[x]$ has finitely many zeros, and for any finite set $V = \{a_1, \dots, a_m\}$, let $f(x) = \prod_{i=1}^{m} (x - a_i)^{l_i}$. with $l_i \geq 1$, then $Z(f) = V$.

(Here we use the fact that $k[x]$ is a PID).

(ii) Every point $(a_1, \dots, a_n) \in \mathbb{A}^n$ is closed. Hence all finite sets are closed.

Def: Let $X \subseteq \mathbb{A}^n$ be a subset. A Zariski topology on X is given by the induced topology from $\mathbb{A}^n$, that is,

$V \subseteq X$ is closed if and only if $V = Y \cap X$ where $Y \subseteq \mathbb{A}^n$ is closed.

Why is Zariski topology is better then Euclidean topology for algebraic geometry?

(i) Works for arbitrary field k ($k = \mathbb{R}$ or $\mathbb{C}$ not required)

(ii) Polynomials $f \in k[x_1, .., x_n]$ define continous functions

$$f : \mathbb{A}^n \to \mathbb{A}^1$$

(Zariski topology on $\mathbb{A}^n$ and $\mathbb{A}^1$).

Ideal associated to a subset of $\mathbb{A}^n$.

Def: Let $X \subseteq \mathbb{A}^n$ be any subset. Then the ideal of X is

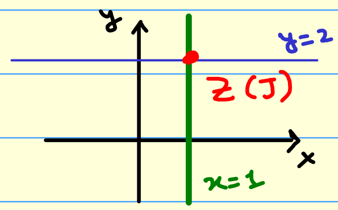
$$I(X) := \{ f \in k[x_1, \dots, x_n] : f(p) = 0 \ \forall \ p \in X \}.$$

(Show that $I(X)$ is indeed an ideal)

Rem: If $X \subseteq Y \subseteq \mathbb{A}^n$, then $I(X) \supseteq I(Y)$.

Eg: Consider ideal $J = \langle (x-1)^2, y-2 \rangle \subseteq k[x,y]$. Then

$X := Z(J) = \{(1,2)\} \subseteq \mathbb{A}^2$



What is the ideal of X?

It is $I(X) = \langle (x-1), (y-2) \rangle \subseteq k[x,y]$ (prove it!)

Observation: $\boxed{I(Z(J)) \neq J}$.

Aim: Find a correspondance

$$\{ \text{ideals in } k[x_1, .., x_n] \} \overset{?}{\longleftrightarrow} \left\{ \begin{array}{l} \text{affine} \\ \text{alg. sets in } \mathbb{A}^n \end{array} \right\}$$

Better notion needed!

Def: Let R be a ring and $J \subseteq R$ be an ideal. The <u>radical</u> of J is defined by

$$\sqrt{J} := \{ f \in R : f^m \in J \text{ for some } m \in \mathbb{N} \}$$

An ideal $J \subseteq R$ is called <u>radical ideal</u> if

$$\sqrt{J} = J .$$

<u>H.W 1</u>: Show that $\sqrt{J}$ is an ideal.

In the example above $J = \langle (x-1)^2, y-2 \rangle$ has radical $I = \sqrt{J} = \langle x-1, y-2 \rangle$.

<u>Lemma</u>: For any $X \subseteq \mathbb{A}^n$, $I(X)$ is a radical ideal.

<u>Proof</u>: Let $f \in k[x_1, .., x_n]$ such that $f^m \in I(X)$. Then for all $p \in X$, $f^m(p) = 0$. This implies $f(p) = 0$ $\forall p \in X$. Hence $f \in I(X)$.

<u>Lemma</u>: For any ideal $J \subseteq k[x_1, .., x_n]$,

$$Z(\sqrt{J}) = Z(J).$$

<u>Proof</u>: "$\subseteq$" Since $J \subseteq \sqrt{J}$, $Z(\sqrt{J}) \subseteq Z(J)$ (by Lemma in Lecture 0).

"$\supseteq$" Let $p \in Z(J)$ and $f \in \sqrt{J}$. Then $f^m \in J$ for $m \in \mathbb{N}$, thus $f^m(p) = 0 \Rightarrow f(p) = 0$. Hence $p \in Z(\sqrt{J})$.

One-one correspondance:

$$\left\{ \begin{array}{l} \text{radical} \\ \text{ideals in } k[x_1,\dots,x_n] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Affine algebraic sets} \\ \text{in } \mathbb{A}^n \end{array} \right\}$$

<u>Maps</u>:

$$J \longrightarrow Z(J)$$

$$I(X) \longleftarrow X$$

well defined by Lemmas above!

<u>Question</u>: Are these maps inverses of each other?

<u>Theorem</u> (Hilbert Nullstellensatz / theorem of zeros)

a) For any affine alg. set $X \subseteq \mathbb{A}^n$, we have $Z(I(X)) = X$

b) For any ideal $J \subseteq k[x_1,\dots,x_n]$, then $I(Z(J)) = \sqrt{J}$.

<u>Proof</u>: a) "$\supseteq$" If $p \in X$, then $f(p) = 0$ $\forall f \in I(X)$.
Hence $p \in Z(I(X))$.

"$\subseteq$" Since X is affine alg. set, $X = Z(J)$ for some ideal $J \subseteq k[x_1,\dots,x_n]$. Note that

$$I(Z(J)) \supseteq J$$

because if $f \in J$, $f(p) = 0$ $\forall p \in X = Z(J)$.

Hence,

$$Z(I(X)) = Z(I(Z(J))) \subseteq X.$$

(b) "$\supseteq$" $\quad f \in \sqrt{J} \Rightarrow f^N \in J$ for some $N \in \mathbb{N}$ $\Rightarrow f^N(p) = 0 \quad \forall p \in Z(J)$

$$\Rightarrow f(p) = 0 \quad \forall p \in Z(J)$$

$$\Rightarrow f \in I(Z(J)).$$

"$\subseteq$" This is the difficult part. We will use weak Hilbert Nullstellensatz (without proof).

(Recall: k is an algraically closed field)

Theorem (weak Hilbert Nullstellensatz): Let $J \subsetneq k[x_1, \ldots, x_n]$ be a proper ideal. Then

$$Z(J) \neq \phi.$$

Rem: $\boxed{\mathrm{I}}$ When $n=1$, let $J = \langle f \rangle \subsetneq k[x]$ be a proper ideal. Then f is not a constant, and

$$Z(J) = Z(f) \neq \phi \quad \text{(since } k \text{ is algebraically closed)}$$

$\boxed{\mathrm{II}}$ $k =$ algebraically closed is necessary: Consider $\langle x^2+1 \rangle \subseteq \mathbb{R}[x]$.

Back to proof of (b): "$\subseteq$" Let $f \in I(Z(J))$, we aim to show that $f^N \in J$ for some $N \in \mathbb{N}$.

Step 1: By Hilbert Basis theorem, $J = \langle f_1, \ldots, f_m \rangle \subseteq k[x_1, \ldots, x_n]$.

Consider the ideal

$$H = \langle f_1, f_2, \ldots, f_m, f \cdot t - 1 \rangle \subseteq k[x_1, \ldots, x_n, t].$$

For points $p \in \mathbb{A}^n$ and $a \in \mathbb{A}^1$, denote $(p,a) \in \mathbb{A}^{n+1}$.

Step 2: If $H \subsetneq k[x_1, \dots, x_n, t]$ is a proper ideal, we apply Weak Hilb. Null. to show that there exists a point $(p,a) \in Z(H) \subseteq \mathbb{A}^{n+1}$.

This impies $\underbrace{f_1(p) = f_2(p) = \cdots = f_m(p) = 0}_{\Updownarrow}$ and $a \cdot f(p) - 1 = 0$,

$$p \in Z(J)$$

Since $f \in I(Z(J))$, $f(p) = 0$ which gives us a contradiction:

$$a \cdot f(p) - 1 = a \cdot 0 - 1 = -1 \neq 0 .$$

Step 3: We conclude that $H = k[x_1, \dots, x_n, t]$. Write

$$1 = g_1 f_1 + \cdots + g_m f_m + g_0 (f \cdot t - 1)$$

where $g_0, g_1, \dots, g_m \in k[x_1, \dots, x_n, t]$.

Consider the homomorphism

fractional field of $k[x_1, \dots, x_n]$

$$\varphi : k[x_1, \dots, x_n, t] \to k(x_1, \dots, x_n)$$

given by

$$\varphi(x_i) = x_i \quad \text{and} \quad \varphi(t) = 1/f .$$

Then

$$1 = \varphi(1) = \varphi(g_1) f_1 + \cdots + \varphi(g_m) f_m + \varphi(g_0) \cdot \underbrace{(f \cdot \varphi(t) - 1)}_{=0}$$

Observe that there exists polynomials $p_1, p_2, \dots, p_m \in k[x_1, \dots, x_n]$ and $N \in \mathbb{N}$ such that

$$\varphi(g_i) = \frac{p_i}{f^N} \quad \text{for all} \quad 1 \leq i \leq m.$$

Thus,

$$1 = \frac{p_1 f_1}{f^N} + \dots + \frac{p_m f_m}{f^N} \in k(x_1, \dots, x_n).$$

$$\Rightarrow \quad f^N = p_1 f_1 + \dots + p_m f_m \in J \subseteq k[x_1, \dots, x_n].$$

$$\Rightarrow \quad f \in \sqrt{J}$$

$\square$