Algebraic Geometry - 0

Notes:

Algebraic Geometry by Andreas Gathmann

Introduction to Algebraic Geometry by Lothar Gottsche

Evaluation

- Quiz : 30% (Quiz 1: March 20, Quiz 2: April 3, Quiz 3: April 17)
- HW : 20% (weekly)
- Final : 50%

Information webpage: https://users.ictp.it/~ssinha1/Algebraic_Geometry.html

Algebraic geometry is an active area of research.

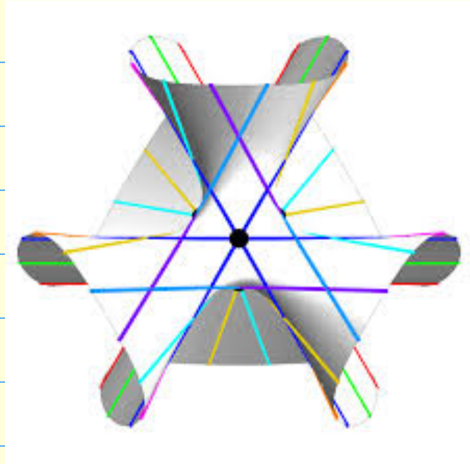
Relations to
- Complex / differential / symplectic geometry
- Number theory
- Representation theory
- Mathematical physics
- Combinatorics
- computer algebra ...

What is algebraic geometry?

linear algebra + algebra + geometry.

Study of polynomial equations in several variables and its geometric structures.

Goal: 27 lines on the cubic surface
(Salmon and Cayley 1849)



For us a <u>ring</u> is a commutative ring with unity 1.

Let k be a field. A k-algebra is a ring R with $k \subseteq R$ as a subring.

Eg: $k[x]$, $k[x_1, \dots, x_n]$ polynomial rings

$k[[x]]$ power series ring

$k[x]/\langle x^2-1 \rangle$, ...

<u>Affine algebraic sets</u>

$\mathbb{A}^n$ = affine n-space $:= \{(a_1, a_2, \dots, a_n) : a_i \in k\}$

<u>Def</u>: Let $S \subseteq k[x_1, \dots, x_n]$ be a set of polynomials. The <u>zero set</u> of S is

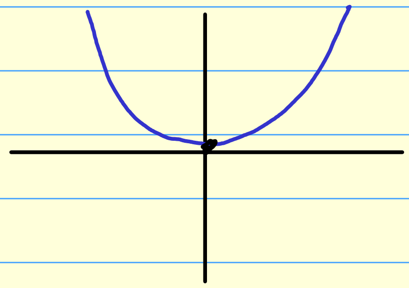
$$Z(S) = \{ p \in \mathbb{A}^n : f(p) = 0 \text{ for all } f \in S \}$$

These are called <u>affine algebraic sets</u>.

Eg: (i) $S = \{0\} \Rightarrow Z(S) = \mathbb{A}^n$, $S = \{1\} \Rightarrow Z(S) = \phi$

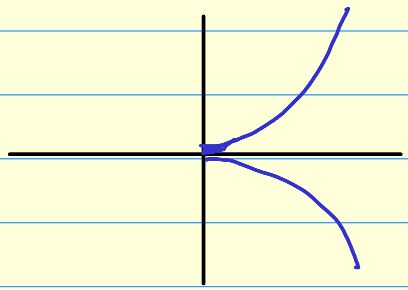
Affine Plane curves

(ii) $Z(y - x^2) \subseteq \mathbb{A}^2$ (conic)

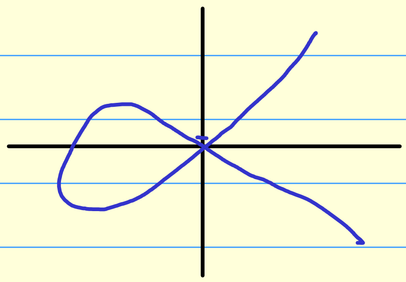


Realization

(iii) $Z(y^2 - x^3) \subseteq \mathbb{A}^2$ (cuspidal cubic)



(iv) $Z(y^2 - (x^3 + x^2)) \subseteq \mathbb{A}^2$



(v) Let $f \in k[x_1, \dots, x_n]$ be a polynomial of degree d.

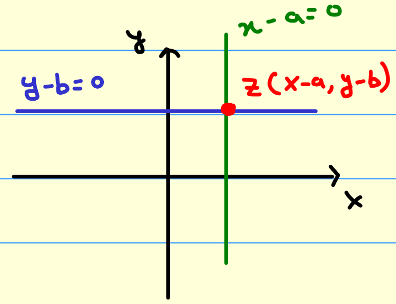
$Z(f) \subseteq \mathbb{A}^n$ is a <u>hypersurface</u> of degree d.

f is a linear (i.e. $d = 1$), these are called hyperplanes.

(vi) $S = \{x_1 - a_1, x_2 - a_2, \dots, x_n - a_n\} \subseteq k[x_1, \dots, x_n]$

$$Z(S) = \{(a_1, \dots, a_n)\} \subseteq \mathbb{A}^n$$

single point



(vii) Set of $n \times n$ matrices is identified with $\mathbb{A}^{n^2}$.

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & & & \vdots \\ x_{n1} & - & \cdots & x_{nn} \end{pmatrix}$$

- $SL_n(k) = Z\left(\det(x_{ij})_{i,j} - 1\right)$ is a degree n hypersurface.

  <u>Affine algeraic set</u> (also a group)

<u>Question</u>: Show that $Gl_n(k)$ is an affine algebraic set.

Let R be a ring.

<u>Def</u>: Let $S \subseteq R$ be a subset. The <u>ideal generated</u> by S is

$$\langle S \rangle := \left\{ \sum_{i=1}^{m} a_i f_i \;:\; m \in \mathbb{N},\ a_i \in R,\ f_i \in S \right\}$$

= the smallest ideal containing S

If $I \subseteq R$ is an ideal such that $\exists\ S = \{f_1, \dots, f_m\} \subseteq R$ such that

$$I = \langle S \rangle,$$

then I is called <u>finitely generated</u>.

Lemma: Let $S, T \subseteq k[x_1, .., x_n]$.

(i) If $S \subseteq T$, then $Z(S) \supseteq Z(T)$

(ii) $Z(S) = Z(\langle S \rangle)$

Proof: (i) If $p \in Z(T)$, then $f(p) = 0$ $\forall f \in T$ and thus for all $f \in S$.

(ii) $Z(\langle S \rangle) \subseteq Z(S)$ ✓

$Z(S) \subseteq Z(\langle S \rangle)$:

Let $p \in Z(S)$ and $g = \sum_{i=1}^{m} h_i f_i \in \langle S \rangle$ with $f_i \in S$. Then

$$g(p) = \sum_{i=1}^{m} h_i(p) \cdot \underbrace{f_i(p)}_{0} = 0$$

Observation: If $S \subseteq k[x_1, .., x_n]$ is too large, there will redundancy

(i) Every ideal $I \subseteq k[x]$ is principal ideal, that is,

$I = \langle f \rangle$ for a polynomial $f \in k[x]$.

- $k[x]$ is principal ided domain (PID).

(ii) $I \subseteq k[x_1, .., x_n]$ may not be principal.

Example: $\langle x-a, y-b \rangle \subseteq k[x,y]$.

Note that $\langle f \rangle$ principal ideals $\longleftrightarrow$ hypersurface $Z(f) \subseteq \mathbb{A}^n$.

Question: Is it possible that $I \subseteq k[x_1, \dots, x_n]$ is not finitely generated?

$\Leftrightarrow$ For an infinite set $S \subseteq k[x_1, \dots, x_n]$, does there always exists $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ such that
$$\langle S \rangle = \langle f_1, \dots, f_m \rangle \ ?$$

$\Leftrightarrow$ Does algebraic set $Z(S) \subseteq \mathbb{A}^n$ require only finitely many relations to define?

Ans: Yes! Hilbert Basis theorem.

Lemma: Let R be a ring. TFAE

(i) Any ideal $I \subseteq R$ can be finitely generated.

(ii) R satisfies the ascending chain condition (ACC), that is, any sequence of ideals
$$I_0 \subseteq I_1 \subseteq \dots \subseteq R$$
becomes stationary. ($\exists\, N > 0$ such that $I_m = I_N$ for all $m \geq N$)

Proof: homework!

We call such a ring Noetherian.

Eg: (i) $R = k$ field. (ii) PID $\Rightarrow$ Noetherian

(iii) Polynomial ring in infinitely many variables $k[x_1, x_2, \dots]$ is not Noetherian.

Theorem (Hilbert Basis): If R is Noetherian, then $R[x]$ is Noetherian.

<u>Corollary</u>: $k[x_1, \dots, x_n]$ is Noetherian.

<u>Proof</u>: Let $I \subseteq R$ be an ideal that is not finitely generated.

<u>Step 1</u>: Let $f_0 \in I$ be an element with minimal degree. Then choose a sequence $f_0, f_1, f_2, \dots$ iteratively such that
$f_n \in I \setminus \langle f_0, f_1, \dots, f_{n-1} \rangle$ has minimal degree for all n.

Hence (i) $\langle f_0 \rangle \subsetneq \langle f_0, f_1 \rangle \subsetneq \langle f_0, f_1, f_2 \rangle \subsetneq \cdots$ is a ascending chain

(ii) $\deg f_0 \leq \deg f_1 \leq \cdots$

(iii) $\deg f_n$ is minimal in $\langle f_0, \dots, f_n \rangle \setminus \langle f_0, \dots, f_{n-1} \rangle \subseteq I / \langle f_0, \dots, f_{n-1} \rangle$

<u>Step 2</u>: Let $a_n =$ leading term of f_n

$\left(f_n(x) = a_{n,d} x^d + a_{n,d-1} x^{d-1} + \cdots \in R[x] \right.$ and $a_n := a_{n,d} \in R$

Consider the ACC

$$\langle a_0 \rangle \subseteq \langle a_0, a_1 \rangle \subseteq \cdots \subseteq R .$$

Since R is Noetherian, there exists $N \geq 0$ such that

$$a_{N+1} \in \langle a_0, a_1, \dots, a_N \rangle$$

<u>Step 3</u>: Let $b_0, b_1, \dots, b_N \in R$ such that

$$\sum_{i=0}^{N} b_i a_i = a_{N+1}$$

Consider the polynomial

$$g(x) = f_{N+1}(x) - \sum_{i=0}^{N} b_i f_i(x) \cdot x^{(\deg f_{N+1} - \deg f_i)}$$

Then

$$\left.\begin{array}{l} \bullet\ \deg(g) < \deg f_{N+1} \\ \bullet\ g \notin \langle f_0, \dots, f_N \rangle \end{array}\right\}$$ Contradicts (iii) mimimality of $\deg f_{N+1}$.