Tautological bundles over Quot schemes on curves

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Outline

Punctual Quot scheme

- C smooth projective curve of genus g.
- $E \to C$ vector bundle with $\operatorname{rank}(E) = N$.

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Definition

Punctual Quot scheme $\mathbf{Quot}_d(E)$ *parameterizes short exact sequence*

$$\{0 \to S \to E \to Q \to 0 : deg(Q) = d; rank(Q) = 0\}.$$

$\mathbf{Quot}_d(E)$ *is a smooth scheme of dimension* Nd.

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$\mathbf{Quot}_d(E)$ *is a smooth scheme of dimension* Nd.

Example

When $E = \mathcal{O}_C$, then $\mathbf{Quot}_d(E) = C^{[d]}$.

Punctual Quot scheme

Several properties of the Punctual Quot schemes has been studied:

- (Bifet'89, Chen'01): Poincare Polynomial
- (Ricolfi'20, Bagnarol-Fantechi-Perroni'20): Motives
- (Biswas-Dhillon-Hurtubise'15): Automorphism group
- (Oprea'22, Oprea-Pandharipande'18): Positivity and Segre classes of tautological bundles
- (Toda'22) S.O.D of the derived category

Punctual Quot Scheme

- (Oprea-**S**'22): Explicit formula for the Euler characteristics of tautological bundles over punctual Quot scheme

Punctual Quot Scheme

- (Oprea-**S**'22): Explicit formula for the Euler characteristics of tautological bundles over punctual Quot scheme
- (Marian-Oprea-Sam'22): Refine the formulas to cohomology in the case of $\mathbb{P}^1$

Definition

We have the following diagram

$$
\begin{array}{ccccccc}
0 \longrightarrow \mathcal{S} & \longrightarrow & pr^*E & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow & & & & \\
 & & C \times \mathbf{Quot}_d(E) & & & & \\
 & \swarrow^{pr} & & \searrow^{\pi} & & & \\
C & & & & \mathbf{Quot}_d(E) & &
\end{array}
$$

,

Universal exact sequence

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For any point $q = [0 \to S \to E \to Q \to 0] \in \textbf{Quot}_d(E)$,

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For any point $q = [0 \to S \to E \to Q \to 0] \in \mathbf{Quot}_d(E)$,

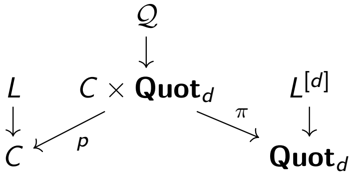
$$\mathcal{S}\big|_{C\times\{q\}} = S \quad \text{and} \quad \mathcal{Q}\big|_{C\times\{q\}} = Q.$$

Tautological Bundles

Definition

Let $L \to C$ be a line bundle. We define the *tautological bundle*

$$L^{[d]} := \pi_*(\mathcal{Q} \otimes pr^* L).$$



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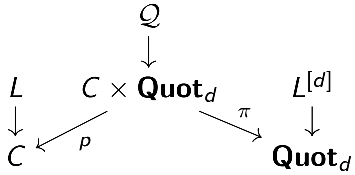

- Note that $L^{[d]}$ is a vector bundle of rank d over $\mathbf{Quot}_d(E)$.

Rationality

Definition

For any line bundles $L_1, \ldots, L_\ell \to C$ and integers $k_1, \ldots, k_\ell \geq 0$,

$$Z_{C,E}\left(L_1, \ldots, L_\ell \,|\, k_1, \ldots, k_\ell\right) = \sum_d q^d \chi\left(\mathbf{Quot}_d(E), \wedge^{k_1} L_1^{[d]} \otimes \cdots \otimes \wedge^{k_\ell} L_\ell^{[d]}\right)$$

- The above K-theoretic series are given by rational functions with pole at $q = 1$.

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- The above K-theoretic series are given by rational functions with pole at $q = 1$.
- Analogous statement is proved for punctual Quot scheme over surfaces by (Arbesfeld-Johnson-Lim-Oprea-Pandharipande '21)
- In this talk, we will be interested in explicit formulas in the curve case.

Results

For any vector bundle $V \to X$, let

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Theorem (Oprea-S'22)

For any line bundle $L \to C$ and vector bundle $E \to C$

$$\sum_{d=0}^{\infty} q^d \chi(\mathbf{Quot}_d(E), \wedge_y L^{[d]}) = \frac{(1+qy)^{\chi(E\otimes L)}}{(1-q)^{\chi(\mathcal{O}_C)}}.$$

- (Marian-Oprea-Sam'22) Determine each cohomology groups in genus 0.

Analogy with Hilbert scheme of Surfaces

Let $X^{[d]}$ be the Hilbert scheme of d points on a smooth surface X,

Theorem (Scala'09, Krug'18)

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where $L^{[d]} \to X^{[d]}$ is defined in a similar way.

- This theorem was proven using Bridgeland-King-Reid correspondence

$$\mathbf{D}^b(X^{[d]}) \cong \mathbf{D}^b_{S_d}(X^d).$$

Conjecture!

Question

Is it true that

$$H^\bullet\left(\mathrm{Quot}_d(E), \wedge^k L^{[d]}\right) = \wedge^k H^\bullet(E \otimes L) \otimes \mathrm{Sym}^{d-k} H^\bullet(\mathcal{O}_C)?$$

For any $\mathbb{Z}_2$ graded vector space $V^\bullet = V_0 \oplus V_1$,

$$\wedge^k V^\bullet = \bigoplus_{i+j=k} \wedge^i V_0 \otimes \mathrm{Sym}^j V_1, \quad \mathrm{Sym}^k V^\bullet = \bigoplus_{i+j=k} \mathrm{Sym}^i V_0 \otimes \wedge^j V_1$$

where the summands have degree j.

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- For $d = 1$, $\mathbf{Quot}_d(E) = \mathbb{P}(E)$.

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- A similar formula holds for Hilbert scheme of points.
- Over $C^{[d]}$ (i.e when $E = \mathcal{O}_C$), the identity was proven using derived categories (Krug'21).
- For $d = 1$, $\mathbf{Quot}_d(E) = \mathbb{P}(E)$.
- For $k = 0$, the formula predicts the known Hodge numbers

$$h^{p,0}(\mathrm{Quot}_d(E)) = \binom{g}{p} \text{ for } p \leq d.$$

The proof can broadly be divided into three steps:

- **Universality:** Using the arguments similar in spirit to the universality result of (Ellingsrud, Göttsche and Lehn), we show that there exists universal series A, B, and C such that

$$\sum_d q^d \chi(\mathbf{Quot}_d, \wedge_y L^{[d]}) = A^{\chi(\mathcal{O}_C)} B^{\deg L} C^{\deg E}.$$

Techniques

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  - A, B, and C depend on N, but not on the triple (C, E, L)
  - Universality reduces the calculations to Quot scheme over $\mathbb{P}^1$.

Techniques

- **Localization:** We use equivariant localization (using a torus action) to find the invariants.
  - (Strømme '85) and (Marian-Oprea '05) used Torus localization to study Quot scheme. However, the combinatorics differs for us.

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  - Let
  $$E = \bigoplus_{i=1}^{N} \mathcal{O}(a_i).$$
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  $\mathbb{C}^*$ acts on E diagonally with distinct weights inducing $\mathbb{C}^*$ action on $\mathbf{Quot}_d(E)$.
  - Fixed loci are products of projective spaces
  $$\mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_N}$$
  where $d_1 + \cdots + d_N = d$.

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  - We use several combinatorial identities, such as **Lagrange-Bürmann formula**, to simplify the expression.
  - Make necessary modifications to realize this expression in terms of determinants.
  - Express the resulting expression as **Schur polynomial** evaluated at roots of a polynomial with coefficients involving q and y.
  - We then use **Jacobi-Trudi identities** to obtain explicit formulas.

Multiple Insertions

The localization calculation enable us to obtain the following:

Theorem (Oprea-S'22)

For any line bundles $M_1, M_2, \ldots, M_r$ and L over C, where $0 \leq r \leq rk\ E - 1$, we have

$$\sum_{d=0}^{\infty} q^d \chi \left(\mathbf{Quot}_d(E), \wedge_y L^{[d]} \otimes_{i=1}^r \left(\wedge_{x_i} M_i^{[d]} \right)^\vee \right)$$

$$= \frac{(1+qy)^{\chi(E \otimes L)}}{(1-q)^{\chi(\mathcal{O}_C)} \prod_{i=1}^r (1 - qx_i y)^{\chi(M_i^\vee \otimes L)}}$$

Symmetric Product

For any vector bundle V, $\mathrm{Sym}_y(V) := \sum_k y^k \, \mathrm{Sym}^k V$.

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For $C = \mathbb{P}^1$ and $d \geq k$, we have

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- (Marian-Oprea-Sam'22) Determine each cohomology groups in genus 0.
- When $\chi(\mathcal{O}_X) = 1$ and $d \geq k$, (Scala'15, Arbesfeld'21)

$$\chi(X^{[d]}, \mathrm{Sym}^k L^{[d]}) = \binom{\chi(L)+k-1}{k}$$

Symmetric Product

Theorem (Oprea-S'22)

When $C = \mathbb{P}^1$ and $\chi = \chi(E \otimes L)$, we have

$$\chi(\mathbf{Quot}_d(E), \mathrm{Sym}_y L^{[d]}) = \sum_{k=0}^{d} \binom{-\chi + d(N+1)}{k} \frac{(-y)^k}{(1-y)^{d(N+1)}}.$$

Symmetric Product

In arbitrary genus, we show that

$$\sum_{d=0}^{\infty} q^d \chi\left(\mathbf{Quot}_d(E), \mathrm{Sym}_y L^{[d]}\right) = \mathsf{A}^{\chi(\mathcal{O}_C)} \cdot \mathsf{B}^{\chi(E \otimes L)}$$

holds true, for two universal power series $\mathsf{A}, \mathsf{B} \in \mathbb{Q}(y)[[q]]$ that depend on N, but not on the triple (C, E, L).

Symmetric Product

Theorem (Oprea-S'22)

We have

$$\mathsf{B} = f\left(\frac{qy}{(1-y)^{N+1}}\right)$$

where $f(z)$ is the solution to the equation

$$f(z)^N - f(z)^{N+1} + z = 0, \qquad f(0) = 1.$$

Example

For instance, in the special case $N = 2$, we obtain

$$f(z) = 1 + \frac{4}{3}\sinh^2\left(\frac{1}{3}\operatorname{arcsinh}\left(\frac{3\sqrt{3z}}{2}\right)\right).$$

Outline

1. Punctual Quot scheme
2. Higher rank quotients and genus 0
3. Further directions

- C smooth projective curve of genus g.
- $E \to C$ vector bundle with $\mathrm{rank}(E) = N$.

Definition

Quot scheme $\mathbf{Quot}_d(E, r)$ *parameterizes short exact sequence*

$$\{0 \to S \to E \to Q \to 0 : deg(Q) = d; rank(Q) = N - r\}$$

Quot scheme

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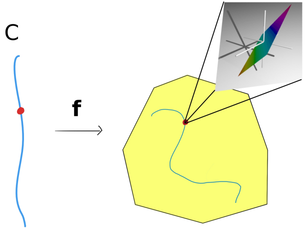
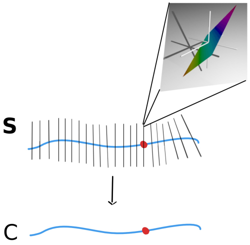
$$\{0 \to S \to E \to Q \to 0 : deg(Q) = d; rank(Q) = N - r\}$$

Example

- When $C = \mathbb{P}^1$ and $E = \mathcal{O}_C^{\oplus N}$ then $\mathbf{Quot}_d(E, r)$ (denoted as $\mathbf{Quot}_d(N, r)$) is smooth.
- Furthermore, when $d = 0$, then $\mathbf{Quot}_d(E, r) = Gr(N, r)$.

Let $E = \mathcal{O}_C^{\oplus N}$.





Maps from C to $Gr(N, r)$

Subbundles of $S \subset \mathcal{O}_C^{\oplus N}$

The Quot scheme compactifies $Mor_d(C, Gr(N, r))$!

History of Quot scheme

- (Witten '93): Quantum cohomology of Grassmannian
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- (Buch-Mihalcea '09) Quantum K-theory of Grassmannian

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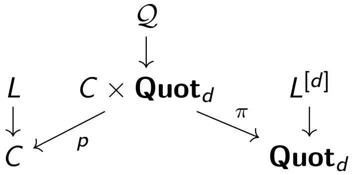

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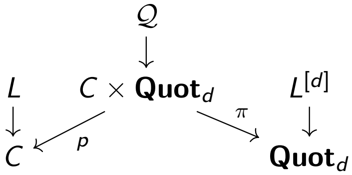

- Note that $L^{[d]}$ may **not** be a vector bundle over $\mathbf{Quot}_d(E, r)$.

Results

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Theorem (Oprea-S'22)

Let $\deg L = \ell \geq -d-1$. *Then*

$$\chi\left(\mathbf{Quot}_d(N,r), \det L^{[d]}\right) = (-1)^{(r-1)d}\left[q^d\right] s_\lambda(z_1, z_2, \ldots, z_N)$$

where

- z_i*'s are the distinct roots of the equation*

$$(z-1)^N - qz^{N-r-1} = 0.$$

- s_λ *is Schur polynomial for* $\lambda = \big(\underbrace{(d+\ell+1), \ldots, (d+\ell+1)}_{r \text{ times}}\big)$

Special cases

- Let $G = Gr(N, r)$, then

$$\chi(G, \mathcal{O}_G(\ell+1)) = s_\lambda(1, \dots, 1), \quad \lambda = \big(\underbrace{(\ell+1), \dots, (\ell+1)}_{r \text{ times}}\big).$$

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- Let $L = \mathcal{O}_C$, then

$$\chi\left(\mathbf{Quot}_d(N, r), \det \mathcal{O}_C^{[d]}\right) = \binom{N}{r-d}.$$

- Let $d > r(\ell+1)$, then

$$\chi\left(\mathbf{Quot}_d(N, r), \det L^{[d]}\right) = 0.$$

General result

Theorem (Oprea-S'22)

Let $\deg L = \ell$ *and* $0 < r < N$. *We have*

$$\chi(\mathbf{Quot}_d(N,r), \wedge_y L^{[d]}) = (-1)^{(r-1)d} \left[q^d\right] \frac{\det(f_i(z_j))}{\det(z_i^{N-j})}.$$

In the numerator $(f_i(z_j))$ *is the* $N \times N$ *matrix with*

$$f_i(z) = \begin{cases} z^{\ell+d+N-i+1} & \text{if } 1 \leq i \leq r \\ z^{N-i}(z+y)^{\ell+1} & \text{if } r+1 \leq i \leq N \end{cases}$$

- z_i*'s are the distinct roots of* $(z-1)^N - q(z+y)z^{N-r-1} = 0$.
- *The denominator is the vandermonde determinant.*

Conjectures of Marian-Oprea-Sam

Conjecture (Marian-Oprea-Sam '22)

Same notations as above.

- *For* $d = (N - r)a + b$, *for all line bundles* $L_1, \ldots, L_m$

$$\chi\left(\mathbf{Quot}_d(N, r), \big(\bigwedge^{k} L_1^{[d]}\big)^\vee \otimes \cdots \otimes \big(\bigwedge^{k} L_m^{[d]}\big)^\vee \right) = 0$$

where $m \leq r - 1$ *and* $0 < k_1 + \cdots k_m \leq d + r(a+1)$.

Universal subbundle

Recall the universal short exact sequence

$$
\begin{array}{ccccccccc}
0 \longrightarrow \mathcal{S} & \longrightarrow & pr^*E & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
& & \downarrow & & & & \\
& & \mathbb{P}^1 \times \mathbf{Quot}_d(N,r) & & & & \\
& \swarrow^{pr} & & \searrow^{\pi} & & & \\
\mathbb{P}^1 & & & & \mathbf{Quot}_d(N,r) & &
\end{array}
$$

Let $\mathcal{S}_x \to \mathbf{Quot}_d(N,r)$ denote the restriction of $\mathcal{S}$ to $\{x\} \times \mathbf{Quot}_d(N,r)$.

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$$
\begin{array}{ccccccc}
0 \longrightarrow \mathcal{S} & \longrightarrow & pr^*E & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow & & & & \\
 & & \mathbb{P}^1 \times \mathbf{Quot}_d(N,r) & & & & \\
 & \swarrow^{pr} & & \searrow^{\pi} & & & \\
\mathbb{P}^1 & & & & \mathbf{Quot}_d(N,r) & &
\end{array}
$$

Let $\mathcal{S}_x \to \mathbf{Quot}_d(N,r)$ denote the restriction of $\mathcal{S}$ to $\{x\} \times \mathbf{Quot}_d(N,r)$.

- The cohomology ring of $\mathbf{Quot}_d(N,r)$ is generated by

$$a_i = c_i(\mathcal{S}_x^\vee) \quad \text{and} \quad f_i = \pi_*(c_i(\mathcal{S}^\vee)).$$

Universal subbundle

Recall the universal short exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow pr^*E \longrightarrow \mathcal{Q} \longrightarrow 0$$

$$\downarrow$$

$$\mathbb{P}^1 \times \mathbf{Quot}_d(N,r)$$

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- The intersection numbers involving the above classes (including the higher genus case) were studied by (Bertram '94, Marian-Oprea '05).

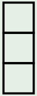
Schur functors

We may apply Schur functors to a vector bundle to obtain new ones.

Example

- For $\lambda = (4)$, we have $\mathbb{S}_\lambda(V) = \mathrm{Sym}^4(V)$

- For $\lambda = (1,1,1)$, we have $\mathbb{S}_\lambda(V) = \wedge^3(V)$



Ongoing work with Ming Zhang

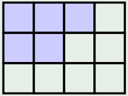
Theorem (Zhang-S)

For any partitions λ and μ contained in the partition $(\underbrace{N-r,\dots,N-r}_{r \text{ times}})$, and $d > 0$,

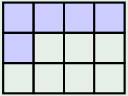
$$\chi\big(\mathbf{Quot}_d(N,r), \det \mathcal{S}_x \otimes \mathbb{S}_\lambda(\mathcal{S}_x) \otimes \mathbb{S}_\mu(\mathcal{S}_x)\big) = 0.$$

Example

When $N = 7$ and $r = 3$, λ and μ are contained in the partition $(4,4,4,4)$.



$\lambda = (3,2)$



$\mu = (4,1)$

Outline

1. Punctual Quot scheme
2. Higher rank quotients and genus 0
3. Further directions

Higher genus

When both genus $g > 0$ and rank of quotient $N - r > 0$, there are two main difficulties:

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Higher genus

When both genus $g > 0$ and rank of quotient $N - r > 0$, there are two main difficulties:

- Quot scheme may not be smooth.
- Universality argument fails.

Virtual Fundamental Class

Theorem (Marian-Oprea'05)

For any C, $0 \leq r \leq N$ and E, the Quot scheme $\mathbf{Quot}_d(E, r)$ admit a 2-term perfect obstruction theory.

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Question

Find a closed-form expression for $\chi^{vir}(\mathbf{Quot}_d(E, r), \wedge_y L^{[d]})$ in all genera.

Isotropic Quot scheme

- $M \to C$ line bundle
- $\sigma : E \otimes E \to M$: non-degenerate bilinear form (symmetric or symplectic)

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Definition

Isotropic Quot scheme is the closed subscheme of $\mathbf{Quot}_d(E, r)$

$$\mathbf{IQ}_d(E, r, \sigma) = \{[0 \to S \to E \to Q \to 0] \in \mathbf{Quot}_d : \sigma\big|_{S \otimes S} = 0\}$$

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- (Kresch-Tamvakis '03) : Over $\mathbb{P}^1$, Lagrangian Quot scheme $\mathbf{IQ}_d(\mathcal{O}^{2n}, n, \sigma)$.
- (Cheong-Choe-Hitching '19 '20) : Lagrangian Quot Scheme is irreducible when $d >> 0$ and studies enumerative invariants.

Over $\mathbb{P}^1$

Y smooth, F vector bundle, $X = \mathrm{Zero}(s)$

$$\begin{array}{ccc} & & F \\ & & s \uparrow\downarrow \\ X & \overset{i}{\hookrightarrow} & Y \end{array}$$

$$i_*[X]^{\mathrm{vir}} = e(F) \cap [Y]$$

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Example

Let $C = \mathbb{P}^1$, then

- $Y = \textbf{Quot}_d$ - smooth
- $X = \textbf{IQ}_d$ - zero locus of a section of a vector bundle

Problem

Note that the isotropic Quot schemes are not smooth even when $C = \mathbb{P}^1$. The virtual K-theoretic invariants are unexplored in these cases.

Problem

Find analogous formula for the virtual K-theoretic invariants and vanishing results for $\chi^{vir}(\mathbf{IQ}_d(E, r, \sigma), \wedge^k L^{[d]})$, where $C = \mathbb{P}^1$.

Perfect Obstruction Theory

Recall

$$0 \longrightarrow \mathcal{S} \longrightarrow pr^*E \longrightarrow \mathcal{Q} \longrightarrow 0$$

$$\downarrow$$

$$C \times \mathbf{IQ}_d$$

$$C \xleftarrow{\;pr\;} C \times \mathbf{IQ}_d \xrightarrow{\;\pi\;} \mathbf{IQ}_d$$

Theorem (S '21)

The isotropic quot scheme $\mathbf{IQ}_d$ admit a perfect 2-term obstruction theory induced by $R\pi_*(B^\bullet)^\vee \to \tau_{\geq -1} L^\bullet_{\mathbf{IQ}_d}$ where

$$B^\bullet = [R\,\mathcal{H}om(\mathcal{S}, \mathcal{Q}) \to \mathcal{H}om(\wedge^2 \mathcal{S}, pr^* M)].$$

Vafa-Intriligator type formula

Let $a_i = c_i(\mathcal{S}|_{\{x\}\times \mathbf{IQ}_d})$ for any $x \in C$.

Theorem (S '21)

When $E = \mathcal{O}^{\oplus N}$, $r = 2$ and $m_1 + 2m_2 = vir\ dim$,

$$\int_{[\mathbf{IQ}_d]^{vir}} a_1^{m_1} a_2^{m_2} = T_{d,g}(N) \sum_{\zeta \neq \pm 1} (1+\zeta)^{m_1+d} \zeta^{m_2} J(\zeta)^{g-1},$$

where the sum is taken over N^{th} roots of unity $\zeta \neq \pm 1$. Here

$$J(\zeta) = -N^2 \zeta^{-1} (1-\zeta)^{-2} (1+\zeta)^{-1}$$

$$T_{d,g}(N) = (-1)^d \frac{N}{2} \sum_{i=0}^{d} \binom{g}{i} (-N)^{-i}.$$

Vafa-Intriligator type formula

- Obtain an explicit formula for the intersection numbers of the form $f_2^\ell a_1^{m_1} a_2^{m_2} \cap [\mathbf{IQ}_d]^{\mathrm{vir}}$, where $f_i = \pi_*(c_i(\mathcal{S}^\vee))$.

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- Construct virtual fundamental class when $\sigma : E \otimes E \to M$ is a symmetric form.
- Obtain Vafa-Intriligator type formula when $r = 1$ and $r = 2$ in symmetric case.
- We show that the above formula match with the Gromov-Ruan-Witten invariants for the symplectic and orthogonal Grassmannians ($SG(N, 2)$ and $OG(N, 2)$).

Problems

The motives of the nested (punctual) Quot schemes of curves were found by (Monavari-Ricolfi '22). We can define the tautological bundles over the nested Quot schemes in a similar fashion.

Problem

Find the Euler characteristics of tautological bundles over nested Quot schemes of curves.

Problems

Dubrovin conjecture implies that the quantum cohomology is generically semi-simple if and only there exist a full exceptional collection.

Problem

Study the quantum cohomology of the punctual Quot scheme of $\mathbb{P}^1$.

Thank you!

QUESTIONS