Intersection Theory of Hyperquot Schemes on Curves

Shubham Sinha

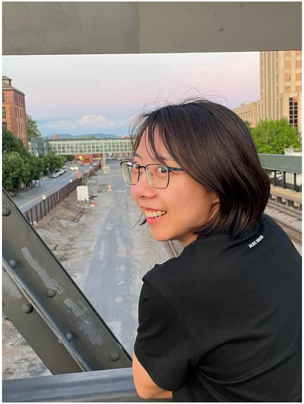
International Centre for Theoretical Physics, Trieste

Sept 24th 2025

Collaborators

Shubham Sinha
ICTP

Riccardo Ontani
Imperial College



Weihong Xu
Caltech

Outline

Quot Scheme

- C smooth projective curve of genus g.
- $V \to C$ vector bundle of rank n.

Definition

Quot scheme $\mathbf{Quot}_d(C, r, V)$ *parameterizes short exact sequence*

$$0 \to E \hookrightarrow V \twoheadrightarrow F \to 0 \qquad \begin{aligned} \operatorname{rank} F &= n - r, \\ \deg F &= d. \end{aligned}$$

Hyperquot Scheme

- Fix tuples of non-negative integers

$$\mathbf{r} = (r_1, r_2, \ldots, r_k) \quad \text{and} \quad \mathbf{d} = (d_1, d_2, \ldots, d_k)$$

Definition

Hyperquot scheme $\mathbf{HQuot_d}(C, \mathbf{r}, V)$, *parametrizes chains of quotient sheaves*

$$V \twoheadrightarrow F_1 \twoheadrightarrow F_2 \twoheadrightarrow \cdots \twoheadrightarrow F_k \qquad \begin{array}{l} \operatorname{rank} F_i = n - r_i, \\ \deg F_i = d_i. \end{array}$$

Hyperquot Scheme

- Fix tuples of non-negative integers

$$\mathbf{r} = (r_1, r_2, \ldots, r_k) \quad \text{and} \quad \mathbf{d} = (d_1, d_2, \ldots, d_k)$$

Definition

Hyperquot scheme $\mathbf{HQuot_d}(C, \mathbf{r}, V)$, *parametrizes chains of subsheaves*

$$E_1 \hookrightarrow E_2 \hookrightarrow \cdots \hookrightarrow E_k \hookrightarrow V \qquad \begin{array}{l} \operatorname{rank} E_i = r_i, \\ \deg E_i = \deg V - d_i. \end{array}$$

Cohomology classes

Consider the universal sequence,

$$0 \to \mathcal{E}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_k \to \pi_C^* V \to \mathcal{F}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{F}_k \to 0,$$

on $\mathbf{HQuot_d} \times C$ where π_C denote the projection map to C.

Consider the universal sequence,

$$0 \to \mathcal{E}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_k \to \pi_C^* V \to \mathcal{F}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{F}_k \to 0,$$

on $\mathbf{HQuot_d} \times C$ where π_C denote the projection map to C.

Fix $p \in C$, and denote rank r_j **vector bundles**

$$\mathcal{E}_{j|p} := \mathcal{E}_j \Big|_{\mathbf{HQuot_d} \times \{p\}} \quad \text{for all } 1 \leq j \leq k.$$

Consider the universal sequence,

$$0 \to \mathcal{E}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_k \to \pi_C^* V \to \mathcal{F}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{F}_k \to 0,$$

on $\mathbf{HQuot_d} \times C$ where π_C denote the projection map to C.

Fix $p \in C$, and denote rank r_j **vector bundles**

$$\mathcal{E}_{j|p} := \mathcal{E}_j\Big|_{\mathbf{HQuot_d} \times \{p\}} \quad \text{for all } 1 \leq j \leq k.$$

Goal: Find top degree intersection numbers of Chern classes of $\mathcal{E}_{j|p}$.

Denote the Chern classes

$$a_{i,j} := c_i(\mathcal{E}^{\vee}_{j|p}) \in H^{2i}(\mathbf{HQuot_d}, \mathbb{Z}).$$

Intersection numbers

Denote the Chern classes

$$a_{i,j} := c_i(\mathcal{E}_{j|p}^{\vee}) \in H^{2i}(\mathbf{HQuot_d}, \mathbb{Z}).$$

We denote a tuple of natural numbers

$$\mathbf{m} = \left\{ m_{i,j} \in \mathbb{N} : \begin{array}{c} 1 \leq j \leq k, \\ 1 \leq i \leq r_j \end{array} \right\} \quad \text{and} \quad R_{\mathbf{m}} = \prod_{j=1}^{k} \prod_{i=1}^{r_j} a_{i,j}^{m_{i,j}}$$

Intersection numbers

Denote the Chern classes

$$a_{i,j} := c_i(\mathcal{E}_{j|p}^{\vee}) \in H^{2i}(\mathbf{HQuot_d}, \mathbb{Z}).$$

We denote a tuple of natural numbers

$$\mathbf{m} = \left\{ m_{i,j} \in \mathbb{N} : \begin{array}{l} 1 \leq j \leq k, \\ 1 \leq i \leq r_j \end{array} \right\} \quad \text{and} \quad R_{\mathbf{m}} = \prod_{j=1}^{k} \prod_{i=1}^{r_j} a_{i,j}^{m_{i,j}}$$

Definition

Let $\mathbf{q} = (q_1, q_2, \ldots, q_k)$ are formal variables. We define

$$\mathbf{B}_{g,\mathbf{r},V}^{R_{\mathbf{m}}}(q_1, q_2, \ldots, q_k) = \sum_{\mathbf{d} \in \mathbb{N}^k} q_1^{d_1} q_2^{d_2} \cdots q_k^{d_k} \int_{[\mathbf{HQuot_d}]^{\mathrm{vir}}} R_{\mathbf{m}}.$$

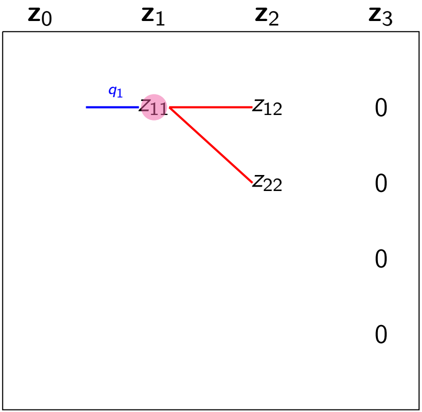
Outline

$\mathbf{r} = (1, 2), n = 4$

$\mathbf{z}_0$	$\mathbf{z}_1$	$\mathbf{z}_2$	$\mathbf{z}_3$
	z_{11}	z_{12}	0
		z_{22}	0
			0
			0

Equations

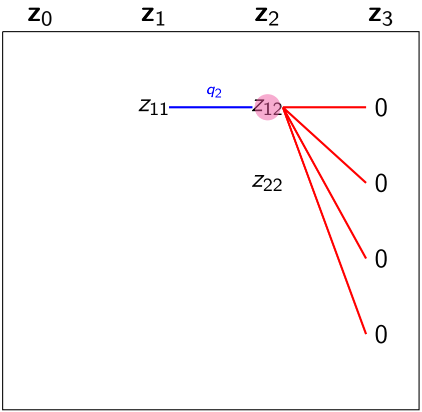
$\mathrm{r} = (1, 2), n = 4$



Equations

$$q_1 = (z_{11} - z_{12})(z_{11} - z_{22})$$

$\mathbf{r} = (1, 2), n = 4$

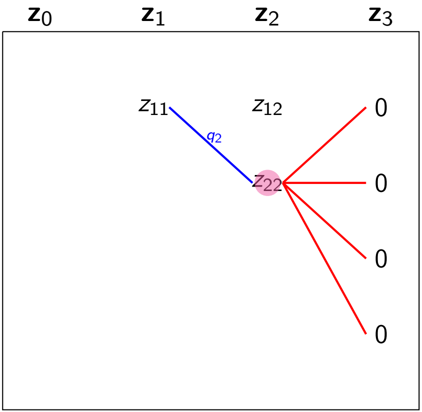


Equations

$$q_1 = (z_{11} - z_{12})(z_{11} - z_{22})$$

$$q_2(z_{12} - z_{11}) = z_{12}^4$$

$\mathrm{r} = (1, 2), n = 4$



Equations

$$q_1 = (z_{11} - z_{12})(z_{11} - z_{22})$$

$$q_2(z_{12} - z_{11}) = z_{12}^4$$

$$q_2(z_{22} - z_{11}) = z_{22}^4$$

$\mathbf{r} = (3,3), n = 3$

$\mathbf{z}_0$	$\mathbf{z}_1$	$\mathbf{z}_2$	$\mathbf{z}_3$
	z_{11}	z_{12}	0
	z_{21}	z_{22}	0
	z_{31}	z_{32}	0

Equations

$$q_1 = (z_{11} - z_{12})(z_{11} - z_{22})(z_{11} - z_{32})$$
$$q_1 = (z_{21} - z_{12})(z_{21} - z_{22})(z_{21} - z_{32})$$
$$q_1 = (z_{31} - z_{12})(z_{31} - z_{22})(z_{31} - z_{32})$$
$$(-q_2)(z_{12} - z_{11})(z_{12} - z_{21})(z_{12} - z_{31}) = z_{12}^3$$
$$(-q_2)(z_{22} - z_{11})(z_{22} - z_{21})(z_{22} - z_{31}) = z_{22}^3$$
$$(-q_2)(z_{32} - z_{11})(z_{32} - z_{21})(z_{32} - z_{31}) = z_{32}^3$$

Consider k tuples of variables $\mathbf{z}_1, \mathbf{z}_2 \ldots, \mathbf{z}_k$ where

$$
\begin{array}{cccccccc}
\emptyset = \mathbf{z}_0 & \mathbf{z}_1 & \mathbf{z}_2 & \cdots & \cdots & \mathbf{z}_k & \mathbf{z}_{k+1} & = \mathbf{0} \in \mathbb{C}^n \\
\hline
 & z_{11} & z_{11} & \cdots & \cdots & z_{11} & 0 & \\
 & \vdots & \vdots & & & \vdots & \vdots & \\
 & z_{r_1,1} & \vdots & & & \vdots & \vdots & \\
 & & z_{r_2,2} & & & \vdots & \vdots & \\
 & & & \ddots & & \vdots & \vdots & \\
 & & & & \ddots & & \vdots & \\
 & & & & & z_{r_2,2} & \vdots & \\
 & & & & & & 0 & \\
\hline
\end{array}
$$

System of equations

Consider k tuples of variables $\mathbf{z}_1, \mathbf{z}_2 \ldots, \mathbf{z}_k$ where

$$
\begin{array}{cccccccc}
\emptyset = \mathbf{z}_0 & \mathbf{z}_1 & \mathbf{z}_2 & \cdots & \cdots & \mathbf{z}_k & \mathbf{z}_{k+1} & = \mathbf{0} \in \mathbb{C}^n \\
 & z_{11} & z_{11} & \cdots & \cdots & z_{11} & 0 & \\
 & \vdots & \vdots & & & \vdots & \vdots & \\
 & z_{r_1,1} & & & & & & \\
 & & z_{r_2,2} & & & & & \\
 & & & \ddots & & & & \\
 & & & & \ddots & & & \\
 & & & & & z_{r_2,2} & & \\
 & & & & & & 0 &
\end{array}
$$

$$P_j(z_{ij}) := (-1)^{r_j - r_{j-1}} q_j \prod_{\alpha \in \mathbf{z}_{j-1}} (z_{ij} - \alpha) + \prod_{\alpha \in \mathbf{z}_{j+1}} (z_{ij} - \alpha).$$

Non-degenerate solutions

Let $(q_1, q_2, \dots, q_k)$ be a k-tuple of non-zero complex numbers. Solve coupled system of equations

$$P_j(z_{i,j}) = 0 \quad \text{for each } 1 \leq j \leq k, \text{and} 1 \leq i \leq r_j.$$

Non-degenerate solutions

Let $(q_1, q_2, \ldots, q_k)$ be a k-tuple of non-zero complex numbers. Solve coupled system of equations

$$P_j(z_{i,j}) = 0 \quad \text{for each } 1 \leq j \leq k, \text{and} 1 \leq i \leq r_j.$$

We say that a solution $(\mathbf{z}_1, \mathbf{z}_2 \ldots, \mathbf{z}_k)$ is *non-degenerate* if

$$\mathbf{z}_j = \begin{bmatrix} z_{1j} \\ \vdots \\ \vdots \\ z_{r_j,j} \end{bmatrix} \qquad \text{has distinct elements for all } j.$$

Number of solution

Proposition (Ontani–S–Xu 25')

For a generic choice of $\mathbf{q} := (q_1, \ldots, q_k) \in (\mathbb{C}^*)^k$, *we have the expected number of non-degenerate solutions*

$$N(\mathbf{q}) = \prod_{j=0}^{k} \frac{r_{j+1}!}{(r_{j+1} - r_j)!}.$$

Outline

Theorem (Ontani–S–Xu 25')

Let $\deg V = 0$.

$$\mathbf{B}^{R_{\mathbf{m}}}_{g,\mathbf{r},V}(q_1, q_2, \ldots, q_k) = \sum_{\mathbf{z}_1,\ldots,\mathbf{z}_k} \prod_{j=1}^{k} \prod_{i=1}^{r_j} e_i(\mathbf{z}_j)^{m_{i,j}} \cdot J^{g-1}(\mathbf{z}_1, \ldots, \mathbf{z}_k)$$

Main theorem

Theorem (Ontani–S–Xu 25')

Let $\deg V = 0$.

$$\mathbf{B}^{R_{\mathbf{m}}}_{g,\mathbf{r},V}(q_1, q_2, \ldots, q_k) = \sum_{\mathbf{z}_1,\ldots,\mathbf{z}_k} \prod_{j=1}^{k} \prod_{i=1}^{r_j} e_i(\mathbf{z}_j)^{m_{i,j}} \cdot J^{g-1}(\mathbf{z}_1, \ldots, \mathbf{z}_k)$$

- *Sum is taken over non-degenerate solutions* $(\mathbf{z}_1, \ldots, \mathbf{z}_k)$*,*

Theorem (Ontani–S–Xu 25')

Let $\deg V = 0$.

$$\mathbf{B}^{R_{\mathbf{m}}}_{g,\mathbf{r},V}(q_1, q_2, \ldots, q_k) = \sum_{\mathbf{z}_1,\ldots,\mathbf{z}_k} \prod_{j=1}^{k} \prod_{i=1}^{r_j} e_i(\mathbf{z}_j)^{m_{i,j}} \cdot J^{g-1}(\mathbf{z}_1, \ldots, \mathbf{z}_k)$$

- *Sum is taken over non-degenerate solutions* $(\mathbf{z}_1, \ldots, \mathbf{z}_k)$*,*
- *and*

$$J(\mathbf{z}_1, \ldots, \mathbf{z}_k) = \det\left(\frac{P_i(z_{i,j})}{\partial z_{i',j'}}\right) \prod_{1\leq j\leq k} \Delta(\mathbf{z}_j)^{-1}.$$

Here $\Delta(X_1, \ldots, X_m) \coloneqq \prod_{a\neq b}(X_a - X_b)$

When $\mathbf{r} = (r)$ and $V = \mathcal{O}_C^{\oplus n}$,

$$\mathbf{HQuot_d} = \mathbf{Quot}_d \,.$$

We recover VI formula, proved by Bertram '94, Seibert-Tian '94, and

When $\mathbf{r} = (r)$ and $V = \mathcal{O}_C^{\oplus n}$,

$$\mathbf{HQuot_d} = \mathbf{Quot}_d\,.$$

We recover VI formula, proved by Bertram '94, Seibert-Tian '94, and

Theorem (Marian–Oprea 05')

Let $m_1 + 2m_2 + \cdots + rm_r = nd + r(n-r)(1-g)$. Then

$$\int_{[\mathbf{Quot}_d]^{\mathrm{vir}}} \prod_{i=1}^{r} c_i(\mathcal{E}_p^\vee)^{m_i} = (-1)^{d(r-1)} \sum_{\zeta} \prod_{i=1}^{r} e_i(\zeta)^{m_i} J^{g-1}(\zeta),$$

where the sum is over all tuples $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_r)$ of distinct nth roots of unity, and

$$J(\zeta) = \prod_{i=1}^{r} n\zeta_i^{n-1} \prod_{1 \le i \neq j \le r} (\zeta_i - \zeta_j)^{-1}.$$

PUNCTUAL HYPERQUOT SCHEME

The *punctual* hyperquot scheme $\mathbf{HQuot_d}(C, V)$ parameterizes successive quotients of zero dimension suppport of V, that is,

$$\mathbf{r} = (n, n, \ldots, n) \quad \text{and} \quad \mathbf{d} = (d_1, d_2, \ldots, d_k).$$

The *punctual* hyperquot scheme $\mathbf{HQuot_d}(C, V)$ parameterizes successive quotients of zero dimension suppport of V, that is,

$$\mathbf{r} = (n, n, \ldots, n) \quad \text{and} \quad \mathbf{d} = (d_1, d_2, \ldots, d_k).$$

- Poincare polynpmial (Chen '01)
- Motivic invariants (Monavari-Ricolfi '22)
- Derived category of punctual quot scheme (Marian-Negut '24)

Corollary (Ontani–S–Xu 25')

$$\sum_{\mathbf{d}} \mathbf{q}^{\mathbf{d}} \int_{[\mathbf{HQuot_d}]} \prod_{i=1}^{k} s_{t_i}(\mathcal{E}_{i|p}) = \prod_{i=1}^{k} \frac{1}{1 - t_i^n \alpha_i},$$

where $s_{t_i}(\mathcal{E}_{i|p})$ are Segre series and

$$\alpha_j = q_1 q_2 \cdots q_j (1 + q_{j+1} + q_{j+1} q_{j+2} + \cdots + q_{j+1} q_{j+2} \cdots q_k).$$

Corollary (Ontani–S–Xu 25')

$$\sum_{\mathbf{d}} \mathbf{q}^{\mathbf{d}} \int_{[\mathbf{HQuot_d}]} \prod_{i=1}^{k} s_{t_i}(\mathcal{E}_{i|p}) = \prod_{i=1}^{k} \frac{1}{1 - t_i^n \alpha_i},$$

where $s_{t_i}(\mathcal{E}_{i|p})$ are Segre series and

$$\alpha_j = q_1 q_2 \cdots q_j (1 + q_{j+1} + q_{j+1} q_{j+2} + \cdots + q_{j+1} q_{j+2} \cdots q_k).$$

For any vector bundle $E \to Y$, we denote

$$s_t(E) = 1 + t s_1(E) + t^2 s_2(E) + t^3 s_3(E) + \cdots$$

where $s_j(E)$ is the j-th Segre class.

$\mathbf{r} = (1, 2)$ AND $n = 3$

Let $\bar{g} = g - 1$. The expected dimension

$$2(d_1 + d_2) - 3\bar{g}.$$

Fix non-negative integers ℓ, m_1, m_2 such that

$$d := \frac{\ell + m_1 + 2m_2 + 3\bar{g}}{2}$$

COROLLARY (ONTANI–S–XU 25')

$$\mathbf{B}^{\ell,\mathbf{m}}_{g,\mathbf{r},V}(q_1, q_2) = 6^g \sum_{j \in \mathbb{Z}} \binom{d - \bar{g} - m_2}{3j - \ell - m_2 + \bar{g}} q_1^{\bar{g}+j} q_2^{d-\bar{g}-j}.$$

This formula also shows the positivity of the above numbers.

$\mathbf{r} = (1, 2)$ and $n = 4$

Proposition (Ontani–S–Xu 25')

Let $\mathbf{r} = (1, 2)$ *and* V *has rank* 4 *and degree zero. Then*

$$\mathbf{B}^{\ell,\mathbf{m}}_{g,\mathbf{r},V}(q_1, q_2) = \sum_{w,s} \left(\frac{w(w - s^2)}{q_2} \right)^{\ell} s^{m_1} w^{m_2} \left(\frac{(3s^2 - 4w)(s^2 + 2w)}{q_2(s^2 - 4w)} \right)^{g-1}$$

where w *and* s *runs over the roots*

$$w^4 = q_1 q_2^2 \quad \text{and} \quad s^3 - 2ws + q_2 = 0.$$

Genus g	$\sum_{\mathbf{d}} q_1^{d1} q_2^{d2} \int_{[\mathbf{HQuot_d}]^{\mathrm{vir}}} 1$
1	6
2	0
3	0
4	0
5	$2^6 \cdot 3^5 q_1^3 q_2^3$
6	0
7	$2^7 \cdot 3^7 (q_1^5 q_2^4 + q_1^4 q_2^5)$
8	0
9	$2^{10} \cdot 3^{10} q_1^6 q_2^6$
10	0
11	$2^{11} \cdot 3^{11} \cdot 5 (q_1^8 q_2^7 + q_1^7 q_2^8)$
12	0
13	$2^{13} \cdot 3^{13} (q_1^{10} q_2^8 + 20 q_1^9 q_2^9 + q_1^8 q_2^{10})$
14	0
15	$2^{15} \cdot 3^{16} \cdot 7 (q_1^{11} q_2^{10} + q_1^{10} q_2^{11})$
16	0
17	$2^{18} \cdot 3^{17} (4 q_1^2 + 35 q_1 q_2 + 4 q_2^2) q_1^{11} q_2^{11}$
18	0
19	$2^{19} \cdot 3^{19} (q_1^2 + 83 q_1 q_2 + q_2^2)(q_1 + q_2) q_1^{12} q_2^{12}$

Outline

Virtual Fundamental Class

- Perfect obstruction theory and virtual cycles (Behrend–Fentachi'97, Li–Tian'98)
- Quot schemes admit a perfect obstruction theory (Marian–Oprea 05').
- Perfect obstruction theory for relative Quot scheme (Gillam 11')
- Perfect obstruction theory for $\mathbf{HQuot_d}(C, \mathbf{r}, V)$:
  When $V = \mathcal{O}_C^{\oplus n}$ (Ciocan-Fontanine–Kim–Maulik 14')
  General V (Monavari–Ricolfi 24')

Recall the universal subsheaves on $\mathbf{HQuot_d} \times C$,

$$0 \to \mathcal{E}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_k \to \pi_C^* V =: \mathcal{E}_{k+1},$$

The obstruction complex of $\mathbf{HQuot_d}(C, \mathbf{r}, V)$ is given by

$$\mathbb{E} := \mathrm{Cone}\left[\bigoplus_{i=1}^{k} \mathcal{H}om_\pi(\mathcal{E}_i, \mathcal{E}_i) \to \bigoplus_{i=1}^{k} \mathcal{H}om_\pi(\mathcal{E}_i, \mathcal{E}_{i+1})\right]^\vee,$$

Here $\mathcal{H}om_\pi$ the composition of $R\pi_*$ and $\mathcal{H}om$ in the derived categories.

$\mathbf{r} = (1, 2, 3), n = 4$

$\mathbf{z}_0$	$\mathbf{z}_1$	$\mathbf{z}_2$	$\mathbf{z}_3$	$\mathbf{z}_4$
	z_{11}	z_{12}	z_{13}	0
		z_{22}	z_{23}	0
			z_{33}	0
				0

Equations

$$q_1 = (z_{11} - z_{12})(z_{11} - z_{22})$$

$$q_2(z_{12} - z_{11}) = (z_{12} - z_{13})(z_{12} - z_{23})(z_{12} - z_{33})$$

$$q_2(z_{22} - z_{11}) = (z_{22} - z_{13})(z_{22} - z_{23})(z_{22} - z_{33})$$

$$q_3(z_{13} - z_{12})(z_{13} - z_{22}) = z_{13}^4$$

$$q_3(z_{23} - z_{12})(z_{23} - z_{22}) = z_{23}^4$$

$$q_3(z_{33} - z_{12})(z_{33} - z_{22}) = z_{33}^4$$

Consequence

Choose $\ell \leq k$.

$$\prod_{i=1}^{\ell} q_i^{r_i} \sum_{\mathbf{d}\in\mathbb{Z}^k} \mathbf{q}^{\mathbf{d}} \int_{[\mathbf{HQuot_d}]^{\mathrm{vir}}} R = \sum_{\mathbf{d}\in\mathbb{Z}^k} \mathbf{q}^{\mathbf{d}} \int_{[\mathbf{HQuot_d}]^{\mathrm{vir}}} R \cdot e(\mathcal{E}_{\ell|p}^{\vee} \otimes \mathcal{E}_{\ell+1|p})$$

Let $\mathbf{z}_\ell$ denote the Chern roots $\mathcal{E}_{\ell|p}^{\vee}$.

$$e(\mathcal{E}_{\ell|p}^{\vee} \otimes \mathcal{E}_{\ell+1|p}) = \prod_{i=1}^{r_\ell} \prod_{\alpha\in\mathbf{z}_{\ell+1}} (z_{i\ell} - \alpha)$$

Compatibility

Proposition (Ontani–S–Xu 25')

Fix $p \in C$ *and* $\delta_\ell = (r_1, \dots, r_\ell, 0, \dots, 0)$ *,*

$$\iota_*[\mathbf{HQuot}_{\mathbf{d}}(C, \mathbf{r}, V)]^{\mathrm{vir}} = e(\mathcal{E}^\vee_{\ell|p} \otimes \mathcal{E}_{\ell+1|p}) \cap [\mathbf{HQuot}_{\mathbf{d}+\delta_\ell}(C, \mathbf{r}, V)]^{\mathrm{vir}}.$$

Consider

$$\iota : \mathbf{HQuot}_{\mathbf{d}}(C, \mathbf{r}, V) \to \mathbf{HQuot}_{\mathbf{d}+\delta_\ell}(C, \mathbf{r}, V)$$

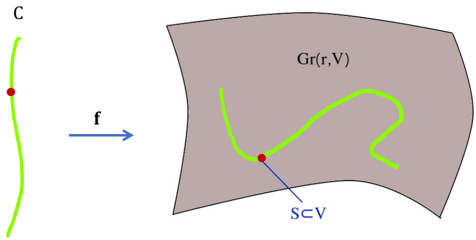
by taking $[E_1 \hookrightarrow \cdots \hookrightarrow E_k \hookrightarrow V]$ to the composition

$$E_1(-p) \hookrightarrow \cdots \hookrightarrow E_\ell(-p) \hookrightarrow E_{\ell+1} \hookrightarrow \cdots \hookrightarrow E_k \hookrightarrow V.$$

Outline

Maps to Grassmannians

Quot scheme $\textbf{Quot}_d(C, r, \mathcal{O}_C^{\oplus n})$ compactifies $\mathrm{Mor}_d(C, Gr(r, n))$



- (Bertram, B-D-W '94): Counting maps using Quot scheme
- (Marian-Oprea '05): Virtual intersection theory and recovered Vafa-Intriligator formula
- (Marian-Oprea '09) Relation with Verlinde numbers on moduli of stable bundles.

The cohomology ring $H^*(\mathrm{Gr}(r,n),\mathbb{Z})$ is generated by

$$\left\{\sigma_i := c_i(S) : \ 1 \leq i \leq r \ \right\}, \quad \sigma_i = [Y_i] \quad \text{Special Schubert cycle}$$

Maps to Grassmannian

The cohomology ring $H^*(\mathrm{Gr}(r,n),\mathbb{Z})$ is generated by

$$\{\sigma_i := c_i(S) : \ 1 \le i \le r \ \}, \quad \sigma_i = [Y_i] \quad \text{Special Schubert cycle}$$

Theorem (Bertram 94')

When $d >> 0$ compared to g, r and n, then VI formula is enumerative: Fix distinct points $p_1, p_2, \dots, p_t \in C$, and special Schubert varieties in general position $Y_{i_1}, Y_{i_2}, \dots, Y_{i_t}$, then

$$\int_{[\mathbf{Quot}_d]} \prod_{s=1}^{t} a_{i_s} = \left\{ f : C \to \mathrm{Gr}(r,n) : \begin{array}{l} \deg f = d; \\ f(p_s) \in Y_{i_s} \quad \forall\, 1 \le s \le t \end{array} \right\}$$

Fix $0 < r_1 < r_2 < \cdots < r_k < n$,

$$\mathrm{Fl}(\mathbf{r}, n) = \{W_1 \subset W_2 \subset \cdots \subset W_k \subset \mathbb{C}^n : \mathrm{rank}\, W_i = r_i\}$$

Consider the universal subsheaves on $\mathrm{Fl}(\mathbf{r}, n)$:

$$0 \to S_1 \to S_2 \to \cdots \to S_k \to \mathcal{O}^{\oplus n}_{\mathrm{Fl}(\mathbf{r},n)}.$$

The cohomology ring $H^*(\mathrm{Fl}(\mathbf{r}, n), \mathbb{Z})$ is generated by

$$\left\{ \sigma_{ij} := c_i(S_j) : \begin{array}{l} 1 \leq j \leq k, \\ 1 \leq i \leq r_j. \end{array} \right\} \quad \sigma_{ij} = [Y_{ij}] \quad \text{Special Schubert cycle}$$

Question

Are virtual integrals over hyperquot scheme enumerative?

Difficulties:

- Multiple degrees $d_1, d_2, \ldots, d_k$, that is

$$H_2(\mathrm{Fl}(\mathbf{r}, n), \mathbb{Z}) \cong \mathbb{Z}^k$$

- Lack of irreducibility results and lack of criterion when

$$[\mathbf{HQuot_d}]^{\mathrm{vir}} = [\mathbf{HQuot_d}]$$

When $C = \mathbb{P}^1$, $\mathbf{HQuot_d}(\mathbb{P}^1, \mathbf{r}, \mathcal{O}_{\mathbb{P}^1}^{\oplus n})$ is smooth and irreducible

- Quantum cohomology of $\mathrm{Fl}(\mathbf{r}, n)$ (Ciocan-Fontanine 99', Chen 03')
- Enumerativity over $\mathbb{P}^1$ (Kim 95')
- Multiple insertions and degenerations (Siebert-Tian 97)
- Genus zero residue formula (Gu–Kalashnikov 24')
- Residue formula using quasimaps (R.Ontani 25')

Result

Theorem (Ontani–S–Xu 25')

When $d_1 >> d_2 >> \cdots >> d_k >> 0$, the following virtual integral over hyperquot scheme is enumerative: Fix distinct points $p_1, p_2, \ldots, p_t \in C$, and special Schubert varieties $Y_{i_1,j_1}, Y_{i_2,j_2}, \ldots Y_{i_t,j_t}$ in general position such that

$$i_s < r_{j_s+1} - r_{j_s-1}$$

then

$$\int_{[\mathbf{HQuot_d}]^{\mathrm{vir}}} \prod_{s=1}^{t} a_{i_s,j_s} = \left\{ f : C \to \mathrm{Fl}(\mathbf{r}, n) : \begin{array}{l} \deg f = (d_1, d_2, \ldots, d_k); \\ f(p_s) \in Y_{i_s,j_s} \quad \forall\, 1 \le s \le t \end{array} \right\}$$

Theorem (B–Daskapoulos–Wentworth 96, Popa–Roth 03')

The Quot scheme is irreducible of expected dimension when $d >> 0$ for fixed g, r, n.

The hyperquot scheme $\mathbf{HQuot_d}$ is irreducible of expected dimension when

$$d_1 >> d_2 >> \cdots >> d_k >> 0, \quad \text{for fixed} \quad g, \mathbf{r}, n.$$

Theorem (Rasul-Sebastian 24')

For $\mathbf{r} = (r_1, r_2)$ and $\mathbf{d} = (d_1, d_2)$, shows irreducibility of $\mathbf{HQuot_d}$ when

$$0 << d_1 << d_2.$$

Thank you!