Homework 5

Due Date: 30/04/2026

Instructions: Please show all your work clearly. You may ask the tutor and discuss questions with other students, but the solution must be written in your own words. Be sure to cite any sources that helped with your solutions (no points will be deducted).

1. Let $0 \neq w \in \Lambda^r k^n$ with $r < n$, and consider the linear map

$$\phi : k^n \longrightarrow \Lambda^{r+1} k^n, \quad v \longmapsto v \wedge w.$$

   Then

$$\mathrm{rank}(\phi) \geq n - r.$$

   Moreover, equality holds if and only if w is a pure tensor, i.e., there exist vectors $v_1, \dots, v_r \in k^n$ such that

$$w = v_1 \wedge \cdots \wedge v_r.$$

2. Consider the Plucker embedding $f : G(r, n) \to \mathbb{P}^{\binom{n}{r}-1}$. Fix a basis $\{e_1, \dots e_n\}$ of k^n, then f is expressed by sending an $n \times r$ matrix A to

$$A \to [w_I := \det A_I]_{I \subset \{1,2,\dots,n\}} \subset \mathbb{P}^{\binom{n}{r}-1},$$

   where for each $I = \{i_1, \dots, i_r\} \subset \{1, 2, \dots, n\}$, A_I is the corresponding $r \times r$ minor.

   Example: Let $r = 2$ and $n = 4$. Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$$

   which is sent to

$$f(A) = [w_{1,2} : w_{1,3} : w_{1,4} : w_{2,3} : w_{2,4} : w_{3,4}] \in \mathbb{P}^5$$

   where each coordinate is calculated as:

$$\begin{aligned} w_{1,2} &= a_{11}a_{22} - a_{21}a_{12} & w_{1,3} &= a_{11}a_{23} - a_{13}a_{21} & w_{1,4} &= a_{11}a_{24} - a_{21}a_{14} \\ w_{2,3} &= a_{12}a_{23} - a_{22}a_{13} & w_{2,4} &= a_{12}a_{24} - a_{22}a_{14} & w_{3,4} &= a_{13}a_{24} - a_{23}a_{14} \end{aligned}$$

   (a) Find explicitly the image $f(G(2, 4)) \subseteq \mathbb{P}^5$.

   (b) For any I, show that

$$U_I := f^{-1}(\{w_I \neq 0\}) \cong \mathbb{A}^{r(n-r)}.$$

   (c) Show that U_I is a affine cover of $G(r, n)$, whose pairwise intersections are non-empty.

(d) Prove that $G(r,n)$ is irreducible of dimension $r(n-r)$.

3. Let $f : X \to Y$ be a morphism of affine varieties. If the induced map

$$f^* : A(Y) \to A(X)$$

is injective, then $f(X)$ is dense in Y.

4. Show that the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point is isomorphic to the blow-up of $\mathbb{P}^2$ at two distinct points. (Complete the sketch of the proof given in the lecture)

5. Let $a = (1:0:0)$, $b = (0:1:0)$, and $c = (0:0:1)$ be the three coordinate points of $\mathbb{P}^2$, and let $U = \mathbb{P}^2 \setminus \{a,b,c\}$. Consider the morphism

$$f : U \to \mathbb{P}^2, \quad (x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_0x_2 : x_0x_1).$$

(a) Show that there is no morphism $\mathbb{P}^2 \to \mathbb{P}^2$ extending f.

(b) Let $\widetilde{\mathbb{P}^2}$ be the blow-up of $\mathbb{P}^2$ at $\{a,b,c\}$. Show that f extends to an isomorphism

$$\tilde{f} : \widetilde{\mathbb{P}^2} \to \widetilde{\mathbb{P}^2}.$$

This isomorphism is called the Cremona transformation.

6. For given $n, d \in \mathbb{N}_{>0}$ consider the Fermat hypersurface

$$X := Z\big(x_0^d + \cdots + x_n^d\big) \subset \mathbb{P}^n.$$

Describe the tangent space T_pX at a point $p \in X$ using the Jacobian criterion. Show that X is smooth for all choices of n, d, and K.