Homework 5

Due Date: 17/04/2026

Instructions: Please show all your work clearly. You may ask the tutor and discuss questions with other students, but the solution must be written in your own words. Be sure to cite any sources that helped with your solutions (no points will be deducted).

1: Consider the polynomial ring $R = k[x_0, x_1, \ldots, x_n]$ as a graded k-algebra $R = \oplus_{d \geq 0} R_d$ where R_d is the span of degree d homogeneous polynomials. Calculate the dimension of the R_d as a k-vector space.

2: (a) Let $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ be the mosphism given by $\phi([s : t]) = [s^2 : st : t^2]$. Show that $X = Im(\phi)$ is a closed subset of $\mathbb{P}^2$, and it isomorphic to $\mathbb{P}^1$.

(b) Calculated the coordinate ring $S(X)$ of X as a projective variety in $\mathbb{P}^2$. Show that $S(X) \ncong k[x, y] = S(\mathbb{P}^1)$. (Unlike the affine case, projective coordinate ring is not invariant under isomorphisms of projective varieties!)

3: (a) (Universal property of products) Let X and Y be affine varieties, and let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the projection morphisms onto the two factors. Then for every affine variety Z and two morphisms $f_X : Z \to X$ and $f_Y : Z \to Y$, there exists a unique morphism

$$f : Z \to X \times Y$$

such that

$$f_X = \pi_X \circ f \quad \text{and} \quad f_Y = \pi_Y \circ f.$$

(b) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be irreducible affine varieties. Prove that their product

$$X \times Y \subset \mathbb{A}^{n+m}$$

is irreducible as well.

(c) Assume $X \subset \mathbb{A}^n$ is an affine variety. Describe explicitly the homomorphism

$$\delta_X^* : A[X \times X] \to A[X]$$

induced by $\delta_X : X \to X \times X,\ x \mapsto (x, x)$.

4: Let $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ be projective varieties. Show that $X \times Y$ is a closed subset of $\mathbb{P}^m \times \mathbb{P}^n$. Conclude using Segre embedding that $X \times Y$ is itself a projective variety.

5: Show that every morphism $f : \mathbb{P}^n \to \mathbb{P}^m$ must be of the form

$$f : \mathbb{P}^n \to \mathbb{P}^m, \quad x \mapsto \big(f_0(x) : \cdots : f_m(x)\big)$$

with $f_0, \ldots, f_m \in K[x_0, \ldots, x_n]$ homogeneous polynomials of the same degree such that

$$V_p(f_0, \ldots, f_m) = \varnothing.$$

Please see Lemma 7.4 in Gathmann's notes.

6: Show by example that not every hypersurface Y in a projective variety X is of the form $V(f)$ for a homogeneous polynomial $f \in S(X)$. (One possibility is to consider the Segre embedding X of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, and $Y = \mathbb{P}^1 \times \{0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$.)