Homework 5

Due Date: 25/11/2024

Instructions: Please show all your work clearly. You may ask the tutor and discuss questions with other students, but the solution must be written in your own words. Be sure to cite any sources that helped with your solutions (no points will be deducted).

1. (a) Let $f_n : U \to \mathbb{C}$ be a sequence of holomorphic functions. Suppose for all compact sets $K \subset U$ and $n \in \mathbb{N}$, there exists positive real number $M_n(K)$ such that $|f_n| \le M_n(K)$ over K and

$$\sum_{n=1}^{\infty} M_n(K) < \infty,$$

then show that $f = \sum_{n=1}^{\infty} f_n$ is an holomorphic function, and

$$f' = \sum_{n=1}^{\infty} f_n'.$$

   (b) Use the above to show that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is holomorphic on $U = \{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$.

2. **Weierstrass $\wp$-function**: Fix two distinct non-zero complex numbers w_1 and w_2 such that $\frac{w_1}{w_2} \notin \mathbb{R}$. Consider the lattice $\Lambda = \{aw_1 + bw_2 : a, b \in \mathbb{Z}\}$. The Weierstrass $\wp$ is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

   (a) Show that $\wp(z)$ is holomorphic on $\mathbb{C} \setminus \Lambda$ using 1(a);
   Hint: Let $A_n := \{aw_1 + bw_2 : |a| + |b| = n\} \subset \Lambda$ and

$$f_n(z) := \sum_{\lambda \in A_n} \left((z-\lambda)^{-2} - \lambda^{-2} \right)$$

   Let K be a compact set contained inside a parallelogram $\{sw_1 + tw_2 : s, t \in \mathbb{R}; |s| + |t| \le k\}$ for some integer k, then show that $||f_n||_K < \frac{C_k}{n^2}$ for some constant C_k (that may also depends on w_1 and w_2). You will also benefit by noting that A_n has exactly $4n$ points.

   (b) Show that $\wp(z)$ has poles of order 2 at each lattice point $\lambda \in \Lambda$;

   (c) Show that $\wp(z)$ is an **elliptic** function, that is,

$$\wp(z) = \wp(z + w_1) = \wp(z + w_2) \quad \forall z \in \mathbb{C}.$$

(d) Using the definition, show that $\wp(z)$ is an even function.

(e) Show that the Laurent expansions around $z = 0$ of $\wp$ and $\wp'$ take the form:

$$\wp(z) = \frac{1}{z^2} + az^2 + bz^4 + \dots$$

$$\wp'(z) = -\frac{2}{z^3} + 2az + 4bz^3 + \dots$$

for certain $a, b \in \mathbb{C}$.

(f) Using undetermined coefficients, show that there exist constants A, B such that:

$$\wp'(z)^2 - \big(4\wp(z)^3 + A\wp(z) + B\big)$$

has no Laurent principal tail at $z = 0$ and vanishes at $z = 0$.

(g) Conclude from (f) and Λ-periodicity that:

$$\wp'(z)^2 - \big(4\wp(z)^3 + A\wp(z) + B\big) = 0.$$

3. Let f be an entire function and let $\mathcal{F}$ denote the family of functions $f(kz)$ for $k \in \mathbb{Z}_{>0}$, defined in the annulus $r_1 < |z| < r_2$ for $r_1, r_2 > 0$. Show that $\mathcal{F}$ is normal if and only if f is constant.

4. Let $\mathcal{F}$ be the family of holomorphic functions in $\Delta(0, 1)$ (the unit disk) such that $f(0) = 1$ and $\operatorname{Re} f > 0$. Show that $\mathcal{F}$ is normal.

5. **Vitali's Theorem** (Conway VII.2.4, page 154). Prove the following statement:

   Let $\{f_n\}$ be a locally bounded sequence of holomorphic functions in $U \subset \mathbb{C}$, and let f be holomorphic in U. If
   $$A = \{z \in U : f_n(z) \to f(z)\}$$
   has a limit point in U, then f_n converges locally uniformly to f in U.

6. **Normal families under composition** (Conway VII.2.7, page 154). Let $U, \Omega \subset \mathbb{C}$ be open and connected. Let $g : \Omega \to \mathbb{C}$ be a holomorphic function that is bounded on bounded sets. Let $\mathcal{F}$ be a normal family of holomorphic functions $f : U \to \Omega$. Show that the family
   $$\{g \circ f : f \in \mathcal{F}\}$$
   is also normal.

Remark on the Weierstrass $\wp$-function

The function $\wp$ is truly very interesting. There are deep connections between $\wp$ (naturally arising in complex analysis) and various fields such as algebraic geometry, number theory, and mathematical physics, which might not be apparent at first glance:
Elliptic curves: The functions $\wp$ and $\wp'$ parametrize the cubic (elliptic) curve:

$$y^2 = 4x^3 + Ax + B$$

for suitable constants A, B. The parametrization is given by $y = \wp'(z)$ and $x = \wp(z)$. The constants A and B depend on the periods ω_1, ω_2 of the $\wp$-function and are expressed in terms of Eisenstein series. These constants are examples of modular forms.