Homework 4

Due Date: 01/04/2026

Instructions: Please show all your work clearly. You may ask the tutor and discuss questions with other students, but the solution must be written in your own words. Be sure to cite any sources that helped with your solutions (no points will be deducted).

1: (a) Show that a point $p \in \mathbb{P}^n$ is a projective variety. Describe the ideal $I(\{p\})$.

(b) Let $k = \mathbb{C}$. Show that the projective space $\mathbb{P}^n$ is a smooth compact differentiable real manifold of dimension $2n$. Hint: Describe the local charts and how to patch them.

2: Let R be a graded ring. Let I and J be homogeneous ideals in R. Show that $I + J$, $I \cap J$ and $I \cdot J$ are homogeneous ideals.

3: Let A be a ring and $I \subset A$ be an ideal. The associated graded ring of A along I is defined by

$$R := \mathrm{gr}_I(A) = \oplus_{d \in \mathbb{N}} R_d$$

where $R_d = I^d / I^{d+1}$. Note that $R_0 = R/I$ is the quotient ring.

(a) Show that R has a natural structure of a graded ring by defining the multiplication

$$(a \bmod I^{d+1}) \cdot (b \bmod I^{e+1}) := ab \bmod I^{d+e+1}.$$

Verify that this is well-defined.

(c) Let $A = k[x]$ and $I = \langle x \rangle$. Compute $\mathrm{gr}_I(A)$ explicitly and identify it with a familiar graded ring.

(d) More generally, let $A = k[x_1, \ldots, x_n]$ and $I = \langle x_1, \ldots, x_n \rangle$. Describe $\mathrm{gr}_I(A)$.

(e) Let $I = \langle p \rangle$ be the prime ideal (for a prime number p) in $A = \mathbb{Z}$. Describe $\mathrm{gr}_I(A)$.

4: (a) Show that every irreducible component of a cone is also a cone.

(b) Let $f \in k[x_0, x_1, \ldots, x_n]$ be homogeneous. Show the irreducible factors of f are homogeneous.

5: A line $L \subset \mathbb{P}^n$ is the projective algebraic set such that the cone $C(L) \subset k^{n+1}$ is a two dimensional vector space.

(a) Show that any two distinct lines in $\mathbb{P}^2$ intersect in exactly one point.

(b) Let $L_1, L_2 \subset \mathbb{P}^3$ be two disjoint lines, and let $a \in \mathbb{P}^3 \backslash (L_1 \cup L_2)$. Show that there is a unique line $L \subset \mathbb{P}^3$ through a that intersects both L_1 and L_2.

(c) Is the corresponding statement for lines and points in $\mathbb{A}^3$ true as well?

6: Consider the cubic curve $C := Z(y^2 z - (x^3 - xz^2)) \subset \mathbb{P}^2$.

(a) Write down the equations of $\phi_i(U_i \cap C)$ in the three affine charts isomorphisms $\phi_i : U_i \cong \mathbb{A}^2$ for $i = 0, 1, 2$ given by $x \neq 0$, $y \neq 0$ and $z \neq 0$ respectively.

(b) Determine the intersection of C with the $Z(x)$, $Z(y)$, $Z(z)$.

7: The *twisted cubic* is the image of the morphism

$$\varphi\colon \mathbb{P}^1 \to \mathbb{P}^3, \qquad [s:t] \mapsto [s^3 : s^2 t : st^2 : t^3].$$

Let $C \subset \mathbb{P}^3$ denote its image.

(a) Show that φ is a well-defined morphism and that it is injective.

(b) Show that C is not contained in any hyperplane of $\mathbb{P}^3$.

(c) Prove that the homogeneous ideal of C is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

(a) Show that there are no two polynomials $f, g \in k[x_0, x_1, x_2, x_3]$ such that $C = Z(f, g)$. Recall that it is possible in the affine version of this problem in HW-2 question 2.