Homework 4

Due Date: 04/11/2024

Instructions: Please show all your work clearly. You may ask the tutor and discuss questions with other students, but the solution must be written in your own words. Be sure to cite any sources that helped with your solutions (no points will be deducted).

1. Characterize all isolated singularities for the following functions. Describe the principal part in the case of a pole or an essential singularity.

   (a) $f(z) = \sin(z)/z$

   (b) $f(z) = \cos(z)/z$

   (c) $f(z) = e^{2z}/(z-1)^2$

2. Let $g(z)$ and $h(z)$ be holomorphic functions on U containing a point a. Suppose $h(z)$ has a simple zero at $z = a$. Show that

$$Res\left(\frac{g}{h}, a\right) = \frac{g(a)}{h'(a)}.$$

   Specialize to the functions $g(z) = (z - \sin z)/z^2$ and $h(z) = \sin z$, to prove that

$$Res\left(\frac{z - \sin z}{z^2 \sin z}, n\pi\right) = \begin{cases} 0 & n = 0 \\ \frac{(-1)^n}{n\pi} & n \neq 0 \end{cases}.$$

3. Evaluate the real integral

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}.$$

4. Let n be a positive integer. Calculate the following residue using residue theorem:

$$\int_0^{2\pi} \cos^{2n}(t)\ dt.$$

   Hint: Write the integral in terms of $z = e^{it}$.

5. Consider the power series expansion

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!}.$$

   The expansion holds for $|z| < 2\pi$, and the coefficients B_k are called Bernoulli numbers.

   Consider the function

$$f(z) = \frac{1}{z^{2n}(e^z - 1)}.$$

Let γ_m denote the boundary of the rectangle with corners $\pm(2m+1)\pi \pm (2m+1)\pi i$. Using the residue theorem, compute the integral

$$\int_{\gamma_m} f(z)\, dz$$

in terms of Bernoulli numbers. Using a suitable estimate of the function f along γ_m, show furthermore that

$$\lim_{m\to\infty} \int_{\gamma_m} f(z)\, dz = 0.$$

Use this to confirm the following formula for the values of the Riemann zeta function:

$$\zeta(2n) = \sum_{p=1}^{\infty} \frac{1}{p^{2n}} = \frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{2(2n)!}.$$