# Homework 3

Due Date: Submission not required!

**Instructions:** Please show all your work clearly. You may ask the tutor and discuss questions with other students, but the solution must be written in your own words. Be sure to cite any sources that helped with your solutions (no points will be deducted).

**1:** Let $X$ be an irreducible affine variety and let $f \in A(X)$. Recall from the lecture that distinguished open set $D(f)$ is an affine variety.

(a) Show that the coordinate ring

$$A(D(f)) = \mathcal{O}_X(D(f)) = A(X)_f.$$

(b) Calculate the coordinate rings of Zariski open sets in $\mathbb{A}^1$.

**2:** Which of the following ringed spaces are isomorphic?

(a) $\mathbb{A}^1 \setminus \{1\}$

(b) $Z(x_1 x_2) \subset \mathbb{A}^2$

(c) $Z(x_1^2 + x_2^2) \subset \mathbb{A}^2$

(d) $Z(x_2 - x_1^2,\ x_3 - x_1^3) \setminus \{0\} \subset \mathbb{A}^3$

(e) $Z(x_2^2 - x_1^3 - x_1^2) \subset \mathbb{A}^2$

(f) $Z(x_1^2 - x_2^2 - 1) \subset \mathbb{A}^2$

**3:** Let $f\colon X \to Y$ be a morphism of affine varieties and $f^*\colon A(Y) \to A(X)$ the corresponding homomorphism of coordinate rings. Are the following statements true or false?

(a) $f$ is surjective if and only if $f^*$ is injective.

(b) $f$ is injective if and only if $f^*$ is surjective.

(c) If $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ is an isomorphism then $f$ is affine linear, i.e. of the form $f(x) = ax + b$ for some $a, b \in k$.

(d) If $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ is an isomorphism then $f$ is affine linear, i.e. it is of the form $f(x) = Ax + b$ for some $A \in \mathrm{Mat}(2 \times 2, k)$ and $b \in k^2$.

**4:** Let $\varphi = (F_1, \ldots, F_n)\colon \mathbb{A}^n \to \mathbb{A}^n$ be an isomorphism. Show that the Jacobian determinant

$$\det \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix}$$

is a nonzero constant polynomial. In the case $k = \mathbb{C}$, the converse is a famous open problem, called the *Jacobian Conjecture*.