Homework 3

Due Date: 21/10/2024

Instructions: Please show all your work clearly. You may ask the tutor and discuss questions with other students, but the solution must be written in your own words. Be sure to cite any sources that helped with your solutions (no points will be deducted).

1. Find all possible values for the integral
$$\int_\gamma \frac{dz}{1+z^2},$$
where γ is any piece-wise C^1 loop in $\mathbb{C}$ not containing $\{i,-i\}$.

2. Let $P:\mathbb{C}\to\mathbb{R}$ be defined by $P(z)=\mathrm{Re}(z)$; show that P is an open map but is not a closed map.

3. Assume f is an entire function and p is a polynomial of degree d. Assume that there exists $M>0$ such that
$$|f(z)|\le|p(z)| \quad \text{for all } |z|\ge M.$$
Show that $f(z)$ is a polynomial of degree at most d.

4. Let $f:\mathbb{C}\to\mathbb{C}$ be an entire function.
   (i) Show that if $\mathrm{Re}(f)$ is bounded (from above or below) then f is constant. You may wish to consider the function $g=e^{\pm f}$.
   (ii) Show that if $\mathrm{Re}(f)\le\mathrm{Im}(f)$, then f is constant.

5. Assume $f:U\to\mathbb{C}$ is a holomorphic function on a simply connected open set U such that $f(z)\neq 0$ for all $z\in U$. Let $n\ge 2$ be an integer. Show that there is a holomorphic function $g:U\to\mathbb{C}$ such that $g(z)^n=f(z)$.

6. Let d be a non-negative integer. A weak composition of d into r parts is the tuple of non-negative integers $(d_1,d_2,\ldots,d_r)$ such that $d_1+\cdots+d_r=d$.
   (a) For $|z|<1$, show that the following Taylor's expansion hold at $z=0$:
$$\frac{1}{(1-z)^r}=\sum_{d=0}^{\infty}\binom{r+d-1}{d}z^d.$$
   (b) Convince yourself that the number of weak composition equals the dimension of the subspace of degree d homogeneous polynomial in $\mathbb{C}[x_1,\ldots,x_r]$. Prove that
$$\#\{(d_1,d_2,\ldots,d_r): d_1+\cdots+d_r=d\}=\binom{d+r-1}{d}.$$

(c) Prove the following identity:

$$\sum_{d_1+\cdots+d_r=d} (1+d_1)\cdot(1+d_2)\cdots(1+d_r) = \binom{2r+d-1}{d},$$

where sum is taken over all weak composition of d.

7. Consider the power series expansion

$$\frac{z}{e^z-1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!}.$$

The expansion holds for $|z| < 2\pi$. The coefficients B_k are called the Bernoulli numbers.

(i) Find the first three non-zero Bernoulli numbers.

(ii) Prove that $B_{2k+1} = 0$ for all $k \geq 1$.

(iii) Show that

$$1^p + 2^p + \cdots + N^p = \frac{1}{p+1}\sum_{j=0}^{p}(-1)^j B_j \binom{p+1}{j} N^{p+1-j}.$$

What does this formula give for $p = 1, 2, 3$?