Homework 2

Due Date: 16/03/2026

Instructions: Please show all your work clearly. You may ask the tutor and discuss questions with other students, but the solution must be written in your own words. Be sure to cite any sources that helped with your solutions (no points will be deducted).

Assumptions: Let k be an algebraically closed field. You may use the fact that $k[x_1, \dots, x_n]$ is a *unique factorization domain* (UFD). Recall that an integral domain R is called UFD if:

(i) every nonzero nonunit element of R can be written as a product of irreducible elements;

(ii) this factorization is unique up to multiplication by units and reordering of the irreducible factors.

For the rest of the problems, you may use also Proposition 2.28 (**Properties of dimension**) in Gathmann's lecture notes.

1: (a) Let $X \subset \mathbb{A}^n$ be an affine variety. A closed subvariety of X is an affine variety $Y \subset \mathbb{A}^n$ with $Y \subset X$. Show that the map

$$Y \longmapsto I_X(Y) := \{ f \in A(X) \mid f|_Y = 0 \}$$

defines a natural bijection between irreducible closed subvarieties (resp. points) of X and prime ideals (resp. maximal ideals) of $A(X)$.

(b) Let $Y \subset X$ be an irreducible closed subvariety, where $X \subset \mathbb{A}^n$ is an affine variety. Let $I_X(Y)$ be the ideal of Y in X as in the previous exercise. Show that $A(Y)$ is isomorphic to $A(X)/I_X(Y)$.

2: Consider $C = \{(t, t^2, t^3) : t \in k\} \subset \mathbb{A}^3$.

(a) Show that is an affine algebraic set, and find the radical ideal $I(C)$.

(b) Show that the coordinate ring $A(C) \cong k[x]$ and deduce $\dim C = 1$.

(c) (Bonus problem) Let $P_1, P_2, P_3 \in k[x]$. Show that any set of the form

$$\{(P_1(t), P_2(t), P_3(t)) : t \in k\} \subset \mathbb{A}^3$$

is an affine algebraic set.

3: (Cuspidal cubic curve) Consider $C = Z(y^2 - x^3) \subset \mathbb{A}^2$.

(a) Show that $I(C) = \langle y^2 - x^3 \rangle$.

(b) Show that the coordinate ring $A(C) \ncong k[t]$, but $\dim C = 1$.

(c) Show that the field of rational functions $K(C) \cong K(\mathbb{A}^1) \cong k(x)$.

4: (Smooth cubic curve) Consider the curve $C = Z(y^2 + x - x^3) \subset \mathbb{A}^2$.

(a) Show that $A(C) \not\cong k[t]$, but $\dim C = 1$.

(b) Show that the field of rational functions

$$K(C) \cong k(x)[y]/\langle y^2 + x - x^3 \rangle,$$

that is, $K(C)$ is a degree two extension of the field $k(x)$.

(c) Show that $K(C) \not\cong k(t)$ as fields. (The field of fractions contains some non-trivial information about affine varieties!)
Hint: You may want to reduce the problem to the following: If $f, g \in k[t]$ are polynomials without common factors and such that there exists a constant $\lambda \neq 0, 1$ for which f, g, $f - g$, and $f + g$ are perfect squares, then f and g must be constant.

5: Consider the affine variety $X = Z(x_1x_4 - x_2x_3) \subset \mathbb{A}^4$, and let $V = Z(x_2, x_4) \subset X$ be a subvariety of X, and let $U = X \backslash V$ be an open set.

(a) Show that X is an irreducible affine variety of dimension 3.

(b) Show that the function $\phi : U \to k$, given by

$$\phi(x_1, x_2, x_3, x_4) = \begin{cases} x_1/x_2 & x_2 \neq 0 \\ x_3/x_4 & x_4 \neq 0 \end{cases},$$

is a well-defined element $\phi \in \mathcal{O}_X(U)$.

(c) Prove that x_1, x_2, x_3, x_4 are irreducible elements in $A(X)$, but the ideals $\langle x_i \rangle$ are not prime.

(d) Show that there are no polynomial functions $f, g \in A(X)$ such that $\phi(p) = f(p)/g(p)$ for all $p \in U$.

6: Show that all regular functions on $\mathbb{A}^2 \setminus \{(0,0)\}$ are given by polynomials.

7: Let $X \subset \mathbb{A}^n$ be an affine variety, and let $a \in X$. Show that

$$\mathcal{O}_{X,a} \cong \mathcal{O}_{\mathbb{A}^n,a} / I(X)\mathcal{O}_{\mathbb{A}^n,a},$$

where $I(X)\mathcal{O}_{\mathbb{A}^n,a}$ denotes the ideal in $\mathcal{O}_{\mathbb{A}^n,a}$ generated by the images of all $f \in I(X)$.