Homework 2

Due Date: 14/10/2024

Instructions: Please show all your work clearly. You may ask the tutor and discuss questions with other students, but the solution must be written in your own words. Be sure to cite any sources that helped with your solutions (no points will be deducted).

Let Log denote the principle branch of logarithm.

1: Let C_R be the circle $|z| = R$ $(R > 1)$, described in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{\mathrm{Log} z}{z^2}\, dz \right| \leq 2\pi \left(\frac{\pi + \ln R}{R} \right),$$

and conclude that the integral tends to zero as $R \to \infty$.

2: (a) Let f and g be holomorphic functions on open sets U and V, respectively. Suppose the image $f(U) \subset V$. Then show that the composition $g \circ f : U \to \mathbb{C}$ is a holomorphic function on U, and derive the chain rule for $(g \circ f)'$.
(b) Using a suitable branch of the logarithm, define the holomorphic function

$$h(z) = \sqrt{\frac{1-z}{1+z}}$$

defined over the domain $U = \mathbb{C} \setminus [-1, 1]$.
(c) Find the derivative $h(z)$ on the open set U.
(d) Show that for all $R > 1$,

$$\int_{C_R} \frac{dz}{(1+z)\sqrt{1-z^2}} = 0,$$

where $\sqrt{1-z^2}$ defined using the same brach of logarithm as in part (b).

3: Assume that $f(z) = c \prod_{k=1}^{\ell} (z - a_k)^{m_k}$ is a polynomial with roots at $a_1, \ldots, a_k$ with multiplicities $m_1, m_2, \ldots, m_k$. Show that for any closed loop γ avoiding $a_1, \ldots, a_k$ we have

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^{\ell} m_k \cdot n(\gamma, a_k).$$

In particular, if R is sufficiently large, and $\gamma(t) = Re^{it}$ for $0 \leq t \leq 2\pi$, show that

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \deg(f).$$

4: Calculate the following integrals using the local form of Cauchy's integral formula:

(i) $\displaystyle\int_{|z|=2} \frac{e^z}{(z-1)(z-3)^2}\,dz$

(ii) $\displaystyle\int_{|z|=2} \frac{\sin z}{z+i}\,dz$

(iii) $\displaystyle\int_{|z|=1} \frac{e^z}{(z-2)^3}\,dz$

5: (Conway IV.2.2) Prove the following analogue of Leibniz's rule.

Let G be an open set and let γ be a piecewise C^1 curve in $\mathbb{C}$. Suppose that $\varphi : \{\gamma\} \times G \to \mathbb{C}$ is a continuous function and define $g : G \to \mathbb{C}$ by

$$g(z) = \int_\gamma \varphi(w, z)\,dw$$

then g is continuous. If $\frac{\partial\varphi}{\partial z}$ exists for each $(w, z) \in \{\gamma\} \times G$ and is continuous, then g is holomorphic and

$$g'(z) = \int_\gamma \frac{\partial\varphi}{\partial z}(w, z)\,dw.$$