Homework 1

Due Date: 09/03/2026

Instructions: Please show all your work clearly. You may ask the tutor and discuss questions with other students, but the solution must be written in your own words. Be sure to cite any sources that helped with your solutions (no points will be deducted).

1: Let R be a commutative ring (with 1). Recall:

- R is *Noetherian* if every ideal of R is finitely generated.
- R satisfies the *ascending chain condition (ACC) on ideals* if every ascending chain

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

stabilizes, i.e. there exists N such that $I_n = I_N$ for all $n \geq N$.

(a) Show that R is Noetherian if and only if R satisfies the ACC on ideals.

(b) If $I \subset R$ is an ideal, then the quotient ring R/I is Noetherian.

2: Let $J \subset R$ be an ideal. Show that the radical

$$\sqrt{J} := \{f \in R : f^m \in J \text{ for some } m \in \mathbb{N}\}$$

is an ideal of R. Determine the radical of the ideal $\langle x^3 - y^6, xy - y^3 \rangle \subset k[x, y]$.

3: Show that the Zariski topology on $\mathbb{A}^2$ is not the product of the Zariski topologies on $\mathbb{A}^1 \times \mathbb{A}^1$.

4: Let $f : \mathbb{A}^n \to \mathbb{A}^m$ be a polynomial map, i.e.

$$f(p) = (f_1(p), \ldots, f_m(p))$$

for $p \in \mathbb{A}^n$, where $f_1, \ldots, f_m$ are polynomials in n variables.

(a) Show that the inverse image $f^{-1}(X) \subset \mathbb{A}^n$ of an affine algebraic set $X \subset \mathbb{A}^m$ is an affine algebraic set.

In particular, polynomial maps are continuous in the Zariski topology.

(b) Show that if $X \subset \mathbb{A}^n$ is an affine algebraic set, then the graph

$$\Gamma = \{(x, f(x)) : x \in X\} \subset \mathbb{A}^{n+m}$$

is an affine algebraic set.

(c) By means of an example, show that the image $f(X) \subset \mathbb{A}^m$ of an affine algebraic set $X \subset \mathbb{A}^n$ may fail to be an affine algebraic set.

In particular, polynomial maps between affine algebraic sets are typically not closed.

5: Let X_1, X_2 be affine algebraic sets in $\mathbb{A}^n$. Assuming the Nullstellensatz if necessary, show that

(a) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$,

(b) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Show by example that taking the radical in (ii) is in general necessary, i.e. find affine algebraic sets X_1, X_2 such that

$$I(X_1 \cap X_2) \neq I(X_1) + I(X_2).$$

6: Let Y be a subspace of a topological space X. Show that Y is irreducible if and only if the closure of Y in X is irreducible.

7: Let X be a topological space, and consider $X = U_1 \cup \cdots \cup U_n$ a finite cover by open sets such that

(a) U_i is irreducible for all i,

(b) $U_i \cap U_j \neq \emptyset$ for all i, j.

Show that X is irreducible.

8: (a) Let $m, n, r \geq 1$, and let $\mathrm{Mat}_{m,n}$ denote the set of $m \times n$ matrices, identified with $\mathbb{A}^{mn}$. Show that the set

$$X_r = \{A \in \mathrm{Mat}_{m,n} : \mathrm{rk}(A) \leq r\}$$

is a Zariski closed subset of $\mathbb{A}^{mn}$.

(b) Show that if X, Y are topological spaces with X irreducible, and $f : X \to Y$ is continuous, then $f(X)$ is irreducible.

(c) Set $r = 1$ in part (i). Show that the set $X_1 \subset \mathbb{A}^{mn}$ of matrices of rank ≤ 1 is irreducible.

(The case of arbitrary r is similar, but you do not have to prove it.)