Homework 1

Due Date: 30/09/2024

Instructions: Please show all your work clearly. You may ask the tutor and discuss questions with other students, but the solution must be written in your own words. Be sure to cite any sources that helped with your solutions (no points will be deducted).

1: Use Cauchy-Riemann equation to determine the domain where the following functions are complex differentiable.

(a) $f(z) = 3x + y + i(3y - x)$

(b) $f(z) = (z^2 - 2)e^{-x}e^{-iy}$

(c) $f(z) = 2xy + i(x^2y^2)$

(d) $f(z) = e^{z^2}$

2: Let a function f be analytic everywhere in an open disk $D \subset \mathbb{C}$. Prove that if $f(z)$ is real-valued for all $z \in D$, then $f(z)$ must be constant throughout D.

3: Let

$$f(z) = \begin{cases} \frac{xy^2(x+iy)}{(x^2+y^4)} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that the real and imaginary part of f satisfy the Cauchy-Riemann equations at $z = 0$, but f is not complex differentiable at $z = 0$. Why doesn't this contradict the results proved in class?

4: Show that $(-1 + \sqrt{3}i)^{3/2} = \pm 2\sqrt{2}$. What determines the sign in the answer?

5: Let $U \subset \mathbb{C}$ be a connected open set (not containing the origin). A logarithm function $\ell : U \to \mathbb{C}$ is a **continuous** function such that

$$\exp(\ell(z)) = z \quad \text{for all } z \in U.$$

Note that we are note assuming that ℓ is complex differentiable.

(i) Show that if ℓ is a complex differentiable function with $\ell'(z) = \frac{1}{z}$, then ℓ must be a logarithm function as defined above, possibly up to a constant. You may wish to begin by differentiating $\exp(\ell(z))$.

(ii) Let $\Delta_1(1)$ be the open disc of radius 1 centered at 1. Using (i), show that the power series

$$L(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(z-1)^k$$

defines a logarithm function on $\Delta_1(1)$.

(iii) Let $a \in \mathbb{C} \setminus \{0\}$ and let b be any logarithm of a. Using (ii), derive that

$$L\left(\frac{z}{a}\right) + b$$

is a logarithm function over the ball $\Delta_{|a|}(a)$ of radius $|a|$ centered at a.

(iv) Show that if ℓ_1 and ℓ_2 are two logarithm functions defined over a connected open set U, then $\ell_1 = \ell_2 + 2\pi i n$ for some integer n.

(v) Using (iii) and (iv), conclude that any logarithm function ℓ is automatically complex differentiable. Show that $\ell'(z) = \frac{1}{z}$.