Quasinormal Modes

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Abstract: A brief review of quasinormal modes, using simple one dimensional potentials.

1 Introduction, initial value problem

Suppose we are interested in solving a wave equation

$$\ddot{\psi} - \psi'' + V(x)\psi = 0, \tag{1}$$

with the initial data $\psi_0(x)$ and $\dot{\psi}_0(x)$, at $t = 0$, where prime is ∂_x and dot is ∂_t. And suppose that

$$V(x \to \pm\infty) = 0, \tag{2}$$

so we have a continuum in the spectrum. The most standard way to solve this problem is to, first, find a complete set of (delta-function) normalizable eigenstates of the Hamiltonian

$$H = -\partial_x^2 + V(x). \tag{3}$$

More explicitly, one finds solutions ϕ_n

$$H\phi_n = \omega_n^2 \phi_n, \qquad \phi_n(t,x) = \phi_n(0,x)e^{-i\omega_n t}. \tag{4}$$

Note ω_n^2 can be negative if there exists a bound state in the spectrum. However, if $V(x) \geq 0$ for all x, then $\omega_n^2 \geq 0$. Next, one decomposes

$$\psi(t,x) = \sum_n a_n \phi_n(t,x). \tag{5}$$

In practice, to find a_n we use another set of solutions $\tilde{\phi}_n(t,x)$ such that its Klein-Gordon inner product with ϕ_m is

$$\left\langle \tilde{\phi}_n | \phi_m \right\rangle = \int dx [\tilde{\phi}_n(t,x)\partial_t \phi_m(t,x) - \phi_m(t,x)\partial_t \tilde{\phi}_n(t,x)] = c_n \delta_{m,n} \tag{6}$$

where $\delta_{m,n}$ is understood as a Dirac-delta for continuous spectrum. Then the solution is

$$\psi(t,x) = \sum_n \frac{1}{c_n} \left\langle \tilde{\phi}_n | \psi_0 \right\rangle \phi_n(t,x). \tag{7}$$

Let's demonstrate this in a simple example:

$$V(x) = v\delta(x), \qquad v > 0. \tag{8}$$

With some hindsight, I choose the following basis for the solutions

$$\phi_k^{\pm} = e^{\mp i|k|t} \begin{cases} e^{ikx} & x \geq 0 \\ (1+\frac{iv}{2k})e^{ikx} - \frac{iv}{2k}e^{-ikx} & x < 0 \end{cases} \tag{9}$$

and

$$\tilde{\phi}_k^{\pm} = e^{\mp i|k|t} \begin{cases} (1+\frac{iv}{2k})e^{-ikx} - \frac{iv}{2k}e^{ikx} & x \geq 0 \\ e^{-ikx} & x < 0 \end{cases} \tag{10}$$

The only nonzero inner products are

$$\left\langle \tilde{\phi}_k^{\mp} | \phi_{k'}^{\pm} \right\rangle = \mp 2i|k| \left(1 + \frac{iv}{2k}\right) 2\pi\delta(k-k'), \tag{11}$$

as I will show next. First, note that

$$\left\langle \tilde{\phi}_n | \phi_m \right\rangle = -i(\omega_m - \omega_n) \int dx \tilde{\phi}_n(x)\phi_m(x), \tag{12}$$

so the Klein-Gordon inner product vanishes if $\omega_n = \omega_m$, or if $\omega_n^2 \neq \omega_m^2$. The latter is a consequence of Hermiticity of H. To find the normalization, and to show that ϕ_k^+ and $\tilde{\phi}_{-k}^-$ are also orthogonal, we calculate

$$\begin{aligned} \int dx \tilde{\phi}_k(x)\phi_{k'}(x) = &\int_0^\infty dx \left[(1+\frac{iv}{2k})e^{-ikx} - \frac{iv}{2k}e^{ikx}\right] e^{ik'x} \\ &+ \int_{-\infty}^0 dx e^{-ikx} \left[(1+\frac{iv}{2k'})e^{ik'x} - \frac{iv}{2k'}e^{ik'x}\right]. \end{aligned} \tag{13}$$

By a rearrangement of terms, we can turn this into

$$\left(1+\frac{iv}{2k}\right) \int_{-\infty}^{\infty} dx e^{i(k'-k)x} + \frac{iv}{2kk'} \left[(k'-k)\int_0^\infty dx e^{i(k'-k)x} - (k+k')\int_0^\infty dx e^{i(k'+k)x}\right]. \tag{14}$$

The square bracket vanishes and from the first term (11) follows.

Now imagine evolving $\psi_0(x) = \delta(x - x_0)$, with $x_0 > 0$, and $\dot{\psi}_0 = 0$, forward in time. The result is

$$\psi(t, x > 0) = \frac{1}{4\pi} \int \frac{dk}{1 + \frac{iv}{2k}} \left[\left(1 + \frac{iv}{2k}\right) e^{-ikx_0} - \frac{iv}{2k} e^{ikx_0} \right] e^{ikx} \left(e^{i|k|t} + e^{-i|k|t}\right) . \tag{15}$$

The last factor can be rewritten as $e^{ikt} + e^{-ikt}$, and we obtain (using the residue theorem)

$$\psi(t, x > 0) = \frac{1}{2}\delta(t - (x - x_0)) + \frac{1}{2}\delta(t + (x - x_0)) - \frac{v}{4} e^{-\frac{v}{2}(t-(x+x_0))} \theta(t - (x + x_0)). \tag{16}$$

The first two terms are just the free propagation; for $x > x_0$ we could eliminate it by using a left-moving ψ_0, rather than the one at rest. The last term is the quasinormal mode. It is the non-normalizable eigenstate of H with out-going boundary conditions at both $x \to \pm\infty$:

$$\phi_q(x) = \begin{cases} e^{i\omega_q x} & x > 0 \\ e^{-i\omega_q x} & x < 0 \end{cases} , \quad \text{with} \quad \omega_q = -\frac{iv}{2} . \tag{17}$$

It captures the effect of propagation in a nontrivial potential, and it is seen only once the initial excitation reaches the barrier at $x = 0$ and gets reflected. (For later use, let me mention that had we placed the potential at $x = a < x_0$ rather than the origin, the quasinormal contribution would arrive starting at $t = x + x_0 - 2a$.)

Hence, we are naturally led to the notion of quasinormal modes as *the non-normalizable eigenstates of the Hamiltonian with appropriate boundary conditions, showing up in the time-evolution problem as poles in the lower-half frequency plane*. I'll say more about this.

Bound States

At this point it would be useful to also discuss the bound states, which are normalizable eigenstates. For instance, in the above example, there would be one bound state if $v < 0$. Unlike Schrödinger equation, whose bound states are surrounding us, the bound states of Klein-Gordon equation have $\omega^2 < 0$, and are spatially localized tachyonic instabilities. But they do not violate causality.

What changes with respect to (15) is that (1) the pole is moved to the upper half plane, and (2) there is extra contribution from the bound state:

$$-\frac{v}{4}\left[e^{\frac{v}{2}(t+x+x_0)} + e^{\frac{v}{2}(-t+x+x_0)}\right] . \tag{18}$$

The pole contributes (now via a counter-clockwise contour, hence the opposite sign) is

$$\frac{v}{4}\left[e^{\frac{v}{2}(t+x+x_0)}\theta(x+x_0+t)+e^{\frac{v}{2}(-t+x+x_0)}\theta(x+x_0-t)\right], \tag{19}$$

which nicely cancels the bound state contribution for $t < x + x_0$, and ensures causality.

2 Initial value as a source

For a generic $V(x)$, finding the complete set $\tilde{\phi}_n$ that satisfy (6) and finding the normalization c_n coefficients can be challenging. Because of the loss of translation-invariance plane waves are not solutions and inner products are nontrivial. On the other hand, time-translation is unbroken and it is natural to use Fourier-transform in time to transform the system into a set of decoupled ODEs, and represent the initial value as a source. For this purpose, we extend the solution to negative t by writing

$$\psi(t,x)=\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dx' G_R(t-t',x,x')J(t',x') \tag{20}$$

where G_R is the retarded Green's function and the following source term is added to the r.h.s. of (1),

$$J(t,x)=\delta(t)\dot{\psi}_0(x)+\dot{\delta}(t)\psi_0(x). \tag{21}$$

This source, implements the initial conditions at $t = 0$. To solve this problem in Fourier space, note that

$$G(\omega,x,x')\equiv\int dt e^{i\omega t}G_R(t,x,x') \tag{22}$$

is analytic in the upper half plane, $\text{Im}\omega > 0$, since $G_R(t < 0, x, x') = 0$ and under the assumption that $G_R(t, x, x')$ does not grow exponentially at late times. In case it does (like when there are bound states), $G(\omega)$ is analytic for $\text{Im}\omega > \Gamma$ where Γ is the largest growth rate.

Therefore, one needs to solve

$$(-\omega^2-\nabla^2+V(x))G(\omega,x,x')=\delta(x-x'), \tag{23}$$

with the boundary condition

$$G(\omega,x\to\infty,x')\propto e^{i\omega x},\qquad G(\omega,x\to-\infty,x')\propto e^{-i\omega t}, \tag{24}$$

which ensures that for $|x| \gg t, x'$ the contour of ω integration in the inverse Fourier transform,

$$G_R(t, x, x') = \int_{i\Gamma^+ - \infty}^{i\Gamma^+ + \infty} \frac{d\omega}{2\pi} e^{-i\omega t} G(\omega, x, x'), \tag{25}$$

can be closed in the upper-half plane, where $G(\omega, x, x')$ is analytic, consistent with the requirement that G_R must vanish outside of the light-cone. Hence, the choice of the boundary condition (24) must be thought of as being equivalent to the choice of the retarded Green's function.

The solution to this problem is

$$G(\omega, x, x') = \frac{1}{W(\omega)} \begin{cases} \phi^+_\omega(x)\phi^-_\omega(x') & x \geq x' \\ \phi^+_\omega(x')\phi^-_\omega(x) & x < x' \end{cases} \tag{26}$$

where the two solutions $\phi_\pm$ satisfy

$$\phi^\pm_\omega(x \to \pm\infty) = e^{\pm i\omega x}, \tag{27}$$

and $W(\omega)$ is the Wronskian of the two solutions (which is x-independent)

$$W(\omega) = \phi^+_\omega(x) \frac{d\phi^-_\omega(x)}{dx} - \phi^-_\omega(x) \frac{d\phi^+_\omega(x)}{dx}. \tag{28}$$

The advantage of this method compared to the standard method of expanding in a complete basis of the Hamiltonian (called Sturm-Liouville problem) is now apparent: $\phi^\pm_\omega$ are uniquely fixed by their boundary condition (27), and once they are known, finding the Wronskian is much easier than finding the inner product.[1] For $x \to \infty$

$$\phi^-_\omega(x) = \frac{1}{T(\omega)} e^{-i\omega x} + \frac{R(\omega)}{T(\omega)} e^{i\omega x} \tag{29}$$

where $T(\omega)$ and $R(\omega)$ are respectively the transmission and reflection coefficients of the left-moving wave. Hence,

$$W(\omega) = \frac{-2i\omega}{T(\omega)}. \tag{30}$$

The quasinormal mode frequencies $\{\omega_n\}$ are the poles of $G(\omega, x, x')$, or zeros of the Wronskian, in the lower-half plane of ω. By residue theorem these complex frequencies characterize the late-time behavior of the solution. The vanishing of the Wronskian implies that the two solutions ϕ^+_ω and ϕ^-_ω are linearly dependent at $\omega = \omega_n$, and thus we have an eigenstate

[1] It would be useful to generalize this recipe to the cases where $V(x \to -\infty) \neq 0$.

of H with outgoing boundary condition at both $x \to \pm\infty$. This is a restrictive boundary condition, so the spectrum is quantized. Since Im $\omega < 0$, the modes are non-normalizable (otherwise they would be bound states).

Let's return to the simple example of a delta-function potential. Incidentally, the right-moving and left-moving solutions are just ϕ_k and $\tilde{\phi}_k$, found above, but with $k \to \omega$. This gives

$$W(\omega) = -2i\omega \left(1 + \frac{iv}{2\omega}\right), \tag{31}$$

and for the localized initial condition, we obtain

$$\psi(t, x > x_0) = \partial_t G_R(t, x, x_0) = \int_{-\infty}^{\infty} \frac{d\omega}{4\pi(1 + iv/2\omega)} e^{-i\omega(t-x)} \left[\left(1 + \frac{iv}{2\omega}\right) e^{-i\omega x_0} - \frac{iv}{2\omega} e^{i\omega x_0}\right]. \tag{32}$$

This reproduces (16). Note that the quasi-normal mode shows up as the zero of the Wronskian and it turns on at $t = x + x_0$.

It would also be instructive to consider the $v < 0$ case, where there is a bound state. Unlike the Sturm-Liouville formulation where the bound state has to be included in the basis, the Green's function knows about the bound state only through the modification of the continuum. Now there is a pole in the upper-half plane and a corresponding exponentially growing mode with rate $|v|/2$. Therefore, the integration contour in (32) has to be shifted above Im $\omega = i|v|/2$, which guarantees the vanishing of the result at early times. For $t > x + x_0$, the contour is closed in the lower-half plane and the pole at $\omega = -iv/2$ reproduces the contribution of the bound state.

Exercise – It is known that modifying black hole horizon can drastically change the spectrum of quasinormal modes [1602.07309]. But if the modification is close enough to the horizon the ringdown signal remains almost unaltered, and well described by the quasinormal modes of the original GR solution. The purpose of this exercise is to see how the original quasinormal modes emerge in the sum over the new modes [1807.04843].

Consider as a 1d toy model the propagation of an initial perturbation at $x_0 > a_1$ in the following double-peak potential

$$V(x) = v\delta(x - a_1) + v\delta(x - a_2). \tag{33}$$

1. Show that the spectrum of quasinormal modes is divided into even and odd modes with respect to reflection around $x = (a_1 - a_2)/2$. Show that the even and odd quasinormal

mode frequencies satisfy, respectively,

$$i + \tan(\omega(a_1 - a_2)/2) = \frac{v}{\omega}, \qquad \text{even} \tag{34}$$

$$i - \cot(\omega(a_1 - a_2)/2) = \frac{v}{\omega}, \qquad \text{odd.} \tag{35}$$

Verify that in the limit $v(a_1 - a_2) \ll 1$ one recovers the quasinormal mode of a single delta-function potential with strength $2v$, i.e. $\omega = -iv$, with the rest of the spectrum at $|\text{Im}\ \omega| = \mathcal{O}(1/(a_1 - a_2)) \gg v$.

In what follows, the regime of interest is $v(a_1 - a_2) \gg 1$. Show that in this regime, the real parts of the solutions are approximately at

$$\text{Re}\ \omega_n \simeq \frac{n\pi}{a_1 - a_2}, \qquad n \in \mathbb{Z}, \tag{36}$$

and the imaginary parts are $|\text{Im}\ \omega_n| \sim 1/(a_1 - a_2) \ll v$.

2. According to the above recipe for finding the retarded Green's function, we need to find the left-moving solution at large positive x. Show that

$$\phi^-_\omega(x \gg a_1) = \frac{R}{T} e^{i\omega x} + \frac{1}{T} e^{-i\omega x} \tag{37}$$

where R/T and $1/T$ are related to R_1, R_2 and T_1, T_2 (the reflection and transmission coefficients from the first and the second barriers) by

$$\frac{R}{T} = \left(\frac{R_1}{T_1 T_2} e^{-2i\omega a_1} + \frac{R_2}{\bar{T}_1 T_2} e^{-2i\omega a_2} \right) \tag{38}$$

and

$$\frac{1}{T} = \left(\frac{1}{T_1 T_2} + \frac{\bar{R}_1 R_2}{\bar{T}_1 T_2} e^{2i\omega(a_1 - a_2)} \right), \tag{39}$$

where $\bar{R}_1(\omega) = R_1(-\omega)$, and $\bar{T}_1(\omega) = T_1(-\omega)$. Show that in our explicit example

$$R_1 = R_2 = \frac{-iv/2\omega}{1 + iv/2\omega}, \qquad T_1 = T_2 = \frac{1}{1 + iv/2\omega}. \tag{40}$$

3. Show that for a left-moving initial condition at x_0 only the right-moving part of ϕ^-_ω is excited, giving

$$\psi(t, x > x_0) = \int d\omega \frac{\left(R_1 e^{-2i\omega a_1} + \frac{T_1}{T_1^*} R_2 e^{-2i\omega a_2} \right)}{-4\pi[1 + T_1 R_2 (\bar{R}_1/\bar{T}_1) \exp(2i\omega(a_1 - a_2))]} e^{-i\omega(t - (x + x_0))}. \tag{41}$$

In the intermediate times $x + x_0 - 2a_1 < t < x + x_0 - 2a_2$ the integration contour in the second term can be closed in the upper half plane where the Green's function is analytic, and thus it gives zero. In contrast, the contour is closed in the lower-half plane in the first term, picking up the residues of the poles. These are the quasinormal modes of the double-peak potential.

4. Show that the imaginary parts satisfy:

$$\text{Im }\omega_n \simeq \frac{1}{4(a_1 - a_2)} \log(|R_1(\omega_n) R_2(\omega_n)|^2), \tag{42}$$

which has a simple interpretation, as the rate at which a wave-packet that is trapped between the two peaks leaks outside.

5. To calculate the residue of the pole, show that we can approximate (when $v(a_1 - a_2) \gg 1$)

$$\frac{d}{d\omega}[1 + T_1 R_2(\bar{R}_1/\bar{T}_1)\exp(2i\omega(a_1 - a_2))]\Big|_{\omega_n} \simeq -2i(a_1 - a_2). \tag{43}$$

Thus, the early signal is

$$\psi_{\text{early}}(t, x > x_0) \simeq \sum_n \frac{-iR_1(\omega_n)}{4i(a_1 - a_2)} e^{i\omega_n(t-(x+x_0-2a_1))}. \tag{44}$$

Neglecting the small imaginary part of ω_n, the sum over the equally spaced Re ω_n can be approximated by an integral for the early time-scales to give

$$\psi_{\text{early}}(t, x > x_0) \simeq \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} R_1 e^{-i\omega(t-(x+x_0-2a_1))}. \tag{45}$$

This is the solution of a single peak potential at a_1, which in our toy example has a quasinormal mode at $\omega = -iv/2$.