Refined curve counting and tropical geometry

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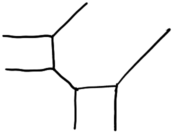
Lecture 3: Floor diagrams, Feynman diagrams and Fock space

plane tropical curve of degree d:
piecewise linear graph Γ immersed in $\mathbb{R}^2$ s.t.

1. the edges e of Γ have rational slope
2. they have weight $w(e) \in \mathbb{Z}_{>0}$
3. **balancing condition:**
   let $p(e)$ primitive integer vector in direction of e;
   for all vertices v of Γ:
   $$\sum_{e \text{ at } v} p(e)w(e) = 0.$$
4. Γ has d unbounded edges in each of the directions $(-1,-1)$, $(1,0)$, $(0,1)$



conic

A **lattice polygon** Δ in $\mathbb{R}^2$ is a polygon with vertices with integer coordinates

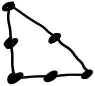
To a convex lattice polygon Δ one can associate a pair $(S(\Delta), L(\Delta))$ of a toric surface and a toric line bundle on S

S is defined by the fan given by the outer normal vectors of Δ

$h^0(S, L) = \#(\Delta \cap \mathbb{Z}^2)$, arithmetic genus $\#int(\Delta \cap \mathbb{Z}^2)$

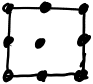
Examples:

1. $(\mathbb{P}^2, \mathcal{O}(d))$



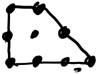
$(\mathbb{P}^2, \mathcal{O}(2))$

2. $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(d_1, d_2))$



$(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,2))$

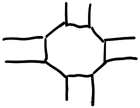
3. Hirzebruch surface $\Sigma_m = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(m))$ on $\mathbb{P}^1$
   F fibre, E section with $E^2 = -m$, $H := E + mF$
   $L := (dH + nF)$



$(\Sigma_1, 2H + F)$

plane tropical curve of degree Δ:
piecewise linear graph Γ immersed in $\mathbb{R}^2$ s.t.

1. the edges e of Γ have rational slope
2. they have weight $w(e) \in \mathbb{Z}_{>0}$
3. **balancing condition:**
   let $p(e)$ primitive integer vector in direction of e;
   for all vertices v of Γ:
   $$\sum_{e \text{ at } v} p(e)w(e) = 0.$$
4. For every edge of Δ (of lattice length n) Γ has n unbounded edges in corresponding outer normal direction



$\mathbb{P}^2$, $O(2,2)$

Known: through $\#(\Delta \cap \mathbb{Z}^2) - 1 - \delta$ general points in $\mathbb{R}^2$, there are finitely many δ-nodal degree Δ tropical curves, all simple
(Simple tropical curves are in particular trivalent)
Count these curves with certain multiplicities
Always use the same principle: for every (trivalent) vertex v of a simple tropical curve Γ define a **vertex multiplicity** $u(v)$.
The multiplicity of Γ is $u(\Gamma) = \prod_{v \text{ vertex}} u(v)$ and the corresponding curve count is

$$u_{\Delta,\delta} := \sum_{\Gamma} u(\Gamma)$$

(sum over all δ-nodal, degree Δ tropical curves through $\#(\Delta \cap \mathbb{Z}^2) - 1 - \delta$ general points in $\mathbb{R}^2$)

Tropical Severi degree $n^{trop}_{\Delta,\delta}$: define vertex multiplicity as

$$m(v) := w(e_1)w(e_2)|\det(p(e_1), p(e_2))|, \qquad m(\Gamma) = \prod_{v \text{ vertex}} m(v)$$

Tropical Welschinger invariants $W^{trop}_{\Delta,\delta}$: define vertex multiplicity as $\omega(v) := \begin{cases} (-1)^{(m(v)-1)/2} & m(v) \text{ odd} \\ 0 & m(v) \text{ even} \end{cases}$

Refined Severi degree $N^{trop}_{\Delta,\delta}(y)$: define vertex multiplicity as $M(v) := [m(v)]_y$ with **quantum number:** $[n]_y := \frac{y^{n/2}-y^{-n/2}}{y^{1/2}-y^{-1/2}}$

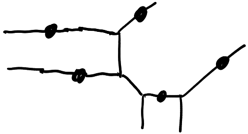
By definition $N^{trop}_{\Delta,\delta}(1) = n^{trop}_{\Delta,\delta} = n_{(S(\Delta),L(\Delta)),\delta}$,

$N^{trop}_{\Delta,\delta}(-1) = W^{trop}_{\Delta,\delta} = W_{(S(\Delta),L(\Delta)),\delta}(P)$

$N^{trop}_{\Delta,\delta}(y)$ is a tropical invariant, i.e. independent of the position of the points.

Vague definition: A configuration $p_1 = (x_1, y_1), \ldots, p_n = (x_n, y_n)$ of points in $\mathbb{R}^2$ is called **vertically stretched** if
$min_{i \neq j}(|y_i - y_j|) \gg max(|x_i - x_j|)$
(Points are stretched out close to vertical line)
Consider them ordered by vertical component
$y_1 < y_2 < \ldots < y_n$
(we turn the drawings so that vertical becomes horizontal)

Let Γ be a δ-nodal degree Δ tropical curve through a vertically stretched configuration of $\#(\Delta \cap \mathbb{Z}) - 1 - \delta$ points. Then Γ has a special shape: a floor decomposition:

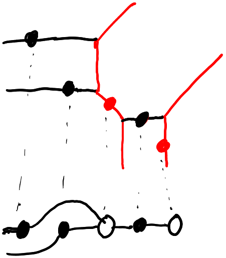


A vertical edge of C is called an **escalator**
A connected component of closure of complement of escalators in Γ is called a **floor**.
The following properties hold:

1. Every floor and every escalator contains precisely one marked point.
2. Only the escalators can have weights different from 1
3. any vertex v has multiplicity $m(v) = 1$, unless it is adjacent to am escalator e, in which case the multiplicity is $m(v) = w(e)$.

To Γ tropical curve through horizontally stretched conf. of points associate marked floor diagram.

escalators: horizontal segments of Γ
floors: conn. comp. of complem. of escalators. One marked point on every floor and escalator
Floor diagram: black vertex for escalator white vertex for floor



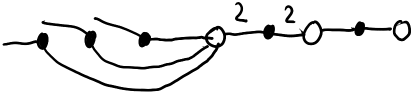
connect if escalator connects to floor
keep weight of escalator

To count tropical curves we can just count floor diagrams

Description of floor diagrams

1. Every bounded edge connects a black and a white vertex
2. Every unbounded edge connects to a black vertex
3. every black vertex is connected to two edges, one incoming (i.e. from left), one outgoing, both of the same weight.
4. white vertices v can have several incoming and outgoing edges the divergence is
$div(v) = \sum_{e-incoming} w(e) - \sum_{e-outgoing} w(e)$.

A floor diagram is δ-nodal of degree d
(i.e. a floor diagrams of δ-nodal tropical curve of degree d) if it has d incoming edges of weight 1, no outgoing edges
$d(d+3)/2 - \delta$ vertices of which d are white of divergence 1



1- nodal curve
of degree 3

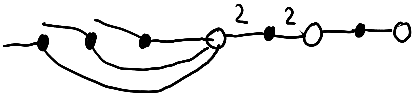
Counting of floor diagrams

Put $m(\Lambda) := \prod_{e \text{ edges}} [w(e)]_y$

By definition:

Proposition

$$N^{trop}_{d,\delta}(y) = \sum_{\delta\text{-nodal floor diagrams } \Lambda \text{ of degree } d} m(\Lambda)$$



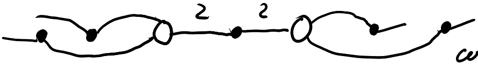
$m(\Lambda) = (y+2+y^{-1})$.

more generally can define δ-nodal floor diagrams of degree Δ for Δ an h-transversal lattice polytope
i.e. slopes of the edges of Δ are 0, ∞ or $1/m$, for m integer

For simplicity deal with $\mathbb{P}^1 \times \mathbb{P}^1$ and Hirzebruch surfaces Σ_m

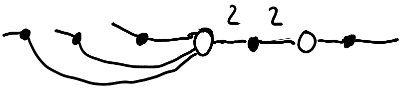
Floor diagrams

$(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(d_1, d_2))$: Λ has d_1 incoming edges and d_1 outgoing edges of weight 1
$(d_1+1)(d_2+1) - 1 - \delta$ vertices of which d_2 are white of divergence 0



1-nodal curve of degree (2,2)

$(\Sigma_m, dH + nF)$, $H^2 = m$, F fibre. Λ has $n + dm$ incoming edges and n outgoing edges of weight 1
$(d+1)(n+1) + m\binom{d}{2} - 1 - \delta$ vertices of which d are white of divergence m.



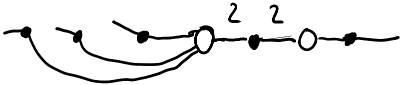
$(\Sigma_1, 2H+F)$, $\delta = 1$

Counting of floor diagrams

Put $m(\Lambda) := \prod_{e \text{ edges}} [w(e)]_y$

Proposition

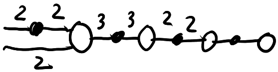
$$N^{trop}_{\Delta,\delta}(y) = \sum_{\delta\text{-nodal floor diagrams } \Lambda \text{ of degree } d} m(\Lambda)$$



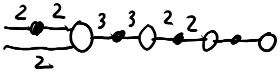
$m(\Lambda) = (y + 2 + y^{-1})$

Floor diagrams with contacts: Can use floor diagrams to prove versions of Caporaso-Harris recursion
For simplicity restrict to case of $\mathbb{P}^2$

Let $\alpha = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$, $\beta = (1^{\beta_1}, 2^{\beta_2}, \ldots)$ partitions
A floor diagram is δ-nodal of degree d, with contact α, β, if has α_i incoming edges of weight i connected to white vertices for all i, β_i incoming edges of weight i conn. to black vertices
no outgoing edges
$d(d+3)/2 - \delta - \|\alpha\| - \|\beta\| + |\beta|$ vertices of which d are white of divergence 1
$(\|\alpha\| = \sum_i i\alpha_i,\ |\alpha| = \sum_i \alpha_i)$



$d=4,\ \delta=3,\ \alpha=(2),\ \beta=(2)$



Counting of floor diagrams

Put $m(\Lambda) := \prod_{e \text{ edges}} [w(e)]_y$

bounded

$[2]_y^3 \, [3]_y^2 = (y^{1/2}+y^{-1/2})(y+2+y^{-1})(y^2+2y+3+2y^{-1}+y^{-2})$

Proposition

$$N^{trop}_{d,\delta}(\alpha,\beta)(y) = \sum_{\delta\text{-nodal floor diagrams } \Lambda \text{ of degree } d, \text{ contact } \alpha,\beta} m(\Lambda)$$

Caporaso-Harris recursion: Remove one by one the vertices from the right.

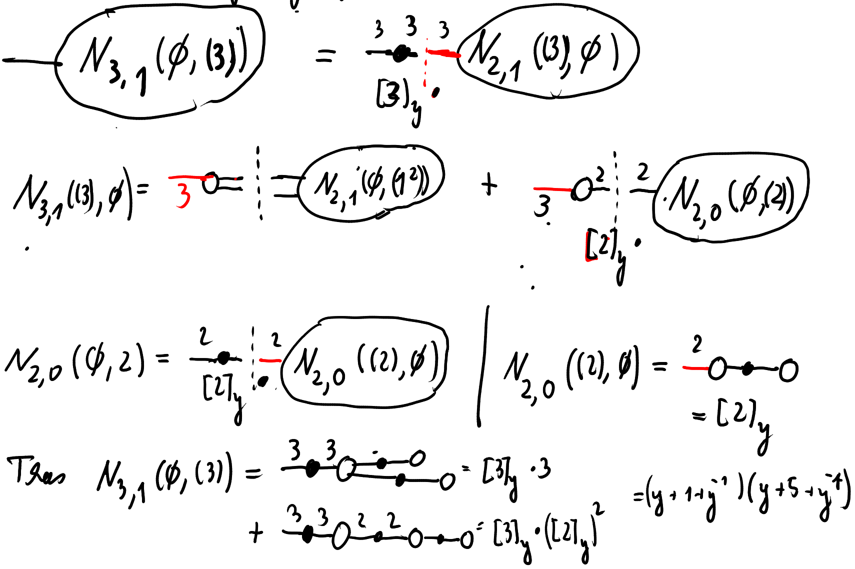
$$N_{d,\delta}(\alpha,\beta)(y) = \sum_{k|\beta_k>0} [k]_y N_{d,\delta}(\alpha+e_k,\beta-e_k)(y)$$
$$+ \sum_{\beta',\alpha',\delta'} \prod_i ([y]_i)^{\beta'_i-\beta_i} \binom{\alpha}{\alpha'}\binom{\beta'}{\beta} N_{d-1,\delta'}(\alpha',\beta')(y)$$

$\alpha = (\alpha_1,\alpha_2,\ldots)$, $\beta = (\beta_1,\beta_2,\ldots)$, $\binom{\alpha}{\beta} = \prod_i \binom{\alpha_i}{\beta_i}$,
$e_k = (0,\ldots,0,1,0,\ldots)$: 1 in position k.
(first sum: remove black vertex,
second sum: remove white vertex)
Second sum is over all $\alpha' \le \alpha$, $\beta' \ge \beta$ and δ' such that

$$\|\alpha'\| + \|\beta'\| = d-1, \quad \delta' = \delta - (d-1) + |\beta'| - |\beta|$$

$y = 1$ gives Caporaso-Harris recursion,
$y = -1$ gives recursion for Welschinger invariants.

Mark: fixed edges (going to white vertices) red.



H deformed Heisenberg algebra gen. by $a_n, b_n, \quad n \in \mathbb{Z}$
$a_{-n},\ b_{-n}$ with $n > 0$ are called **creation operators**
$a_n,\ b_n$ with $n > 0$ are called **annihilation operators**
commutation relations

$$[a_n, a_m] = 0 = [b_n, b_m], \quad [a_n, b_m] = [n]_y \delta_{n,-m}, \quad [n]_y = \frac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}}$$

Fock space: F generated by **creation operators** a_{-n}, b_{-n}
acting on vacuum vector $v_\emptyset$
elements of F are $f\, v_\emptyset$, where f is a polynomial
(with coefficients in $y^{\pm 1/2}$ in the a_{-n}, b_{-n})
H-module by $a_n v_\emptyset := 0$, $b_n v_\emptyset := 0$ for $n \geq 0$
(concatenate and apply commutation relations) e.g.
$a_2(a_{-1}a_{-2}v_\emptyset) = a_{-1}(a_{-2}a_2 + [2]_y a_{-1})v_\emptyset = (y^{1/2} + y^{-1/2})a_{-1}v_\emptyset.$

Basis paramtr. by pairs of partitions

$\mu = (1^{\mu_1}, 2^{\mu_2}, \ldots)$, $\nu = (1^{\nu_1}, 2^{\nu_2}, \ldots)$
$a_\mu := \prod_i \frac{a_i^{\mu_i}}{\mu_i!}$, $a_{-\mu} := \prod_i \frac{a_{-i}^{\mu_i}}{\mu_i!}$, similarly for b_ν, $b_{-\nu}$
$v_{\mu,\nu} := a_{-\mu} b_{-\nu} v_\emptyset$ basis for F

inner product $\langle v_\emptyset | v_\emptyset \rangle = 1$; a_n, b_n adjoint to a_{-n}, b_{-n}

Explicitely $\langle v_{\mu,\nu} | v_{\mu',\nu'} \rangle = \left(\prod_i \frac{([i]_y)^{\mu_i}}{\mu_i!} \right) \left(\prod_i \frac{([i]_y)^{\nu_i}}{\nu_i!} \right) \delta_{\mu,\nu'} \delta_{\nu,\mu'}$.

Expression for refined Severi degrees in terms of Heisenberg algebra:
Case of $\mathbb{P}^2$:

$$H(t) := \sum_{k>0} b_k b_{-k} + t \sum_{\|\mu\|=\|\nu\|-1} a_\nu a_{-\mu}$$

$\|\mu\| := \sum_i i\mu_i;$ sum includes $\mu = \emptyset$

Theorem

$$N^{trop}_{d,\delta}(y) = \langle v_{(1^d),\emptyset} | \mathrm{Coeff}_{t^d} H(t)^{d(d+3)/2-\delta} v_\emptyset \rangle$$

Generating function

$$\sum_{d\geq 0} \sum_{\delta\geq 0} \frac{t^d q^{d(d+3)/2-\delta}}{(d(d+3)/2-\delta)!} N^{trop}_{d,\delta}(y) = \langle v_\emptyset | \exp(qH(t)) \exp(a_{-1}) v_\emptyset \rangle$$

Case of Hirzebruch surface Σ_m:

$$H_m(t) := \sum_{k>0} b_k b_{-k} + t \sum_{\|\mu\|=\|\nu\|-m} a_\nu a_{-\mu}$$

$\|\mu\| := \sum_i i\mu_i;$ sum includes $\mu = \emptyset$

Theorem

$$N^{trop}_{(\Sigma,dH+nF),\delta}(y) = \langle v_{(1^{dm+n}),\emptyset} | \mathrm{Coeff}_{t^d} H_m(t)^{(d+1)(\frac{d}{2}+n+1)-1-\delta} v_{(1^n),\emptyset} \rangle$$

Idea of proof: Feynman diagrams = floor diagrams

Feynman diagrams: To monomial M in $b_k b_{-k}$, $a_\nu a_{-\mu}$ and inner product $\langle v_{(1^n),\emptyset} | M v_{(1^l),\emptyset} \rangle$ associate diagrams:

- black vertex for $b_k b_{-k}$, with one incoming and one outgoing edge of weight k
- white vertex for $a_\nu a_{-\mu}$, with weights of incoming vertices given by ν weight of outgoing vertices by μ. e.g. for $a_{(1^2,2)} a_{-(1^3)}$ 
- write vertices in order they are in the monomial
- connect the vertices, all half edges are connected except for n incoming edges of weight 1 and l outgoing vertices of weight 1
- edges connect only vertices of different colour, and the weights match

$(b_1 b_{-1})^2 a_{(1^2)} a_{-1} b_1 b_{-1} a_1$ 

- connect the vertices, all half edges are connected except for n incoming edges of weight 1 and l outgoing vertices of weight 1
- edges connect only vertices of different colour, and the weights match

count the diagrams for M with multiplicity $m(\Gamma) := \prod_{e \text{ edges}} [w(e)]_y$.

Proposition (Wicks Theorem)

$$\langle v_{(1^n),\emptyset} | M v_{(1^l),\emptyset} \rangle = \sum_{\Gamma \text{ Graphs for } M} m(\Gamma)$$

Idea of proof of proposition: Can write $v_{(1^n),\emptyset} = a_{-(1^n)} v_\emptyset$, thus
$\langle v_{(1^n),\emptyset} | M v_{(1^l),\emptyset}\rangle = \langle v_\emptyset | a_{(1^n)} M\, a_{-(1^l)} v_\emptyset\rangle$.
This allows to reduce to case $n = l = 0$.
(1) Now let N be any monomial in the a_i, b_j, $i, j \in \mathbb{Z}_{\neq 0}$
Assign diagrams to $\langle v_\emptyset | N v_\emptyset\rangle$.

- For each a_i put white vertex, with incoming edge of weight i (if $i > 0$) and outgoing edge of weight $-i$ (if $i < 0$)
- For each b_i put black vertex, with incoming edge of weight i (if $i > 0$) and outgoing edge of weight $-i$ (if $i < 0$)
- a diagram for $\langle v_\emptyset | N v_\emptyset\rangle$ is a diagram with these vertices, such that the total diagram has no incoming and no outgoing edges.

Count these diagrams with multiplicity $m(\Gamma) := \prod_{e \text{ edges}} [w(e)]_y$.
Remark: $\langle v_\emptyset | N v_\emptyset\rangle = \sum_{\Gamma \text{ Graphs for } N} m(\Gamma)$

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Compute by applying commutation relations to move annihilation operators a_n, b_n, $n > 0$ to the right, and creation operators to left
We get many summands. They are only nonzero where none of the creation and annihilation operators survive
We get e.g. $U a_n b_{-n} V = U b_{-n} a_n V + [n]_y UV$
do not connect the vertices for b_n, a_n for the first summand, connect them for the second summand
We get a nonzero result only if every vertex is connected to a other one

Let $M = m_1 \cdots m_l$ monomial in the $(a_{-\mu}a_\nu)$, $(b_{-k}b_k)$, b_{-k}
Assume M contains factors $a_{-\mu^s}a_{\nu^s}$ for $s = 1, \ldots, n$
Then $M = \frac{1}{\prod_{s=1}^n \mu^s!\nu^s!}N$, where N is obtained from M by replacing the $a_{\nu^s}a_{-\mu^s}$ by $\left(\prod_j a_j^{\nu_j^s}\right)\left(\prod_j a_{-j}^{\mu_j^s}\right)$.

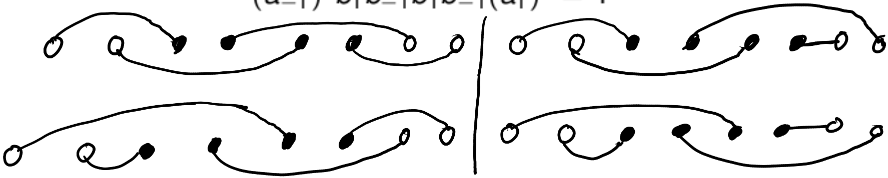
The Feynman diagrams for M are obtained from the diagrams for N by

- replacing all vertices corresponding to each $(a_{\nu^s}a_{-\mu^s})$ by one white vertex
- replacing the two vertices corresponding to $b_{-k}b_k$ by one black vertex

This maps $\prod_{s=1}^n \mu^s!\nu^s!$ graphs corresponding to the reorderings of the factors in each $\prod_j(a_{-j})^{\mu_j^s} \cdot \prod_j(a_j)^{\nu_j^s}$ to equivalent Feynman diagrams for M

Idea of proof: Feynman diagrams = floor diagrams

$$(a_{-1})^2 b_1 b_{-1} b_1 b_{-1} (a_1)^2 = 4$$



$$a_{(-1^2)} b_1 b_{-1} b_1 b_{-1} a_{(1^2)} = 1$$

Claim: floor diagrams = Feynman diagrams Do this just in case of $\mathbb{P}^2$.

Recall: $H(t) := \sum_{k>0} b_k b_{-k} + t \sum_{\|\mu\|=\|\nu\|-1} a_\nu a_{-\mu}$

We claimed: $N^{trop}_{d,\delta}(y) = \langle v_{(1^d),\emptyset} | \mathrm{Coeff}_{t^d} H(t)^{d(d+3)/2-\delta} v_\emptyset \rangle$

Corresponding Feynman diagrams:

- $d(d+3)/2 - \delta$-vertices (every factor $H(t)$ adds one vertex)
- of which d are white (the $a_\nu a_{-\mu}$ go with the t in $H(t)$
- all the white vertices have divergence 1 (because of condition $\|\mu\| = \|\nu\| - 1$).
- We have d incoming edges and no outgoing edges of weight 1 (because we have $\langle v_{(1^d),\emptyset}|$)
- edges connect vertices of different colour, and the weights match

This precisely was our description of the δ-nodal floor diagrams of degree d.

A floor diagram is δ-nodal of degree d
(i.e. a floor diagrams of δ-nodal tropical curve of degree d) if it has d incoming edges, of weight 1, no incoming edges
$d(d+3)/2-\delta$ vertices of which d are white of divergence 1