Refined curve counting and tropical geometry

Lothar Göttsche

Lecture 2: Refined Severi degrees

Severi degree: $n_{d,\delta} = \#\Big\{\delta - \text{nodal, degree } d \text{ curves through } (d+3)d/2 - \delta \text{ gen. points}\Big\}$

General surface: S proj. alg. surface, L line bundle on S

$$|L| = \{C = Z(s) \mid s \text{ section of } L\} = \mathbb{P}^{h^0(L)-1}$$

$\mathbb{P}^\delta \subset |L|$ general δ-dimensional linear subspace

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Assume L is sufficiently ample with respect to δ.
$S^{[n]}$ =Hilbert scheme of points on S
$\mathcal{C} = \{(p,[C])|p \in C\} \subset S \times \mathbb{P}^\delta$ universal curve
$\mathcal{C}^{[n]} = \{([Z],[C])|Z \subset C\} \subset S^{[n]} \times \mathbb{P}^\delta$ relative Hilbert scheme
χ_{-y}-genus $\chi_{-y}(X) = \sum_{p,q}(-1)^{p+q}h^{p,q}(X)y^q$

Write $\sum_{n\geq 0} \chi_{-y}(\mathcal{C}^{[n]})t^n = \sum_{l\geq 0}^{\delta} N_l^{\mathcal{C}}(y)t^l((1-t)(1-yt))^{g(L)-l-1}$
$g(L)$ =genus of smooth curve in $|L|$
Refined invariants $N^{(S,L),\delta}(y) := N_\delta^{\mathcal{C}}(y)/y^\delta$. By definition $N^{(S,L),\delta}(1) = n_{(S,L),\delta}$.

Conjecture

$$\sum_{\delta\geq 0} N^{(S,L),\delta}(y)t^{\delta} = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{\chi(\mathcal{O}_S)}.$$

for universal power series $A_1, A_2, A_3, A_4 \in \mathbb{Q}[y^{\pm 1}][[t]]$
A_1 *and* A_4 *are expressed in terms of modular forms.*

Theorem

The conjecture is true if

1. *S is an abelian or a K3 surface*
2. *modulo* t^{11} *for all surfaces*

Know: $N^{(S,L),\delta}(1) = n_{(S,L),\delta}$ for L sufficiently ample
What is the meaning at other values of y?

What do the refined invariants count?

The claim is that for toric surfaces this has to do with real algebraic geometry and tropical geometry

Welschinger invariants:

Let S real algebraic surface; complex conj. τ maps S to S
real algebraic curve $=$ curve C such that $\tau(C) = C$
Real locus of C: $C^{\mathbb{R}} = C^{\tau}$
P configuration of dim $|L| - \delta$ real points of S
Welschinger invariants: $W_{(S,L),\delta}(P) = \sum_{C} (-1)^{s(C)}$

sum is over all real δ-nodal curves C in $|L|$ though P
$s(C) = \#\{\text{isolated nodes of } C\}$ i.e local equation x^2+y^2 .

$s(C)=3$

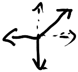
These invariants depend in general on the point configuration, via walls and chambers. These are in general not the actual Welschinger invariants, which are deformation invariants, but we use the name by abuse of notation.

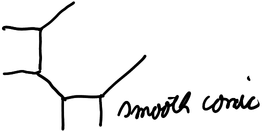
What I say below is for toric surfaces, for simplicity restrict to $\mathbb{P}^2$

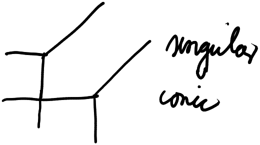
Tropical geometry: piecewise linear version of algebraic geometry

Real and complex algebraic curves can be counted by counting piecewise linear objects: the tropical curves

plane tropical curve of degree d:
piecewise linear graph Γ immersed in $\mathbb{R}^2$ s.t.

1. the edges e of Γ have rational slope
2. they have weight $w(e) \in \mathbb{Z}_{>0}$
3. **balancing condition:**
   let $p(e)$ primitive integer vector in direction of e;
   for all vertices v of Γ:
   $$\sum_{e \text{ at } v} p(e)w(e) = 0.$$
   
4. Γ has d unbounded edges in each of the directions $(-1,-1)$, $(1,0)$, $(0,1)$

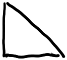




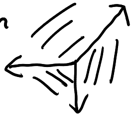
A **lattice polygon** Δ in $\mathbb{R}^2$ is a polygon with vertices with integer coordinates
To a convex lattice polygon Δ one can associate a pair $(S(\Delta), L(\Delta))$ of a toric surface and a toric line bundle on S

S is defined by the fan given by the outer normal vectors of Δ
$h^0(S, L) = \#(\Delta \cap \mathbb{Z}^2)$, arithmetic genus $\#int(\Delta \cap \mathbb{Z}^2)$

$\mathbb{P}^2$

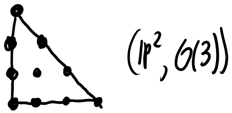


Fan

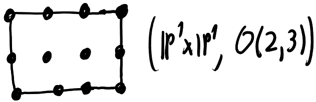


Examples:

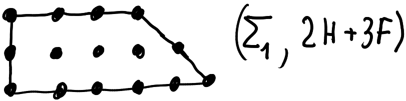
1. $(\mathbb{P}^2, \mathcal{O}(d))$



2. $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(d_1, d_2))$

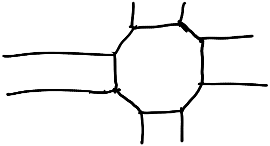


3. Hirzebruch surface $\Sigma_m = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(m))$ on $\mathbb{P}^1$
   F fibre, E section with $E^2 = -m$, $H := E + mF$
   $L := (dH + nF)$



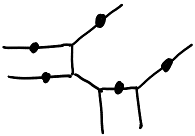
plane tropical curve of degree Δ:
piecewise linear graph Γ immersed in $\mathbb{R}^2$ s.t.

1. the edges e of Γ have rational slope
2. they have weight $w(e) \in \mathbb{Z}_{>0}$
3. **balancing condition:**
   let $p(e)$ primitive integer vector in direction of e;
   for all vertices v of Γ:
   $$\sum_{e \text{ at } v} p(e)w(e) = 0.$$
4. For every edge of Δ (of lattice length n) Γ has n unbounded edges in corresponding outer normal direction



curve of degree (2,2) on $\mathbb{P}^1 \times \mathbb{P}^1$

There is a notion of simple tropical curve
(these can be thought of as analogues of nodal curves)
Simple tropical curves are in particular trivalent
The **genus** $g(\Gamma)$ of a tropical curve $\Gamma \to \mathbb{R}^2$ is $h^1(\Gamma) - h^0(\Gamma) + 1$
The number of nodes of a simple tropical curve of degree Δ is $\#int(\Delta) - g(\Gamma)$
Known: through $\#(\Delta \cap \mathbb{Z}^2) - 1 - \delta$ general points in $\mathbb{R}^2$, there are finitely many δ-nodal degree Δ tropical curves, all simple



smooth conic through 5 gen. points

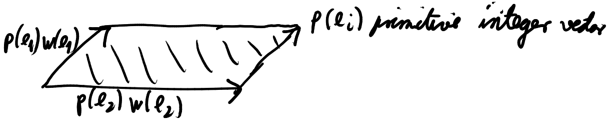
Count these curves with certain multiplicities
Always use the same principle:
for every (trivalent) vertex v of a simple tropical curve Γ define a **vertex multiplicity** $u(v)$.
The multiplicity of Γ is $u(\Gamma) = \prod_{v \text{ vertex}} u(v)$ and the corresponding curve count is

$$u(\Delta, \delta) := \sum_{\Gamma} u(\Gamma)$$

(sum over all δ-nodal, degree Δ tropical curves through $\#(\Delta \cap \mathbb{Z}^2) - 1 - \delta$ general points in $\mathbb{R}^2$)

Tropical Severi degree: Let Γ simple tropical curve, v vertex, e_1, e_2, e_3 edges at v, define vertex multiplicity as

$$m(v) := w(e_1)w(e_2)|\det(p(e_1), p(e_2))|, \qquad m(\Gamma) = \prod_{v \text{ vertex}} m(v)$$



Tropical Severi degree: $n^{trop}_{\Delta,\delta} := \sum_{\Gamma} m(\Gamma)$

sum over all δ-nodal, degree Δ tropical curves through $\#(\Delta \cap \mathbb{Z}^2) - 1 - \delta$ general points in $\mathbb{R}^2$.

Let Γ simple tropical curve, v vertex

$$\omega(v) := \begin{cases} (-1)^{(m(v)-1)/2} & m(v) \text{ odd} \\ 0 & m(v) \text{ even} \end{cases}$$

$$\omega(\Gamma) = \prod_{v \text{ vertex}} \omega(v)$$

Tropical Welschinger inv.: $W^{trop}_{\Delta,\delta} := \sum_{\Gamma} \omega(\Gamma)$

sum over all δ-nodal, degree Δ tropical curves through $\#(\Delta \cap \mathbb{Z}^2) - 1 - \delta$ general points in $\mathbb{R}^2$.

Mikhalkin: The Severi degree is equal to the tropical Severi degree and the Welschinger invariants are equal to the tropical Welschinger invariants.

$$n_{d,\delta} = n^{trop}_{d,\delta}, \qquad n_{d,\delta} = n^{trop}_{d,\delta}$$

$$W_{d,\delta}(P) = W^{trop}_{d,\delta}, \qquad W_{(S(\Delta),L(\Delta)),\delta}(P) = W^{trop}_{\Delta,\delta}$$

(the second for suitable P)
We know, for $L(\Delta)$ sufficiently ample $N^{\Delta,\delta}(1) = n_{S(\Delta),L(\Delta)),\delta}$

Conjecture

For $d \geq \delta/3 + 1$ $N^{d,\delta}(-1) = W^{trop}_{d,\delta}$

Is there a tropical invariant $\mathrm{N}^{trop}_{d,\delta}(y)$, that interpolates between Severi degree and Welschinger invariant?

quantum number: $[n]_y := \frac{y^{n/2}-y^{-n/2}}{y^{1/2}-y^{-1/2}}$ $= y^{(n-1)/2} + y^{(n-3)/2} + \dots + y^{-(n-1)/2}$

By definition $[n]_1 = n$, $[n]_{-1} = \begin{cases} (-1)^{(n-1)/2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

Let Γ simple tropical curve, v vertex

$$M(v) := [m(v)]_y, \qquad M(\Gamma) = \prod_{v \text{ vertex}} M(v)$$

Refined Severi degree: $N^{trop}_{d,\delta}(y) := \sum_\Gamma M(\Gamma)$, sum as above

By definition $N^{trop}_{d,\delta}(1) = n^{trop}_{d,\delta} = n_{d,\delta}$,

$N^{trop}_{d,\delta}(-1) = W^{trop}_{d,\delta} = W_{d,\delta}(P)$

Itenberg-Mikhalkin: $N^{trop}_{d,\delta}(y)$ is a tropical invariant, i.e. independent of the position of the points.

Conjecture

For $d \geq \delta/2 + 1$, $N^{d,\delta}(y) = N^{trop}_{d,\delta}(y)$

The above conjectures specialize to

Conjecture

1. $N^{trop}_{d,\delta}(y) = N^{d,\delta}(y)$ *for* $d \geq \delta$
2. $N^{d,\delta}(y)$ *is a polynomial in* $d, y^{\pm 1}$
3. $$\sum_{\delta \geq 0} N^{d,\delta}(y) t^{\delta} = B_1^{d^2} B_2^d B_3$$

   for universal power series $B_i \in \mathbb{Q}[y^{\pm 1}][[t]]$. (given before)

Theorem

1. *There exist* **refined node polynomials** $N_\delta(d,y) \in \mathbb{Q}[d, y^{\pm 1}]$ *with* $N_\delta(d,y) = N^{trop}_{d,\delta}(y)$ *for* $d \geq \delta$.
2. $N_\delta(d,y) = N^{d,\delta}(y)$, *for* $\delta \leq 10$.

Theorem

$$\sum_{\delta \geq 0} N_\delta(d,y) t^\delta = \overline{B}_1^{d^2} \overline{B}_2^{d} \overline{B}_3$$

for power series $\overline{B}_i \in \mathbb{Q}[y^{\pm 1}][[t]]$.

Rest of the conjecture is $B_i = \overline{B}_i$ for $i = 1, 2, 3$.

Note that in particular the specialization to $y = -1$ gives a corresponding result for the Welschinger invariants.

The method to prove this is floor diagrams and combinatorics
These have been introduced by Brugallé and Mikhalkin to study the $n^{trop}_{d,\delta}$, and used e.g. by Block, Liu, to prove the analogue of the above results for $n^{trop}_{d,\delta}$