Refined curve counting and tropical geometry

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Lecture 1: Curve counting and refined curve counting

Aim: count (singular) curves on algebraic surfaces

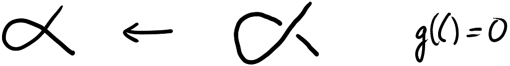
What does this mean?

$C \subset \mathbb{P}^n$ projective curve over $\mathbb{C}$

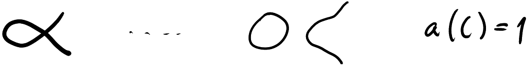
If C is smooth, $g(C) = \text{genus} = \#\text{handles}$

If C singular,

$g(C) = \text{geometric genus} = \text{genus of normalization.}$

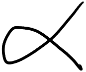


$a(C) > g(C)$ genus of smooth deformation



Simplest singularity = node =transversal self intersection

$a(C) - g(C) = 1$



Severi degrees: count nodal curves in $\mathbb{P}^2$
C plane curve of degree d
$C = Z(F) \subset \mathbb{P}^2$, $F \in \mathbb{C}[x_0, x_1, x_2]$ homog. of degree d

$$\{\text{curves of degree d in } \mathbb{P}^2\} = \mathbb{P}^{(d+3)d/2}.$$

A node imposes one condition on curves of degree d
passing through a gen. point imposes one condition
Severi degree: $n_{d,\delta} = \#\Big\{\delta - \text{nodal, degree } d \text{ curves in } \mathbb{P}^2$

$$\text{through } (d+3)d/2 - \delta \text{ gen. points}\Big\}$$

(same as number of curves of genus $\binom{d-1}{2} - \delta$)
$n_{d,0} = 1$, $n_{d,1} = 3(d-1)^2$ (Steiner 1848).

General surface: S proj. alg. surface, L line bundle on S

$$|L| = \{ C = Z(s) \mid s \text{ section of } L \} = \mathbb{P}^{h^0(L)-1}$$

$\mathbb{P}^\delta \subset |L|$ general δ-dimensional linear subspace

Severi degree: $n_{(S,L),\delta} := \#\{\delta\text{-nodal curves in } \mathbb{P}^\delta \subset |L|\}$

There are different ways to count curves in varieties
Usually there are several steps:

1. Find the correct compact moduli space M, parametrizing the curves and the degenerations one wants to allow.
2. On wants to count curves satisfying certain conditions, this will be an intersection number on M.
3. Often one needs to use a virtual fundamental class and count "virtual numbers of curves".

(0) Severi degrees: curves are elements of $|\mathcal{O}(d)|$ or $|L|$
Count curves with given genus through correct number of points as points in proj. space $\mathbb{P}^{h^0(L)-1}$

(1) Gromov-Witten invariants: Count maps $f: C \to X$.
Look at moduli space

$$M_g(X,\beta) = \{(C,f) \text{ stable map } f: C \to X,\ C \text{ nodal curve of genus } g,\ f_*([C]) = \beta \in H^2(X,\mathbb{Z})\}$$

(2) Pandharipande-Thomas invariants:
These count possibly degenerate curves in $C \subset X$, by counting their structure sheaves on $\mathcal{O}_X \to \mathcal{O}_C$

P-T moduli space:

$$P_n(X,\beta) := \Big\{(F,s) \,\Big|\, F \text{ pure 1-dimensional sheaf on } X,\ s: \mathcal{O}_X \to F \text{ section},\ dim(coker(s)) = 0,\ c_2(F) = \beta, \chi(F) = n\Big\}$$

Conjectural PT-GW correspondence:
PT and GW invariants conjectured equivalent (generating functions related by explicit change of variables)

Why are the Severi degrees interesting?
(1) Classical old question
(2) Rel. to other invariants and moduli spaces
(Pandharipande-Thomas-invariants, Gromov-Witten-inv)
(3) Relation to physics (string theory).

String theory also gives refined invariants.
$n_{(S,L),\delta}$ should be Euler numbers of some moduli space M
The refined invariants something like Betti numbers.

Severi degree: $n_{d,\delta} = \#\Big\{\delta - \text{nodal, degree } d \text{ curves in } \mathbb{P}^2$
$\text{through } (d+3)d/2 - \delta \text{ gen. points}\Big\}$
$n_{(S,L),\delta} := \#\{\delta\text{-nodal curves in } \mathbb{P}^\delta \subset |L|\}$
Are there closed formulas for the $n_{(S,L),\delta}$ in terms of the "topological data" L^2, LK_S K_S^2, $c_2(S)$? It is easy to see that this cannot be true without conditions: L must be sufficiently ample with respect to δ. Formulas for $n_{(S,L),\delta}$ as polynomial in $L^2, LK_S, K_S^2, c_2(S)$ computed by Avritzer-Vainsencher for $\delta \leq 6$ and by Kleimann-Piene for larger δ.

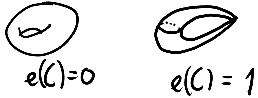
Conjecture (G'97)

1. *There exists a universal polyn.* $n^{(S,L),\delta}$ *in* $L^2, LK_S, K_S^2, c_2(S)$ *computing* $n_{(S,L),\delta}$ *for L sufficiently ample with respect to* δ *(more below)*
2. $$\sum_{\delta\geq 0} n^{(S,L),\delta} t^\delta = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{\chi(\mathcal{O}_S)}.$$
   for universal power series $A_1, A_2, A_3, A_4 \in \mathbb{Q}[[t]]$
   (in particular $\sum_{\delta\geq 0} n_{d,\delta} t^\delta \equiv A_1^{d^2} A_2^{-3d} A_3^9 A_4$. *modulo* $t^{(2d-2)}$*)*

Proven by Tzeng, Kool-Shende-Thomas
KST obtain $n^{(S,L),\delta}$ from generating function of Euler numbers of Hilbert schemes of points on the universal curve $\mathcal{C}/\mathbb{P}^\delta$
This Hilbert scheme is a Pandharipande-Thomas moduli space so $n^{(S,L),\delta}$ is closely related to PT-invariants

Example: 1-nodal plane curves of degree d
$\mathcal{C} \to \mathbb{P}^1$ total space general pencil of curves of degree d in $\mathbb{P}^2$
contains finite number of 1-nodal curves
Euler number of smooth curve of degree d is $3d - d^2$.
a node increases Euler number by 1. Thus

$$e(\mathcal{C}) = 2(3d - d^2) + n_{d,1}$$



$\mathcal{C}$ is blowup of $\mathbb{P}^2$ at d^2 intersection points of $Z(F)$ and $Z(G)$
$F(x_0, x_1, x_2)$, $G(x_1, x_2, x_2)$ gen. polynomials of degree d.
Euler number $e(\mathcal{C}) = 3 + d^2$. Thus

$$n_{d,1} = 3 + d^2 + 2(d^2 - 3d) = 3(d-1)^2$$

To make argument work for larger number of nodes replace universal curve by relative Hilbert scheme of points

X projective variety. Hilbert scheme $X^{[n]}$ of n points on X parametrizes zero dimensional subschemes of length n on X, i.e. generically sets of n points on X.
On a smooth curve C a subscheme of length n is a set of n points counted with multiplicity.
If S is a smooth projective surface, then $S^{[n]}$ is smooth projective variety of dimension $2n$.
For any line bundle $L \in Pic(S)$ have a tautological vector bundle $L^{[n]}$ of rank n on $S^{[n]}$ with fibre $L^{[n]}([Z]) = H^0(L|_Z)$.

Recall $L \in Pic(S)$ suff. ample (e.g .δ-very ample, sufficient: $\mathcal{C}^{[n]}$ below is smooth),
$\mathbb{P}^\delta \subset |L|$ general linear subspace
$\mathcal{C} := \{(p,[C]) \in S \times \mathbb{P}^\delta \mid p \in C\}$ universal curve
(fibred over $\mathbb{P}^\delta$ with fibre C over $[C]$)
$\mathcal{C}^{[n]} := \{([Z],[C]) \in S^{[n]} \times \mathbb{P}^\delta | Z \subset C\}$ rel. Hilbert scheme
(fibred over $\mathbb{P}^\delta$ with fibre $C^{[n]}$ over $[C]$)

KST show:

1. $\exists_{n_l^{\mathcal{C}} \in \mathbb{Z}}$, $l = 0, \ldots, \delta$ s.th.
$$\sum_{n \geq 0} e(\mathcal{C}^{[n]}) q^n = \sum_{l=0}^{\delta} n_l^{\mathcal{C}} t^l (1-t)^{2g(L)-2l-2}.$$
$g(L)$ genus of nonsingular curve in $|L|$
$$e(X) = \sum_{i=0}^{2dim(X)} (-1)^i rk(H^i(X,\mathbb{Z}))$$ topological Euler number
2. $n_{(S,L),\delta} = n_\delta^{\mathcal{C}}$.

Why does this prove the conjecture?
Assume L is δ-very ample (or just $\mathcal{C}^{[n]}$ is smooth) .
$e(\mathcal{C}^{[n]})$ is tautological intersection number on $S^{[n]}$:
$L^{[n]}$ tautological vector bundle on $S^{[n]}$, $L^{[n]}([Z]) = H^0(L|_Z)$.
Let H pullback of $\mathcal{O}(1)$ from $\mathbb{P}^\delta$.
$L^{[n]} \boxtimes H$ has section s with zero set $Z(s) = \mathcal{C}^{[n]} \subset S^{[n]} \times \mathbb{P}^\delta$
This allows to compute $e(\mathcal{C}^{[n]})$ as intersection number on $S^{[n]}$:

$$e(\mathcal{C}^{[n]}) = \int_{S^{[n]} \times \mathbb{P}^\delta} \frac{c(T_{S^{[n]}}) c_n(L^{[n]} \boxtimes H)}{c(L^{[n]} \boxtimes H)}$$

($c(E) = 1 + c_1(E) + \ldots + c_{rk(E)}(E)$ Chern class).

$$e(\mathcal{C}^{[n]}) = \int_{S^{[n]}\times\mathbb{P}^\delta} \frac{c(T_{S^{[n]}})c_n(L^{[n]}\boxtimes H)}{c(L^{[n]}\boxtimes H)}$$

($c(E) = 1 + c_1(E) + \ldots + c_{rk(E)}(E)$ Chern class).
Ellingsrud-G-Lehn: such "tautological" integrals are always given by universal polynomials in $L^2, LK_S, K_S^2, c_2(S)$. By

$$\sum_{n\geq 0} e(\mathcal{C}^{[n]})q^n = \sum_{l=0}^{\delta} n_l^{\mathcal{C}} t^l (1-t)^{2g(L)-2l-2},$$

this gives for L sufficiently ample that the number $n_\delta^{\mathcal{C}}$ is a universal polynomial in $L^2, LK_S, K_S^2, c_2(S)$. Denote this polynomial by $n^{(S,L),\delta}$, for all (S,L), then $n^{(S,L),\delta} = n_{(S,L),\delta}$ for L sufficiently ample.

Give refinement replacing Euler number by χ_{-y}-genus

$S^{[n]}$ =Hilbert scheme of points on S
$\mathcal{C} = \{(p, [C]) | p \in C\} \subset S \times \mathbb{P}^\delta$ universal curve
$\mathcal{C}^{[n]} = \{([Z], [C]) | Z \subset C\} \subset S^{[n]} \times \mathbb{P}^\delta$ relative Hilbert scheme
χ_{-y}-genus $\chi_{-y}(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X) y^q$

Write

$$\sum_{n \geq 0} \chi_{-y}(\mathcal{C}^{[n]}) t^n = \sum_{l \geq 0}^{\infty} N_l^{\mathcal{C}}(y) t^l ((1-t)(1-yt))^{g(L)-l-1}$$

$g(L)$ =genus of smooth curve in $|L|$
By similar argument to above $N_\delta^{\mathcal{C}}(y)$ is universal polynomial in $L^2, LK_S, K_S^2, c_2(S)$ if L is δ-very ample.
Refined invariants: $N^{(S,L),\delta}(y)$ polynomial in $L^2, LK_S, K_S^2, c_2(S)$ s.th. $\mathrm{N}^{(S,L),\delta}(y) := N_\delta^{\mathcal{C}}(y)/y^\delta$ for L sufficiently ample.

Replaced Euler number by χ_{-y}-genus (combin. of Hodge numbers) obtain refined invariants $N^{(S,L),\delta}(y) \in \mathbb{Z}[y, y^{-1}]$, with $N^{(S,L),\delta}(1) = n^{(S,L),\delta}$
Denote them $N^{d,\delta}(y)$ in the case of $\mathbb{P}^2$

Conjecture

$$\sum_{\delta \geq 0} N^{(S,L),\delta}(y) t^\delta = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{\chi(\mathcal{O}_S)}.$$

for universal power series $A_1, A_2, A_3, A_4 \in \mathbb{Q}[y^{\pm 1}][[t]]$
A_1 and A_4 are expressed in terms of modular forms.

Generating function for refined invariants $N^{(S,L),\delta}(y)$. $D := q\frac{q}{dq}$

$\Delta(y,q) = q\prod_{n\geq 1}(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2$,

$\widetilde{DG}_2(y,q) = \sum_{n\geq 1} q^n\Big(\sum_{d|n}\frac{n}{d}\frac{(y^{d/2}-y^{-d/2})^2}{(y^{1/2}-y^{-1/2})^2}\Big)$

$B_1(y,q) = 1 - q - (y+3+y^{-1})q^2 + \ldots$,

$B_2(y,q) = 1 + (y+3+y^{-1})q + (y^2+y^{-2})q^2 + \ldots$

Conjecture

$$\sum_\delta \overline{N}^{(S,L),\delta}(y)(\widetilde{DG}_2(y,q))^\delta = \frac{(\widetilde{DG}_2(y,q)/q)^{\chi(L)}B_1(y,q)^{K_X^2}B_2(y,q)^{LK_X}}{\big(\Delta(y,q)\,D\widetilde{DG}_2(y,q)/q^2\big)^{\chi(\mathcal{O}_X)/2}}$$

Putting $y=1$ recovers the old conjecture

Reformulation without variable change: $D := q\frac{q}{dq}$

$\Delta(y,q) = q\prod_{n\geq 1}(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2$,

$\widetilde{DG}_2(y,q) = \sum_{n\geq 1} q^n\Big(\sum_{d|n}\frac{n}{d}\frac{(y^{d/2}-y^{-d/2})^2}{(y^{1/2}-y^{-1/2})^2}\Big)$

$B_1(y,q) = 1 - q - (y+3+y^{-1})q^2 + \ldots$,

$B_2(y,q) = 1 + (y+3+y^{-1})q + (y^2+y^{-2})q^2 + \ldots$

$r := \chi(L) - 1 - \delta$ number of point conditions for δ-nodal curves in $|L|$ (i.e. count δ-nodal curves through r gen. points on S). Then

$$N^{(S,L),\delta}(y) = \mathrm{Coeff}_{q^{(L^2-LK_S)/2}}\left[\widetilde{DG}_2(y,q))^r\frac{D\widetilde{DG}_2(y,q)B_1(y,q)^{K_X^2}B_2(y,q)^{LK_X}}{(\Delta(y,q)\,D\widetilde{DG}_2(y,q))^{\chi(\mathcal{O}_X)/2}}\right]$$

Residue formula:

$$f(q) = \sum_{\ell\geq 0} g(q)^\ell\,\mathrm{Coeff}_{q^0}\left(\frac{f(q)\,Dg(q)}{g(q)^{\ell+1}}\right)$$

Theorem

The conjecture is true in the following cases:

1. *For $y = 1$ (this is the old conjecture which was proved).*
2. *S is an abelian or a K3 surface*
3. *modulo q^{11} for all surfaces (i.e. for up to 11 nodes)*

Check of Conjecture: $\chi_{-y}(\mathcal{C}^{[n]})$ computed by very similar integral on $S^{[n]}$ as $e(\mathcal{C}^{[n]})$.
EGL: coeff. of $\chi_{-y}(\mathcal{C}^{[n]})$ are univ. polyn. in $L^2, LK_S, K_S^2, c_2(S)$.
$\Longrightarrow$ determined by values for
$(S, L) = (\mathbb{P}^2, \mathcal{O}), (\mathbb{P}^2, \mathcal{O}(1)), (\mathbb{P}^2, \mathcal{O}(-1)), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O})$.
These are toric surface: action of $T = \mathbb{C}^* \times \mathbb{C}^*$ on S with finitely many fixpoints. Action lifts to $S^{[n]}$ with finitely many fixpoints $p_1, \ldots, p_e$.
Bott Residue formula: Integral for $\chi_{-y}(\mathcal{C}^{[n]})$ on $S^{[n]}$ can be computed in terms of the weights of action of T on the fibres $T_{S^{[n]}}(p_i)$, $L^{[n]}(p_i)$. Programmed on computer:
Result: Conjecture is true modulo q^{11}.

Case of a K3 surface: Let S $K3$-surface, e.g. quartic in $\mathbb{P}^3$, L primitive line bundle on S
Write $M^S_{L^2/2,k}(y) := N^{(S,L),g(L)-k}(y)$ (genus k curves in $|L|$ through k points).

Conjecture says: $\sum_{d\geq 0} M^S_{d,k}(y)q^d = \dfrac{\widetilde{DG}_2(y,q)^k}{\Delta(y,q)}$

Case of an abelian surface Let A abelian surface, L primitive line bundle on A Write
$M^A_{L^2/2,k}(y) := N^{(A,L),g(L)-k-2}(y)$ (genus $k+2$ curves in $|L|$ through k points)

Conjecture says: $\sum_{d\geq 0} \widetilde{M}^A_{d,k}(y)q^d = \widetilde{DG}_2(y,q)^k D\widetilde{DG}_2(y,q)$

By the multiplicativity of the generating functions,
it is enough to show these for $k = 0$. In this case the relative Hilbert scheme is isomorphic to a moduli space of pairs, parametrising sheaves of dimension 0 with one section.
In the case of K3 surfaces this was studied by Kawai and Yoshioka, giving the formula.
A similar, a bit more complicated argument gives the result for abelian surfaces.

Know: $N^{(S,L),\delta}(1) = n_{(S,L),\delta}$ for L sufficiently ample
What is the meaning at other values of y?

What do the refined invariants count?

The claim is that for toric surfaces this has to do with real algebraic geometry and tropical geometry