Refined curve counting and tropical geometry

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(in order of appearance)

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Aim: count (singular) curves on algebraic surfaces

What does this mean?

$C \subset \mathbb{P}^n$ projective curve over $\mathbb{C}$

If C is smooth, $g(C) = \text{genus} = \#\text{handles}$

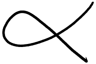
If C singular,

$g(C) = \text{geometric genus} = \text{genus of normalization.}$

$a(C) > g(C)$ genus of smooth deformation

Simplest singularity = node =transversal self intersection

$a(C) - g(C) = 1$



Severi degrees: count nodal curves in $\mathbb{P}^2$
C plane curve of degree d
$C = Z(F) \subset \mathbb{P}^2$, $F \in \mathbb{C}[x_0, x_1, x_2]$ homog. of degree d

$$\{\text{curves of degree d}\} = \mathbb{P}^{(d+3)d/2}.$$

A node imposes one condition on curves of degree d
passing through a gen. point imposes one condition
Severi degree: $n_{d,\delta} = \#\Big\{\delta - \text{nodal, degree } d \text{ curves}$

$$\text{through } (d+3)d/2 - \delta \text{ gen. points}\Big\}$$

(same as number of curves of genus $\binom{d-1}{2} - \delta$)
$n_{d,0} = 1$, $n_{d,1} = 3(d-1)^2$ (Steiner 1848).

General surface: S proj. alg. surface, L line bundle on S

$$|L| = \{ C = Z(s) \mid s \text{ section of } L \} = \mathbb{P}^{h^0(L)-1}$$

$\mathbb{P}^\delta \subset |L|$ general δ-dimensional linear subspace

Severi degree: $n_{(S,L),\delta} := \#\{\delta\text{-nodal curves in } \mathbb{P}^\delta\}$

Why are these interesting?

(1) Classical old question

(2) Rel. to other invariants and moduli spaces

(Pandharipande-Thomas-invariants Gromov-Witten-inv)

(3) Relation to physics (string theory).

String theory also gives refined invariants.

$n_{(S,L),\delta}$ should be Euler numbers of some moduli space M

The refined invariants something like Betti numbers.

Conjecture ('97)

1. *There exists a universal polyn.* $n_\delta^{(S,L)}$ *in* $L^2, LK_S, K_S^2, c_2(S)$ *computing* $n_{(S,L),\delta}$ *for L sufficiently ample*
2. $$\sum_{\delta\geq 0} n_\delta^{(S,L)} t^\delta = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{\chi(\mathcal{O}_S)}.$$
   for universal power series $A_1, A_2, A_3, A_4 \in \mathbb{Q}[[t]]$
   (in particular $\sum_{\delta\geq 0} n_{d,\delta} t^\delta \equiv A_1^{d^2} A_2^{-3d} A_3^9 A_4$. *modulo* t^δ*)*

Proven by Tzeng, Kool-Shende-Thomas
KST obtain $n_\delta^{(S,L)}$ from generating function of Euler numbers of Hilbert schemes of points on the universal curve $\mathcal{C}/\mathbb{P}^\delta$
This Hilbert scheme is a Pandharipande-Thomas moduli space so $n_\delta^{(S,L)}$ is closely related to PT-invariants

Give refinement replacing Euler number by χ_{-y}-genus
$S^{[n]}$ =Hilbert scheme of points on S
$\mathcal{C} = \{(p,[C]) | p \in C\} \subset S \times \mathbb{P}^\delta$ universal curve
$\mathcal{C}^{[n]} = \{([Z],[C]) | Z \subset C\} \subset S^{[n]} \times \mathbb{P}^\delta$ relative Hilbert scheme
χ_{-y}-genus $\chi_{-y}(X) = \sum_{p,q}(-1)^{p+q} h^{p,q}(X) y^q$

Write

$$\sum_{n\geq 0} \chi_{-y}(\mathcal{C}^{[n]}) t^n = \sum_{l\geq 0}^{\delta} N_l^{\mathcal{C}}(y) t^l ((1-t)(1-yt))^{g(L)-l-1}$$

$g(L)$ =genus of smooth curve in $|L|$
Refined invariants $N_\delta^{(S,L)}(y) := N_\delta^{\mathcal{C}}(y)$.

Replaced Euler number by χ_{-y}-genus (combin. of Hodge numbers) obtain refined invariants $N_\delta^{(S,L)}(y) \in \mathbb{Z}[y, y^{-1}]$, with $N_\delta^{(S,L)}(1) = n_\delta^{(S,L)}$
Denote them $N_\delta^d(y)$ in the case of $\mathbb{P}^2$

Conjecture

$$\sum_{\delta \geq 0} N_\delta^{(S,L)}(y) t^\delta = A_1^{L^2} A_2^{LK_S} A_3^{K_S^2} A_4^{\chi(\mathcal{O}_S)}.$$

for universal power series $A_1, A_2, A_3, A_4 \in \mathbb{Q}[y^{\pm 1}][[t]]$
A_1 *and* A_4 *are expressed in terms of modular forms.*

Theorem

The conjecture is true if

1. S *is an abelian or a K3 surface*
2. *modulo* t^{11} *for all surfaces*

Know: $N_{\delta}^{(S,L)}(-1) = n_{(S,L),\delta}$ for L sufficiently ample
What is the meaning at other values of y?

What do the refined invariants count?

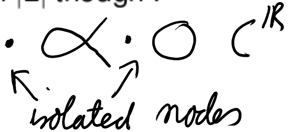
The claim is that for toric surfaces this has to do with real algebraic geometry and tropical geometry

Welschinger invariants:
Show: $N_\delta^{(S,L)}(y)$ related to real algebraic and tropical geometry

Let S real algebraic surface; complex conj. τ maps S to S
real algebraic curve = curve C such that $\tau(C) = C$
Real locus of C: $C^{\mathbb{R}} = C^{\tau}$
P configuration of dim $|L| - \delta$ real points of S
Welschinger invariants: $W_{(S,L),\delta}(P) = \sum_C (-1)^{s(C)}$

sum is over all real δ-nodal curves C in $|L|$ though P

$s(C) = \#\{\text{isolated nodes of } C\}$



What I say below is for toric surfaces, for simplicity restrict to $\mathbb{P}^2$

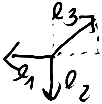
Tropical geometry: piecewise linear version of algebraic geometry

Real and complex algebraic curves can be counted by counting piecewise linear objects: the tropical curves

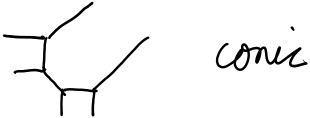
plane tropical curve of degree d:
piecewise linear graph Γ immersed in $\mathbb{R}^2$ s.t.

1. the edges e of Γ have rational slope
2. they have weight $w(e) \in \mathbb{Z}_{>0}$
3. **balancing condition:**
   let $p(e)$ primitive integer vector in direction of e;
   for all vertices v of Γ:

$$\sum_{e \text{ at } v} p(e)w(e) = 0.$$

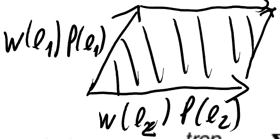


4. Γ has d unbounded edges in each of the directions $(-1,-1)$, $(1,0)$, $(0,1)$



There is notion of number of nodes of tropical curve Γ
A **simple** tropical curve is trivalent
Known: through $d(d+3)/2-\delta$ general points in $\mathbb{R}^2$, there are finitely many δ-nodal degree d tropical curves, all simple
Tropical Severi degree: Let Γ simple tropical curve, v vertex, e_1, e_2, e_3 edges at v

$$m(v) := w(e_1)w(e_2)|\det(p(e_1), p(e_2))|, \qquad m(\Gamma) = \prod_{v \text{ vertex}} m(v)$$



Tropical Severi degree: $n^{trop}_{d,\delta} := \sum_{\Gamma} m(\Gamma)$

sum over all δ-nodal, degree d tropical curves through $d(d+3)/2-\delta$ general points in $\mathbb{R}^2$.

Let Γ simple tropical curve, v vertex

$$\omega(v) := \begin{cases} (-1)^{(m(v)-1)/2} & m(v) \text{ odd} \\ 0 & m(v) \text{ even} \end{cases}$$

$$\omega(\Gamma) = \prod_{v \text{ vertex}} \omega(v)$$

Tropical Welschinger inv.: $W^{trop}_{d,\delta} := \sum_{\Gamma} \omega(\Gamma)$

sum over all δ-nodal, degree d tropical curves through $d(d+3)/2 - \delta$ general points in $\mathbb{R}^2$.

Mikhalkin: The Severi degree is equal to the tropical Severi degree and the Welschinger invariants are equal to the tropical Welschinger invariants.

$$n_{d,\delta} = n^{trop}_{d,\delta}$$

$$W_{d,\delta}(P) = W^{trop}_{d,\delta}$$

(the second for suitable P)
We know, for $d \geq \delta$ sufficiently ample $N^d_\delta(1) = n_{d,\delta}$

Conjecture

For $d \geq \delta/3 + 1$ $N^d_\delta(-1) = W^{trop}_{d,\delta}$

quantum number: $[n]_y := \frac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}}$

By definition $[n]_1 = n$, $[n]_{-1} = \begin{cases} (-1)^{(n-1)/2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

Let Γ simple tropical curve, v vertex

$$M(v) := [m(v)]_y, \qquad M(\Gamma) = \prod_{v \text{ vertex}} M(v)$$

Refined Severi degree: $N^{trop}_{d,\delta}(y) := \sum_\Gamma M(\Gamma)$ sum as above

By definition $N^{trop}_{d,\delta}(1) = n^{trop}_{d,\delta} = n_{d,\delta}$,
$N^{trop}_{d,\delta}(-1) = W^{trop}_{d,\delta} = W_{d,\delta}(P)$

Conjecture

For $d \geq \delta/2 + 1$, $N^d_\delta(y) = N^{trop}_{d,\delta}(y)$

The above conjectures specialize to

Conjecture

1. $N^{trop}_{d,\delta}(y) = N^d_\delta(y)$ *for* $d \geq \delta$
2. $N^d_\delta(y)$ *is a polynomial in* $d, y^{\pm 1}$
3. $$\sum_{\delta \geq 0} N^d_\delta(y) t^\delta = B_1^{d^2} B_2^d B_3$$

   for universal power series $B_i \in \mathbb{Q}[y^{\pm 1}][[t]]$.

Theorem

1. *There exist* **refined node polynomials** $N_\delta(d,y)\in\mathbb{Q}[d,y^{\pm 1}]$ *with* $N_\delta(d,y)=N^{trop}_{d,\delta}(y)$ *for* $d\geq\delta$.
2. $N_\delta(d,y)=N^d_\delta(y)$, *for* $\delta\leq 10$.

Theorem

$$\sum_{\delta\geq 0} N_\delta(d,y)t^\delta=\overline{B}_1^{d^2}\overline{B}_2^{d}\overline{B}_3$$

for power series $\overline{B}_i\in\mathbb{Q}[y^{\pm 1}][[t]]$.

Rest of the conjecture is $B_i=\overline{B}_i$ for $i=1,2,3$.

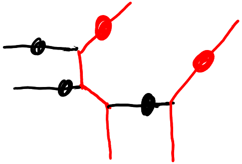
Note that in particular the specialization to $y=-1$ gives a corresponding result for the Welschinger invariants.

The method to prove this is floor diagrams and combinatorics
These have been introduced by Brugallé and Mikhalkin to study the $n^{trop}_{d,\delta}$, and used e.g. by Block, Liu, to prove the analogue of the above results for $n^{trop}_{d,\delta}$

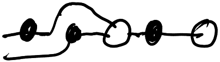
To Γ tropical curve through horizontally stretched conf. of points associate marked floor diagram.

escalators: horizontal segments of Γ

floors: conn. comp. of complem. of escalators. One marked point on every floor and escalator



Floor diagram: black vertex for escalator
white vertex for floor
connect if escalator connects to floor, keep weight



Put $m(\Lambda) := \prod_{e \text{ edges}} [w(e)]_y$

Proposition

$$N^{trop}_{d,\delta}(y) = \sum_{\Lambda \text{ floor diagrams}} m(\Lambda)$$

H deformed Heisenberg algebra gen. by $a_n, b_n, \quad n \in \mathbb{Z}$
$a_{-n},\ b_{-n}$ with $n > 0$ are called **creation operators**
$a_n,\ b_n$ with $n > 0$ are called **annihilation operators**
commutation relations

$$[a_n, a_m] = 0 = [b_n, b_m], \qquad [a_n, b_m] = [n]_y \delta_{n,-m}$$

Fock space: F generated by **creation operators** a_{-n}, b_{-n}
acting on vacuum vector $v_\emptyset$
H-module by $a_n v_\emptyset := 0$, $b_n v_\emptyset := 0$ for $n \geq 0$
(concatenate and apply commutation relations)

Basis paramtr. by pairs of partitions

$\mu = (1^{\mu_1}, 2^{\mu_2}, \ldots)$, $\nu = (1^{\nu_1}, 2^{\nu_2}, \ldots)$

$a_\mu := \prod_i \frac{a_i^{\mu_i}}{\mu_i!}$, $a_{-\mu} := \prod_i \frac{a_{-i}^{\mu_i}}{\mu_i!}$, similarly for b_ν, $b_{-\nu}$

$v_{\mu,\nu} := a_{-\mu} b_{-\nu} v_\emptyset$ basis for F

inner product $\langle v_\emptyset | v_\emptyset \rangle = 1$; a_n, b_n adjoint to a_{-n}, b_{-n}.

Expression for refined Severi degrees in terms of Heisenberg algebra:

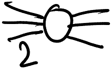
$$H(t) := \sum_{k>0} b_k b_{-k} + t \sum_{\|\mu\|=\|\nu\|-1} a_\nu a_{-\mu}$$

$\|\mu\| := \sum_i i\mu_i$; sum includes $\mu = \emptyset$

Theorem

$$N^{trop}_{d,\delta}(y) = \langle v_\emptyset | \mathrm{Coeff}_{t^d} H(t)^{d(d+3)/2-\delta} v_{(1^d),\emptyset} \rangle$$

Feynman diagrams: To each monomial M in the $b_k b_{-k}$, $a_\nu a_{-\mu}$ associate diagrams:

- for $b_k b_{-k}$ write  e.g. for $a_{(1^2,2)} a_{-(1^3)}$ 
- write vertices in order they are in the monomial
- connect all vertices so that edges connect only vertices of different colour, and the weights match
  $(b_1 b_{-1})^2 a_{(1^2)} a_{-1} b_1 b_{-1} a_1$

count the diagrams with multiplicity $m(\Gamma) := \prod_{e \text{ edges}} [w(e)]_y$.

Proposition (Wicks Theorem)

$$\langle v_\emptyset | M v_\emptyset \rangle = \sum_{\Gamma \text{ Graphs for } M} m(\Gamma)$$

Claim: floor diagrams = Feynman diagrams