Virtual topological invariants of moduli spaces of sheaves on surfaces

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(0) Introduction

We work over $\mathbb{C}$. Study topology of some moduli spaces.

What is a moduli space?

Algebraic variety M that parametrizes naturally certain interesting objects in algebraic geometry

"parametrizes": $M(\mathbb{C}) = \{\text{points of } M\} \overset{\text{bijection}}{\longleftrightarrow} \{\text{Objects in question}\}$

"naturally": If $\begin{matrix} \mathcal{E} \\ \downarrow \pi \\ T \end{matrix}$ family of objects parametrized by scheme T $\Longrightarrow$ map.

$$\begin{aligned} X &\longrightarrow M \\ x &\longmapsto [\pi^{-1}(x)] \end{aligned}$$

morphism of schemes.

__Example__: Hilbert scheme of points

$S^{[n]} = \mathcal{H}ilb^n S$ on a smooth surface S.

$S^{[n]} = \{ [Z] \subset S \mid$ zero dimensional subschemes of degree n on $S \}$

__General point__: $Z = \{p_1, \dots, p_n\} \subset S$

Points can come together

$[Z] \in S^{[n]}$ is given by ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_S$ s.th. $\mathcal{O}_Z = \mathcal{O}_S / \mathcal{I}_Z$ has finite support.

and $\deg(Z) = \dim_{\mathbb{C}} \mathcal{O}_Z = \sum_{p \in \mathrm{supp}(Z)} \dim_{\mathbb{C}} (\mathcal{O}_{Z,p}) = n.$

__Examples__: $S^{[0]} = pt = [\emptyset]$.

$S^{[1]} = S = \{ [p] \mid p \in S \}$.

Related to symmetric power.

$$S^{(n)} = \mathrm{Sym}^n(S) = S^n / \mathfrak{S}_n = \left\{ \sum n_i p_i \;\middle|\; \begin{array}{l} p_i \in S \text{ distinct} \\ \sum n_i = n \end{array} \right\}.$$

__Hilbert Chow morphism__:

$$\pi : S^{[n]} \longrightarrow S^{(n)} ; \; Z \longmapsto \mathrm{supp}(Z) = \sum_{p \in \mathrm{supp}(Z)} \dim(\mathcal{O}_{Z,p}) \cdot p.$$

Theorem (Fogarty): Let S smooth projective surface.

(a) $S^{[n]}$ is nonsingular projective of dim $2n$.

(b) $\pi : S^{[n]} \to S^{(n)}$ is resolution of singularities.

Topological invariants:

Let $e(X) = \sum_{i=0}^{\dim_{\mathbb{R}} X} (-1)^i \dim H^i(X, \mathbb{R})$ topological Euler number.

Theorem (G)
$$\sum_{n \geq 0} e(S^{[n]}) t^n = \frac{1}{\prod_{n>0} (1-t^n)^{e(S)}}$$
$$= 1 + e(S)t + \frac{1}{2}\left(e(S)^2 + 3e(S)\right)t + \ldots$$

Aim: Get a similar formula for moduli spaces of sheaves on S

(1) Moduli spaces of sheaves on surfaces

Let S smooth projective surface over $\mathbb{C}$.
Let H ample line bundle on S.
Fix $r \in \mathbb{Z}_{>0}$, $c_1 \in H^2(S,\mathbb{Z})$, $c_2 \in H^4(S,\mathbb{Z}) \cong \mathbb{Z}$.

(Recall the Chern classes of a vector bundle or sheaf E on X are classes
$$c(E) = 1 + c_1(E) + c_2(E) + \dots \text{ on } X, \quad c_i(E) \in H^{2i}(X,\mathbb{Z})$$
measuring how far E is from being trivial
i.e. $c(\mathcal{O}_X^{\oplus r}) = 1.$).

Want to study moduli space of rank r torsion free coherent sheaves on S with
$c_1(E) = c_1$, $c_2(E) = c_2$

<u>Bad news</u>: No such moduli space exists; too many sheaves.

<u>Solution</u>: Make a moduli space of "good" (i.e. semistable) sheaves

Definition: Let $\mathcal{E}$ be a coherent sheaf on scheme X.

Denote $H^i(X,\mathcal{E})$ i^{th} sheaf cohomology.

The holomorphic Euler characteristic of $\mathcal{E}$ is

$$\chi(X,\mathcal{E}) := \sum_{i=0}^{\dim X} (-1)^i \; h^i(S,\mathcal{E})$$

Now let $\mathcal{E}$ be a coherent sheaf on S, H on S ample.

The Hilbert polynomial of $\mathcal{E}$ is

$$P_H(\mathcal{E},m) = \chi(X, \mathcal{E}\otimes H^{\otimes m}) \quad \text{(polynomial in } m)$$

Definition: A torsion free coherent sheaf $\mathcal{E}$ on S is called H-semistable if for all subsheaves

$$0 \neq \mathcal{F} \subsetneq \mathcal{E}$$

we have $\dfrac{P_H(\mathcal{F},m)}{rk(\mathcal{F})} \leq \dfrac{P_H(\mathcal{E},m)}{rk(\mathcal{E})}$ for all $m \gg 0$.

$\mathcal{E}$ is called stable if inequality is always strict.

<u>Theorem</u> (Maruyama – Gieseker)

There exists a (coarse) moduli space $M_S^H(r, c_1, c_2)$ of H semistable torsion-free coherent sheaves E on S of rank r with Chern classes $c_1(E) = c_1$, $c_2(E) = c_2$.

$M_S^H(r, c_1, c_2)$ is projective. The open subset $M_S^H(r, c_1, c_2)^s \subset M_S^H(r, c_1, c_2)$ parametrizes H-stable sheaves

<u>Expected dimension:</u> Assume $b_1(S) = h^1(S, \mathbb{Q}) = 0$

$M = M_S^H(r, c_1, c_2)$ is usually very similar but has an <u>expected dimension.</u>

$$\mathrm{vd}(M) = 2r c_2 - (r-1) c_1^2 - (r^2-1)\chi(\mathcal{O}_S)$$

<u>Expected dimension</u>: $\mathrm{vd}(M)$ = dimension of M if it was good.

[Kuranishi]: locally in the analytic topology $M_S^H(r, c_1, c_2)$ is the zero set of holom. map.

$$\mathbb{C}^m \xrightarrow{K} \mathbb{C}^k \quad \text{with} \quad vd(M) = m - k.$$

If K was smooth M would be nonsingular of dim $vd(M)$.

(2) Vafa-Witten formula

S projective alg surface with ample line bundle H.

Assume $b_1(S) = 0$, $p_g(S) = \dim H^0(S, K_S) > 0$.

(e.g. S K3 surface, S elliptic surface, S surface of general type).

Choose H ample and c_1, c_2 s.th.

$$M_S^H(c_1, c_2) := M_S^H(c_1, c_2)^S$$

Write $K_S^2 = \int_S K_S^2 \in \mathbb{Z}$. Assume there is a connected curve in $|K_S|$.

Vafa-Witten formula: Put $\eta(x) = \prod_{n>0}(1-x^n) = 1 - x - x^2 + x^5 + \ldots$

(modular forms) $\Theta(x) = \sum_{n \in \mathbb{Z}} x^{n^2} = (1 + 2x + 2x^4 + \ldots)$

Put $$\Psi_S(x) := 8\left(\frac{1}{2\bar{\eta}(x^2)^{12}}\right)^{\chi(\mathcal{O}_S)}\left(\frac{2\bar{\eta}(x^4)}{\theta_3(x)}\right)^{K_S^2}$$

<u>Vafa-Witten formula</u> : $e(M_S^H(c_1,c_2)) = \mathrm{Coeff}_{x^{vd(M)}}(\Psi_S(x))$

Actual Vafa-Witten formula is more general.

One considers invariants of Higgs-Moduli space

$$M_S^{Higgs}(c_1,c_2) = \{ E \xrightarrow{\phi} E\otimes K_S \}$$

$M_S^H(c_1,c_2)$ is the connected component where $\phi=0$.

<u>Want to interpret, check and generalize this formula.</u>

(3) Virtual topological invariants

$M = M_S^H(c_1,c_2)$ is usually singular, not of dimension $vd(M)$,

But it is virtually smooth of dimension $vd(M)$

Practically it means one can define virtual analogues of all invariants of smooth projective varieties

Behave as if M was nonsingular of dimension vd.

Technically this means M has a 1-perfect obstruction theory.

<u>Definition</u>: Let M be a scheme with an embedding
$$M \xhookrightarrow[i]{} X \text{ into a smooth scheme}$$
$I = I_{M/X}$ ideal sheaf.

A perfect obstruction theory on M is a complex $E^\bullet := [E^{-1} \xrightarrow{d} E^0]$ of vector bundles on M with morphism of complexes

$$\begin{tikzcd} E^{-1} \arrow[r, "d"] \arrow[d, "\varphi"] & E^0 \arrow[d, "\varphi"] \\ I/I^2 \arrow[r, "d"] & \Omega_X|_M \end{tikzcd}$$

s.th (1) $\varphi : \operatorname{Coker}(d) \to \operatorname{Coker}(d)$ is an iso.
(2) φ $\operatorname{Ker}(d) \to \operatorname{Ker}(d)$ is surjective.

Denote $vd(M) := rk(E^0) - rk(E^1)$ the expected dim of $[M, E]$

<u>Theorem</u> (Behrend-Fantechi, Li-Tian)
Let M be a scheme with a 1-perfect obstruction theory

(1) M has a virtual fundamental class
$[M]^{vir} \in H_{2vd(M)}(M)$

(2) M has a virtual structure sheaf
$\mathcal{O}_M^{vir} \in K_0(M)$

<u>Review of K-groups</u>: Let X variety

$K^0(X) = \{$ formal linear combinations
$a_1 E_1 + \dots + a_n E_n \mid n \geq 0,\ a_i \in \mathbb{Z},\ E_i$ vector bundles on $X \} / \sim$

Equivalence relation $F \sim E + G$ whenever
$0 \to E \to F \to G \to 0$ is exact.

$K_0(X)$ is the same with vector bundles replaced by coherent sheaves.

Chern classes are defined for elements of $K^0(M)$

$$c\left(\sum_{i=1}^n a_i E_i\right) = \prod_{i=1}^n c(E_i)^{a_i}$$

Definition: The virtual tangent bundle of M is $T_M^{vir} = E_0 - E_1 \in K^0(M)$, $E_i := (E^{-i})^\vee$.

The virtual Euler number of M is

$$e^{vir}(M) := \int_{[M]^{vir}} c_{vd}(T_M^{vir})$$

The case of moduli spaces of sheaves

Let $\mathcal{E}/_{S\times M_S^H(r,c_1,c_2)}$ be a universal sheaf, i.e.

$\mathcal{E}|_{S\times[E]} = E$ for all E in M, $\pi: S\times M \to M$ proj.

The dual of the obstruction theory for M is

$$R\pi_* R\mathcal{H}om(\mathcal{E},\mathcal{E})_0[1] \in D^b(M).$$

traceless

One can represent $R\pi_* R^* \mathcal{H}om(\mathcal{E},\mathcal{E})_0$ by a complex $E^0 \to E^1$ of vector bundles on M.

This gives the following

(1) M has virtual fundamental class
$[M]^{vir} \in H_{2vd}(M)$

(2) M has virtual tangent bundle $T_M^{vir} = E^0 - E^1 \in K^0(M)$

Can define virtual Euler number.

$$e^{vir}(M) = \int_{[M]^{vir}} c_{vd}^{vir}(M) \in \mathbb{Z}.$$

(3) M has virtual structure sheaf $\mathcal{O}_M^{vir} \in K_0(M)$.

For any vector bundle V on M define

$$\chi^{vir}(M,V) := \chi(M, V \otimes \mathcal{O}_M^{vir})$$

Conjecture: Vafa-Witten conjecture holds for virtual Euler number.

(assume $b_1(S)=0$, $p_g(S)>0$, H-stable $=$ H semistable
$|K_S|$ contains a smooth connected curve)

Then $$e^{vir}(M_S^H(c_1,c_2)) = \mathrm{Coeff}_{x^{vd}}\left(8 \left(\frac{1}{2\bar{\eta}(x^2)} \right)^{\chi(\mathcal{O}_S)} \left(\frac{2\bar{\eta}(x^4)}{\theta(x)} \right)^{K_S^2} \right)$$

Check this conjecture and refine it

Refinements of conjecture:

Replace $e(M)$ by finer topological invariants.

χ_{-y}-genus: X smooth projective variety, has Hodge numbers $h^{p,q}(X) = h^p(X, \Omega_X^q)$

The χ_{-y}-genus of X is

$$\chi_{-y}(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X)\, y^q = \sum_q (-y)^q \chi(X, \Omega_X^q)$$

Easy to see $\chi_{-y}(X)\big|_{y=1} = e(X)$.

e.g. K3 surface $\chi_{-y}(S) = 2 + 20y + 2y^2$; $e(S) = 24$.

Vertical χ_{-y}-genus

Now let M vertically smooth.

Set $\Omega_M^{p,\mathrm{vir}} = \Lambda^p (T_M^{\mathrm{vir}})^\vee$ vertical p-forms.

The virtual χ_{-y}-genus of M is

$$\chi_{-y}^{\mathrm{vir}}(M) := y^{-\mathrm{vd}(M)/2} \sum_p (-y)^p \chi^{\mathrm{vir}}(M, \Omega_M^{p,\mathrm{vir}})$$

<u>Can show</u>: $\chi^{vir}_{-y}(M)\big|_{y=1} = e^{vir}(M)$.

Conjecture for virtual χ_{-y}-genus

Put $\Theta(x,y) := \sum_{n\in\mathbb{Z}} x^{n^2} y^n$, $\tilde{\eta}(x) := \prod_{n>0}(1-x^n)$

$$\Psi_S(x,y) := 2^{3-\chi(\mathcal{O}_S)+K_S^2}\left(\frac{1}{\prod(1-x^{2n})^{10}(1-x^{2n}y)(1-x^{2n}/y)}\right)^{\chi(\mathcal{O}_S)}$$

$$\cdot\left(\frac{\tilde{\eta}(x^4)^2}{\Theta(x,y^{1/2})}\right)^{K_S^2}$$

<u>Conjecture</u>: S surface with $b_1(S)=0$, $p_g(S)>0$, $|K_S|$ contains smooth irred curve, $M_S^H(c_1,c_2) = M_S^H(c_1,c_2)^s$

Then $\chi^{vir}_{-y}(M_S^H(c_1,c_2)) = \mathrm{Coeff}_{x^{vd(M)}}(\Psi_S(x,y))$.

(4)

Mochizuki's formula

The main tool is Mochizuki's formula computing intersection numbers on moduli spaces of sheaves $M_S^H(c_1, c_2)$ in terms of intersection numbers on Hilbert schemes, and Seiberg-Witten invariants.

Seiberg Witten invariants

Seiberg-Witten invariants are C^∞ invariants of 4 manifolds for projective algebraic surfaces they are quite simple

Let S smooth projective surface, with $p_g(S)>0$, $b_1(S)=0$.

(a) Seiberg Witten invariants: $SW: H^2(S,\mathbb{Z}) \to \mathbb{Z}$
$a \mapsto SW(a)$

If $SW(a)\neq 0$ a is called a Seiberg-Witten class.

Facts: (1) If S is a K3 surface the only SW class is 0, with $SW(0)=1$.

(2) If $|K_S|$ contains a smooth irreducible curve SW classes are $0, K_S$ with $SW(0)=1$, $SW(K_S)=(-1)^{\chi(\mathcal{O}_S)}$

(3) If $\hat{S} \xrightarrow{\pi} S$ is the blowup of S in a point with exceptional divisor E, SW classes of $\hat{S}$ are $\{\pi^* a + E \mid a \text{ SW class of } S\}$.

(b) Setup for Mochizuki's formula

S projective surface with $b_1(S)=0$, $p_g(S)>0$.

$$S \xleftarrow{q} S\times S^{[n_1]}\times S^{[n_2]} \xrightarrow{p} S^{[n_1]}\times S^{[n_2]}$$

Universal sheaves:

Let $Z_n(S) = \{(x,Z)\in S\times S^{[n]} \mid x\in Z\}$ univ. subscheme.

· $I_{Z_n(S)/S\times S^{[n]}}$ univ. ideal sheaf,

For $L\in Pic(S)$ put

$$I_i(L) := p^*_{S\times S^{[n_1]}}\left(I_{Z_{n_i}(S)}\otimes p_S^* L\right) \qquad i=1,2.$$

sheaf on $S\times S^{[n_1]}\times S^{[n_2]}$ with $I_i(L)\big|_{S\times[Z_1]\times[Z_2]} = I_{Z_i}\otimes L$.

Tautological sheaves:

$$\begin{array}{ccccc} & & Z_n(S) & & \\ & \swarrow^{q} & & \searrow^{p} & \\ S & & & & S^{[n]} \end{array}$$

$\mathcal{O}^{[n]}(L) := p_* q^* L$ is vector bundle of rk n on S.

with fibre $\mathcal{O}^{[n]}(L)(Z) = H^0(Z, L|_Z)$

Define $\mathcal{O}_1(L), \mathcal{O}_2(L)$ on $S^{[n_1]} \times S^{[n_2]}$ by

$\mathcal{O}_i(L) := p_i^* \mathcal{O}^{[n_i]}(L)$.

Classes on $M = M_S^H(c_1, c_2)$.

Assume on $S \times M$ we have universal sheaf.

$$\begin{array}{ccccc} & & S\times M & & \\ & \swarrow & & \searrow & \\ S & & & & M \end{array}$$

For $\alpha \in H^k(S)$ put $\tau_i(\alpha) := \pi_{M*}(c_{i+2}(\mathcal{E}) \cdot \pi_S^* \alpha)$
$\in H^{2i+k}(S, \mathbb{Z})$

Let $P(\mathcal{E})$ be a polynomial in the

$(\tau_i(\alpha))_{i \geq 0,\ \alpha \in H^*(S)}$.

<u>Mochizuki's formula</u>: Compute

$\int_{[M]^{vir}} P(\mathcal{E})$ in terms of intersection numbers on

$S^{[n_1]} \times S^{[n_2]}$

For a_1, a_2 in $\mathrm{Pic}(S)$

$\Psi(a_1, a_2, n_1, n_2, s)$ = Expression in the Chern classes of $I_1(a_1), I_2(a_2), \mathcal{O}_1(a_1), \mathcal{O}_2(a_2)$ depending on P.

Put $$A(a_1, a_2, n, s) := \sum_{n_1+n_2=n} \int_{S^{[n_1]}\times S^{[n_2]}} \Psi(a_1, a_2, n_1, n_2, s) \in \mathbb{Q}[[s]]$$

<u>Theorem</u> (Mochizuki): Assume: (1) All $E \in M_S^H(c_1, c_2)$ are H stable

(2) $\chi(S, E) > 0 \quad \forall E \in M_S^H(c_1, c_2)$.

For every P as above.

$$\int_{[M_S^H(c_1,c_2)]^{vir}} P(\mathcal{E}) = \sum_{\substack{c_1 = a_1 + a_2 \\ a_1 H < a_2 H}} SW(a_1)\,\mathrm{Coeff}_{s^0}\left(A(a_1, a_2, c_2 - a_1 a_2, s)\right)$$

How does this help us?

We know almost nothing about $M_S^H(c_1, c_2)$, so cannot compute any intersection numbers on it.

Mochizuki's formula replaces the simpler $P(\mathcal{E})$ by a much more terrible expression on $S^{[n_1]} \times S^{[n_2]}$.

But while before it was <u>impossible</u> to compute anything, now it is only very complicated.

(5) Application to virtual Euler number

Now take $P(\mathcal{E}) = c_{vd}(M) = c_{vd}(T_M^{vir})$.

Where $T_M^{vir} = -R\pi_* R\mathcal{H}om(\mathcal{E}, \mathcal{E})_0$.

Applying Grothendieck-Riemann Roch to projection $\pi: S \times M \to M$ gives

$c_{vd}(T_M^{vir})$ is a polynomial in the $\tau_i(\alpha)$.

So Mochizuki's formula applies.

Now we will assume $P(\mathcal{E}) = c_{vd}(T_M^{vir})$.
Generalizations will be considered later.

Thus to compute

$$e^{vir}(M) := \int_{[M_S^H(c_1,c_2)]^{vir}} c_{vd}(T_M^{vir}) =$$

$$= \sum_{\substack{c_1 = a_1 + a_2 \\ a_1 H < a_2 H}} SW(a_1)\, \mathrm{Coeff}_{s^0}\left[A(a_1, a_2, c_2 - a_1 a_2, s)\right].$$

Still seems impossible to compute, but can show some nice properties

Cobordism invariance and multiplicativity

Write down generating function.

$$Z_S'(a_1, a_2, s, q) := \sum_{n \geq 0} A(a_1, a_2, n, s)\, q^n$$

<u>Proposition</u>: There is a polynomial $\tilde{p}$ in $a^2, ab, b^2, aK_S, bK_S, K_S^2, \chi(\mathcal{O}_S)$ s.th. for all surfaces S all $a, b \in \mathrm{Pic}(S)$ we have

$$A(a, b, n, s) = \tilde{p}(a^2, ab, b^2, aK_S, bK_S, K_S^2, \chi(\mathcal{O}_S))$$

This is a modification of an argument of Ellingsrud-G-Zehn, using inductive scheme for understanding $S^{[n]}$

based on.

<u>Theorem</u>: Let $Z_n(S) = \{(x,Z) \in S\times S^{[n]} \mid x \in Z\}$ univ. subscheme

$$S^{[n,n+1]} = \{(Z,W) \in S^{[n]} \times S^{[n+1]} \mid Z \subset W\}$$

incidence variety.

Then $S^{[n,n+1]}$ is the (smooth) blowup of $S\times S^{[n]}$ along $Z_n(S)$.

Can make inductive scheme.

$$\begin{tikzcd}[column sep=small] & S^{[n-1,n]} \arrow[dl, "q_n"'] \arrow[dr, "p_{n-1}"] & & S\times S^{[n-1,n-2]} \arrow[dl] & \cdots\cdots & \arrow[d, "p_1"] \\ S^{[n]} & & S\times S^{[n-1]} & & & S^n \end{tikzcd}$$

For $\alpha \in H^*(S^{[n]})$

$$\int_{S^{[n]}} \alpha = \frac{1}{n} \int_{S^{[n-1,n]}} q_n^* \alpha = \frac{1}{n!} \int_{S\times S^{[n-1]}} p_{n-1*} q_n^* \alpha$$

$$= \cdots = \int_{S^n} p_{1*} q_2^* \cdots p_{n-1*} q_n^* \alpha.$$

This leads to

<u>Theorem</u> (E-G-L): Let P be a polynomial in the Chern classes of $\mathcal{O}^{[n]}(L)$, and $T_{S^{[n]}}$.
$\exists$ polyn. $\tilde{P}$ in $L^2, LK_S, K_S^2, c_2(S)$ s.th.

$$\int_{S^{[n]}} P = \tilde{P}.$$

Method of proof can be adapted to our situation.

<u>Multiplicativity</u>.

Write

$$Z'(a,b,s,q) := (2s)^{\chi(\mathcal{O}_S)} \left(\frac{2s}{1+2s}\right)^{\chi(S,b-a)} \left(\frac{-2s}{1-2s}\right)^{\chi(S,a-b)} \cdot Z(a,b,s,q)$$

$\Rightarrow Z_S(a,b,s,q)$ is a power series in q starting with 1.

<u>Theorem</u>: There are power series $A_1, \dots A_7 \in \mathbb{Q}((s))[[q]]$ with

$$Z_S(a,b,s,q) = A_1^{a^2} A_2^{ab} A_3^{b^2} A_4^{aK_S} A_5^{bK_S} A_6^{K_S^2} A_7^{\chi(\mathcal{O}_S)}.$$

Reduction to case of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$

$Z_S(a,b,s,q)$ depends only on the 7 tuple
$v(S,a,b) := (a^2, ab, b^2, aK_S, bK_S, K_S^2, \chi(\mathcal{O}_S))$

By multiplicativity in order to compute
$Z_S(a,b,s,q)$ for all S, $a,b \in \mathrm{Pic}(S)$
enough to compute for 7 triples (S_i, a_i, b_i)
such that the $v(S_i, a_i, b_i)$ are linearly independent
in $\mathbb{Q}^7$.

Choose: $(\mathbb{P}^2, (\mathcal{O},\mathcal{O}), (\mathcal{O}(1),\mathcal{O}), (\mathcal{O},\mathcal{O}(1)), (\mathcal{O}(1),\mathcal{O}(1)))$
$(\mathbb{P}^1 \times \mathbb{P}^1, (\mathcal{O},\mathcal{O}), (\mathcal{O}(1,0),\mathcal{O}), (\mathcal{O},\mathcal{O}(1,0)))$.

Note: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ are toric surfaces, i.e. have
an action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many
fixpoints.
$\Longrightarrow$ Can compute $Z_S(a,b,s,q)$ by equivariant
localization.

Equivariant localization

Let X smooth proj. variety with action of $T = (\mathbb{C}^*)^2$ with finitely many fixpoints $p_1, \dots p_e$.

Let E be a T-equivariant vectorbundle on X,

for p one of the fixpoints the fibre $E(p)$ is a vector space with linear T-action $\Rightarrow$

$E(p)$ has a Basis of eigenvectors

$$E(p) = \bigoplus_{i=1}^{r} \mathbb{C} \cdot v_i \qquad \text{with } (t_1, t_2) \cdot v_i = t_1^{n_i} \cdot t_2^{m_i} v_i$$

The weight of v_i is $w(v_i) := n_i \varepsilon_1 + m_i \varepsilon_2$

Localized equivariant Chern class of E at p.

$$c^T(E)(p) = (1 + c_1^T(E)(p) + \dots c_r^T(E)(p)).$$
$$= \prod_{i=1}^{r} (1 + w(v_i)) \in \mathbb{Z}[\varepsilon]. \qquad c_i^T = \text{part of degree } i \text{ in } \varepsilon.$$

Theorem (Bott residue formula)

Let E T-equivariant vector bundle on X.

$P(c_i(E))_{i=1}^{r}$ pol. in the Chern classes of E of weight $d = \dim(X)$.

$$\Longrightarrow \int_{[X]} P(c_i(E)) = \sum_{j=1}^{\ell} \frac{P(c_i^T(E)(p_j))}{c_d^T(T_{p_i}X)}.$$

Localization on Hilbert schemes of points

S smooth projective toric surface, i.e. $T=(\mathbb{C}^*)^2$ acts on S with finitely many fixpoints. $p_1, \dots p_\ell$

For simplicity $S=\mathbb{P}^2$, have local coordinates x_i, y_i
$p_0=[1,0,1]$, $p_1=[0,1,0]$, $p_2=[0,0,1]$ which are eigenvectors for T-action

The action lifts to $\mathbb{P}^{2[n]}$ by $t\cdot[Z]=[tZ]$
($t:\mathbb{P}^2\to\mathbb{P}^2$ iso!)

If $Z\in(\mathbb{P}^2)^{[n]}$ fixpoint for T action, then

$Z = Z_0 \sqcup Z_1 \sqcup Z_2$, supp $(Z_i)=p_i$, len $(Z_i)=n_i$, $n_1+n_2+n_3=n$

and in local coord. x_i, y_i ideal I_{Z_i} is gen. by monomials i.e if $m=n_i$

$$I_{Z_i} = (y^{m_0}, x y^{m_1}, \dots, x^r y^{m_r}, x^{r+1})$$

Where $(m_0, m_1, \dots m_r)$ is a partition of m.

Thus bijection: Fixpoints $Z\in((\mathbb{P}^2)^{[n]})^T \longleftrightarrow$ triples ν_0, ν_1, ν_3 of partitions.

Apply Bott residue to this

(1) Fibers at fixpoints of all univ sheaves we considered are described in terms of partitions.

(2) Formula is evaluated as a combinatorial formula in terms of partitions

e.g. Assume Z supported in $(0,0)$, action by $w(x) = \varepsilon_1$, $w(y) = \varepsilon_2$, $I_Z = (y^{n_0}, y^{n_1}x, \dots)$

Basis of $\mathcal{O}_Z$ is of Eigenv.

$$\begin{array}{llll}
y^{n_0-1} & & & \\
\vdots & xy^{n_1-1} & & \\
y & xy & \cdots & xy^{n_r-1} \\
1 & x & & x^r
\end{array}$$

Then $c^T(\mathcal{O}_{Z_n}(z)) = \prod_{i=1}^{r} \prod_{j=0}^{n_{i-1}} (1 + i\varepsilon_2 + j\varepsilon_1)$

Similar, but more complicated for the other bundles

All this can be programmed on the computer

<u>Result</u>: Conjectures are true for K3 surfaces, double covers of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, elliptic surfaces, hypersurfaces in $\mathbb{P}^3$ ... up to high dimension of the moduli space.