Hypergeometric families of Calabi-Yau manifolds

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Hypergeometric Weight Systems

By a hypergeometric weight system we will mean a formal linear combination

$$\gamma = \sum_{\nu \geq 1} \gamma_{\nu}[\nu], \quad (0.1)$$

where $\gamma_{\nu} \in \mathbb{Z}$ are zero for all but finitely many $\nu$, satisfying the following two conditions

$$(i) \quad \sum_{\nu \geq 1} \nu \gamma_{\nu} = 0$$

$$(ii) \quad d = d(\gamma) := -\sum_{\nu \geq 1} \gamma_{\nu} > 0 \quad (0.2)$$

We denote by $\Gamma$ the set of all such $\gamma$. Note that $\Gamma$ is a cone, i.e. if $\gamma, \gamma' \in \Gamma$ then $\gamma + \gamma' \in \Gamma$. We call $d$ the dimension of the weight system $\gamma$.

To $\gamma \in \Gamma$ we associate the hypergeometric function

$$u(\lambda) := \sum_{n \geq 0} u_n \lambda^n \quad (0.3)$$

where

$$u_n = \prod_{\nu \geq 1} (\nu n)!^{\gamma_{\nu}}.$$ 

It is easy to check that for some minimal $r$ we have

$$u(\lambda) = {}_r F_{r-1} \left( \begin{array}{c} \alpha_1 \cdots \alpha_r \\ \beta_1 \cdots \beta_r \end{array} \, \frac{\lambda}{\lambda_0} \right),$$

where

$$\lambda_0^{-1} := \prod_{\nu \geq 1} \nu^{\nu \gamma_{\nu}}.$$ 

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and $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r < 1$ and $0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_r < 1$ are two sets of $r$ rational numbers. We call $\lambda_0 = \lambda_0(\gamma)$ and $r = r(\gamma)$, respectively, the special point and the rank of $\gamma$.

The condition 0.2 (i) precisely guarantees that the linear differential equation (of order $r$) satisfied by $u$ has only regular singularities.

By Stirling

$$u_n \sim \frac{\sqrt{D}}{(2\pi n)^{d/2}} \lambda_0^n$$

where

$$D := \prod_{\nu \geq 1} \nu^{\gamma_{\nu}}.$$ 

It will be convenient to introduce

$$P(t) := \prod_{\nu \geq 1} (1 - t^\nu)^{\gamma_{\nu}}$$

the Poincaré series of $\gamma$.

Note that

$$d = \text{order of pole of } P \text{ at } t = 1 \quad (0.4)$$

and

$$D = \prod_{\nu \geq 1} \left( \frac{1 - t^\nu}{1 - t} \right)^{\gamma_{\nu}} \bigg|_{t = 1} = (-1)^d \left[ (t - 1)^d P(t) \right]_{t = 1},$$

is $(-1)^d$ times its leading coefficient at $t = 1$. If we write

$$P(t) = (-1)^d \frac{A(t)}{B(t)}$$

in lowest terms, with $A, B \in \mathbb{Z}[t]$ monic and relatively prime then $r = \deg A = \deg B$ and

$$A(t) = \prod_{j=1}^r (t - e^{2\pi i \alpha_j}), \quad B(t) = \prod_{j=1}^r (t - e^{2\pi i \beta_j}),$$

with the $\alpha$’s and $\beta$’s as above.

1 Integrality

Our goal is to obtain $\gamma \in \Gamma$ such that the truncation of $u$

$$\sum_{n=0}^{p-1} u_n \lambda^n \mod p \quad (1.1)$$

for a prime $p$ is related to the number of points over $\mathbb{F}_p$ of some family of varieties $X_\lambda$. In order to even make sense of this expression we need some integrality assumption on $u_n$. The simplest such assumption would be

$$u_n \in \mathbb{Z} \quad \text{for all } n = 0, 1, 2, \ldots \quad (1.2)$$

and as we will see below this is in fact necessary if we want the truncation of $u$ modulo $p$ for all sufficiently large primes $p$. 
Let us denote by $\Gamma_{\text{int}} \subset \Gamma$ all weight systems satisfying (1.2), which we will call integral; they clearly form a sub-cone of $\Gamma$. There is an obvious collection of elements in $\Gamma_{\text{int}}$, namely, the multinomials

$$\gamma = [w_0 + \cdots + w_d] - [w_0] - \cdots - [w_d], \quad w_0, \cdots, w_d \in \mathbb{N}.$$ 

Let $\Gamma_{\text{mon}}$ be the cone spanned by the multinomials. We have then

$$\Gamma_{\text{mon}} \subset \Gamma_{\text{int}}.$$ 

It is interesting and perhaps at first surprising that these two cones are not in fact the same. A classical example

$$\gamma = [30] + [1] - [6] - [10] - [15] \in \Gamma_{\text{int}} \setminus \Gamma_{\text{mon}}$$

was used by Chebychev in his work on the distribution of prime numbers.

There is a beautiful criterion due to Landau [8] for checking whether $\gamma \in \Gamma_{\text{int}}$.

**Proposition 1** We have

$$u_n \in \mathbb{Z} \quad \text{for all } n = 0, 1, 2, \cdots$$

if and only if

$$\mathcal{L}(x) \geq 0 \quad \text{for all } x \in \mathbb{R}.$$ 

**Proof** Fix a prime $p$ and let $v_p$ denote the corresponding valuation. By the well know formula for $v_p(n!)$ we have

$$v_p(u_n) = \sum_{\nu \geq 1} \gamma_{\nu} \sum_{k \geq 1} \left\lfloor \frac{\nu n}{p^k} \right\rfloor.$$ 

Combining $x = [x] + \{x\}$ with 0.2 (i) we find that

$$v_p(u_n) = \sum_{k \geq 1} \mathcal{L}\left(\frac{n}{p^k}\right). \quad (1.3)$$

It follows that if $\mathcal{L}(x) \geq 0$ then $v_p(u_n) \geq 0$ for all $p$ and hence $u_n \in \mathbb{Z}$.

For some $\delta > 0$ we have that $\mathcal{L}(x) = 0$ for $x \in [0, \delta)$ (proposition 3); hence, for all primes $p$ such that $p\delta > 1$, $v_p(u_n) = \mathcal{L}(n/p)$ for $n \leq p$. Since $\mathcal{L}$ is right continuous it follows that $u_n \in \mathbb{Z}$ implies the non-negativity of $\mathcal{L}$. \qed

We can be a bit more precise about the way in which the $u_n$ may fail to be integral.

**Proposition 2** If $\gamma$ is not integral then for all $p$ sufficiently large there exists an $0 \leq n < p$ such $v_p(u_n) < 0$.

**Proof** If $\gamma$ is not integral the by the previous proposition $\mathcal{L}$ is negative at some point. Since $\mathcal{L}$ is locally constant (proposition 3) it follows that $\mathcal{L}$ is negative in some interval of length say $\alpha > 0$. As in the proof of the previous proposition, for all $p\delta > 1$ we have $v_p(u_n) = \mathcal{L}(n/p)$ for $0 \leq n < p$. It follows that for all $p$ with $p\delta > 1$ and $p\alpha > 1$ we have $v_p(u_n) < 0$ for some $0 \leq n < p$. \qed
We summarize in the next proposition a number of simple properties of \( \mathcal{L} \); we leave the proofs to the reader (more details will appear in [10]). We call the \textit{support} \( N \) of \( \gamma \in \Gamma \) those \( \nu \in \mathbb{N} \) for which \( \gamma_{\nu} \neq 0 \). For \( z \in \mathbb{C} \) with \( |z| < 1 \) we let \( \log(1-z) \) be the standard branch of the logarithm vanishing at \( z = 0 \) and define

\[
\log(P(z)) = \sum_{\nu \geq 1} \gamma_{\nu} \log(1-z^{\nu})
\]

**Proposition 3**

1. The regularity condition 0.2 (i) is equivalent to \( \mathcal{L} \) being locally constant.
2. For all \( x \)
   \[
   \mathcal{L}(x) = \frac{d}{2} - \lim_{t \to 1^-} \Re \left[ \log P(te^{2\pi ix}) \right].
   \]
3. \( \mathcal{L} \) is right continuous with discontinuity points exactly at \( x \equiv \alpha_j \mod 1 \) or \( x \equiv \beta_j \mod 1 \) for some \( j = 1, \ldots, r \).
4. More precisely,
   \[
   \mathcal{L}(x) = \#\{j \mid \alpha_j \leq x\} - \#\{j \mid 0 < \beta_j \leq x\}.
   \]
5. \( \mathcal{L} \) takes only integer values.
6. \[
   \int_0^1 \mathcal{L}(x) \, dx = \frac{1}{2}d, \quad \lim_{x \to 1^-} \mathcal{L}(x) = d, \quad \lim_{x \to 0^+} \mathcal{L}(x) = 0. \tag{1.4}
   \]
   In particular, for a general non-zero \( \gamma = \sum_{\nu \geq 1} \gamma_{\nu}[\nu] \), the conditions 0.2 (i) and \( \mathcal{L}(x) \geq 0 \) imply 0.2 (ii); i.e., integrality implies positive dimension.
7. If we fix the support \( N \) of \( \gamma \in \Gamma \) the finitely many conditions
   \[
   \mathcal{L} \left( \frac{k}{N} \right) \geq 0, \quad k = 0, 1, \ldots, N-1, \tag{1.5}
   \]
   where \( N \) is the lcm of the numbers in \( N \), is equivalent to \( \gamma \in \Gamma_{\text{int}} \).
8. Away from the discontinuity points of \( \mathcal{L} \) we have
   \[
   \mathcal{L}(-x) = d - \mathcal{L}(x) \tag{1.6}
   \]
   and, in particular, for all \( x \)
   \[
   \mathcal{L}(x) \leq d, \quad \text{if } \gamma \in \Gamma_{\text{int}}. \tag{1.7}
   \]

### 1.1 Examples of \( \gamma \in \Gamma_{\text{int}} \setminus \Gamma_{\text{mon}} \)

(i) A computer search reveals some simple examples of \( \gamma \in \Gamma_{\text{int}} \setminus \Gamma_{\text{mon}} \)

\[
\]

Note that by (1.5) integrality can be checked in finite time; in fact, if we fix the support \( N \) of \( \gamma \) the condition \( \gamma \in \Gamma_{\text{int}} \) is defined by finitely many inequalities on the \( \gamma_{\nu} \) for \( \nu \in N \). We can solve this system completely using, for example, the computer package PORTA [www.zib.de/Optimization/Software/Porta](http://www.zib.de/Optimization/Software/Porta).
Here is a list of 22 generators of the cone of integral $\gamma$ with support $N = 1, 2, 3, 5, 6, 10, 15, 30.$

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(The fourth vector from the bottom is Chebychev’s example.)

(ii) We may also exhibit an infinite sequence $\gamma(n) \in \Gamma_{\text{int}} \setminus \Gamma_{\text{mon}}$ for $n = 3, 4, \ldots$ of dimension $n - 2.$ Define recursively

\[ p_1 := 2 \]
\[ p_k := p_{k-1} \cdots p_2 p_1 + 1, \quad k = 1, \ldots, n - 1 \]
\[ p_n := p_{n-1} \cdots p_2 p_1 - 1 \]

and set

\[ \gamma(n) := [p_1 \cdots p_n] + [1] - \sum_{j=1}^{n} [p_1 \cdots \overline{p_j} \cdots p_n], \]

where a bar indicates that that argument should be omitted. For $n = 3$ we have $p_1 = 2, p_2 = 3, p_3 = 5$ so that $\gamma(3)$ is Chebychev’s example.

It is easy to show by induction that

\[ \sum_{j=1}^{n} \frac{1}{p_j} = 1 + \frac{1}{p_1 \cdots p_n} \quad (1.8) \]
It is clear that $\gamma(n)$ is not in $\Gamma_{\text{mon}}$; to check that indeed $\gamma(n) \in \Gamma_{\text{int}}$ we use Landau’s criterion. By (1.8) we have

$$\mathcal{L}\left(\frac{k}{p_1 \cdots p_n}\right) = k - \sum_{j=1}^{n} \left\lfloor \frac{k}{p_j} \right\rfloor$$

(1.9)

We need only check that the value of $\mathcal{L}$ in (1.9) is non-negative for all $0 < k < p_1 \cdots p_n$. First note that we cannot have $\{ \frac{k}{p_j} \} = 0$ for all $j = 1, 2, \ldots, n$. Otherwise, $k$ would be divisible by $p_j$ for all $j = 1, 2, \ldots, n$ and therefore divisible by their product $p_1 \cdots p_n$, since the $p_j$’s are clearly pairwise relatively prime, contradicting that $k < p_1 \cdots p_n$. Say $1 \leq l \leq n$ is such that $\{ \frac{k}{p_l} \} > 0$. Then by (1.9) and (1.8)

$$\mathcal{L}\left(\frac{k}{p_1 \cdots p_n}\right) \geq k - \sum_{j=1}^{n} \frac{k}{p_j} + \frac{1}{p_l} = \frac{1}{p_l} - \frac{1}{p_1 \cdots p_n} \geq 0$$

(iii) We may use the following theorem of Eisenstein (see [7] for a modern treatment and further references). If

$$f(t) = \sum_{n \geq 0} f_n t^n \in \mathbb{Q}[[t]]$$

is the Taylor expansion of an algebraic function then there exist an $N \in \mathbb{N}$ such that $N^n f_n \in \mathbb{Z}$ for all $n = 0, 1, \ldots$

When is a hypergeometric function algebraic? It is possible to describe the hypergeometric differential equations for which all of its solutions are algebraic. Schwartz famously did this for the case of rank $r = 2$. More recently Beukers and Heckman [4] did the general case. Scanning their list we find a number of examples of hypergeometric functions of the type (0.3) we are considering (these correspond to the monodromy group being defined over $\mathbb{Q}$), including, once again, Chebychev’s case; in fact, we discover that for Chebychev’s example the monodromy group of the corresponding differential equation is the Weyl group of the $E_8$ lattice!

We will return to the algebraic hypergeometric functions in a later publication [10] but let us mention one of the results. A general converse of Eisenstein’s theorem is blatantly false; nevertheless, we have the following.

**Theorem 1** Let $\gamma \in \Gamma$ be a hypergeometric weight system. Then the associated hypergeometric function (0.3) is algebraic if and only if $\gamma$ is integral of dimension $d = 1$.

2 Cases with dimension equal rank

By (0.4) we have

$$d \leq r.$$  

We will consider the $\gamma \in \Gamma$ were equality holds, which precisely means that $\lambda = 0$ is a point of maximal unipotency for the hypergeometric equation satisfied by $u$.

If $d = r$ then

$$P(t) = \frac{A(t)}{(t - 1)^d}$$

(2.1)

where

$$A(t) = \prod_{n \geq 2} \Phi_n(t)^{\gamma_n}$$
with $\Phi_n$ the $n$-th cyclotomic polynomial and $e_n$ non-negative integers zero for all but finitely many $n$.

It follows that

$$d = r = \sum_{n \geq 2} e_n \phi(n)$$

(2.2)

where $\phi$ is Euler’s phi-function.

On the other hand we have

$$\Phi_n(t) = \prod_{m | n} (t^m - 1)^{\mu(n/m)},$$

where $\mu$ is the Möbius function, and therefore any solution to (2.2) gives rise via (2.1) to the Poincaré series of some $\gamma \in \Gamma$ with $d = r$.

It follows that $d = r = \sum_{n \geq 2} e_n \phi(n)$ (2.2) where $\phi$ is Euler’s phi-function. On the other hand we have

$$\Phi_n(t) = \prod_{m | n} (t^m - 1)^{\mu(n/m)},$$

where $\mu$ is the Möbius function, and therefore any solution to (2.2) gives rise via (2.1) to the Poincaré series of some $\gamma \in \Gamma$ with $d = r$.

It follows that the $\gamma \in \Gamma$ with $d = r$ form a cone $\Gamma_{uni}$ generated by

$$\phi(n)[1] - \sum_{m | n} \mu \left( \frac{n}{m} \right) [m], \quad n = 2, \ldots$$

It is easy to verify using Landau’s criterion (proposition 3, 4.) that $\Gamma_{uni} \subset \Gamma_{int}$.

Since there are only finitely many $n$’s with $\phi(n)$ less than a given bound we see that there are finitely many $\gamma \in \Gamma_{uni}$ of fixed dimension. For small $d$ these are easy to enumerate. For example, $n = 2, 3, 4, 5, 6, 8, 10, 12$ are all $n > 2$ with $\phi(n) \leq 4$, the respective values of $\phi$ being $1, 2, 2, 4, 2, 4, 4, 4$. We now describe all cases with $d = r \leq 4$.

For $d = 1$ we only have one case

$$\gamma = [2] - 2[1], \quad P(t) = \frac{t + 1}{t - 1} = \frac{t^2 - 1}{(t - 1)^2}$$

corresponding to

$$u_n = \binom{2n}{n}, \quad u(\lambda) = (1 - 4\lambda)^{-\frac{1}{2}}.$$

For $d = 2$ we obtain

$$\gamma = [2] - 2[1] \quad a \quad b \quad \lambda_0^{-1}$$

$$\begin{array}{ccc}
\end{array}$$

where

$$u(\lambda) = {}_2F_1 \left( \frac{a}{1} \frac{b}{1} \frac{\lambda}{\lambda_0} \right)$$

Note that since $\phi(n)$ is even for all $n > 2$ any solution of (2.2) for $d$ odd arises from a solution for $d - 1$ by adding an extra $1 = \phi(2)$. In terms of the weight systems, if $\gamma$ has $d = r$ odd then

$$\gamma = \gamma_0 + [2] - 2[1],$$

where $\gamma_0 \in \Gamma_{uni}$ has dimension and rank $d - 1$. In particular, we get a description of all $\gamma \in \Gamma_{uni}$ with $d = r = 3$ from those with $d = r = 2$.

There are 14 cases with $d = 4$ and these are listed in the first column of the table at the end of the paper.
As it happens all cases of $\gamma \in \Gamma_{\text{uni}}$ with $d \leq 4$ are also in $\Gamma_{\text{mon}}$ (this is not true for general $d$; for example, $[30] + [5] + [3] + [2] - [15] - 10 - [6] - 9[1]$ is integral with $d = r = 8$ but is not monomial). We can associate to them a one-parameter family of CY hypersurfaces in a weighted projective space of dimension $d$ with $u(\lambda)$ as one of its periods. More precisely, for $d = 2, 3, 4$ we obtain families of elliptic curves, K3 surfaces, and CY threefolds respectively. The 14 families of CY threefolds, except for the last in the table, are discussed in a paper of Batyrev and Straten [2].

3 The case of threefolds

Let $X_\lambda$ be one of the families of threefolds associated via toric geometry to a $\gamma \in \Gamma_{\text{uni}}$ of dimension 4. Corresponding to the Picard-Fuchs equation for $u$ there is a factor $R_0(t)$ (see [6] for more details) of the numerator of the zeta function of $X_\lambda$. For $\lambda \neq \lambda_0$ this factor is of degree 4 but at the special point $\lambda = \lambda_0$, where $X_\lambda$ becomes singular, we have

$$R_0(t) = (1 - \left(\frac{D}{p}\right) pt)(1 - a_p t + p^3 t^2).$$

One can verify, the same way that Schoen [11] did it for the standard quintic $[5] - 5[1]$ using powerful results of Faltings and Serre, that $a_p$ is the $p$-th coefficient of a certain modular form of weight 4 for a congruence subgroup of $SL_2(\mathbb{Z})$. (This feature of rigid CY threefolds has been discussed by several people, including N. Yui and H. Verrill.)

We identified these modular forms by computing $a_p$ for several $p$’s using the $p$-adic formulas of [5], [6] and then comparing with the tables of W. Stein [www.math.harvard.edu/~was] (Actually, some of the forms had bigger level than those tabulated there and we had to generate the modular forms with an a-priori guess for the level.) The resulting data is tabulated at the end of the paper ($N$ is the level of the modular form, the last columns list $a_p$ for $p = 2, 3, \cdots, 11$).

In the process something interesting emerged: the congruence between the truncation (1.1) of the period $u$ at $\lambda = \lambda_0$ actually appears to hold mod $p^3$ rather than just mod $p$. This phenomenon of super-congruence was first observed by Beukers [3] in connection with the numbers Apéry used in his proof of the irrationality of $\zeta(3)$. There is by now a quite extensive literature on questions of this kind (see for example [1]).

Precisely, we find (numerically) that for all primes $p$ not dividing $\lambda_0^{-1}$

$$\sum_{n=0}^{p-1} a_n \lambda_0^{-n} \equiv a_p \mod p^3.$$

The super-congruences also appear to hold for smaller dimensions. For example, for the case $\gamma = 2[2] - 4[1]$ with $d = r = 2$ we find (again numerically) that for odd $p$

$$\sum_{n=0}^{p-1} \binom{2n}{n} 16^{-n} \equiv \left(\frac{-4}{p}\right) \mod p^2.$$

These have been now proved by E. Mortenson [9].
## References


