1. Introduction

Let \( f(z) \in S_2(p) \) be a newform of weight two, prime level \( p \). If \( f(z) = \sum_{m=1}^{\infty} a_m q^m \), where \( q = e^{2\pi i z} \), and \( D \) is a fundamental discriminant, we define the twisted \( L \)-function

\[
L(f, D, s) = \sum_{m=1}^{\infty} a_m m^{-s} \left( \frac{D}{m} \right).
\]

It will be convenient to also allow \( D = 1 \) as a fundamental discriminant, in which case we write simply \( L(f, s) \) for \( L(f, 1, s) \).

In this paper we consider the question of computing the twisted central values \( \{ L(f, D, 1) : |D| \leq x \} \) for some \( x \).

It is well known that the fact that \( f \) is an eigenform for the Fricke involution yields a rapidly convergent series for \( L(f, D, 1) \). Computing \( L(f, D, 1) \) by means of this series, which we call the standard method, takes time very roughly proportional to \( |D| \) and therefore time very roughly proportional to \( x^2 \) to compute \( L(f, D, 1) \) for \( |D| \leq x \). We will see that this can be improved to \( x^{3/2} \) by using an explicit version of Waldspurger’s theorem; this theorem relates the central values \( L(f, D, 1) \) to the \( |D| \)-th Fourier coefficient of weight \( 3/2 \) modular forms in Shimura correspondence with \( f \).

Concretely, the formulas we use have the basic form

\[
(1.1) \quad L(f, D, 1) = \star \kappa \pm \frac{|c_{\pm}(|D|)|^2}{\sqrt{|D|}}, \quad \text{sign}(D) = \pm,
\]

Key words and phrases. Waldspurger correspondence, Half integral weight forms, Special values of \( L \)-functions.
where \( \star = 1 \) if \( p \nmid D \), \( \star = 2 \) if \( p \mid D \), \( \kappa_\pm > 0 \) is a constant independent of \( D \) and \( c_\pm(|D|) \) is \(|D|\)-th Fourier coefficient of a certain modular form \( g_\pm \) of weight \( 3/2 \).

Gross [Gr] proves such a formula, and gives an explicit construction of the corresponding form \( g_- \), in the case that \( L(f, 1) \neq 0 \) (which holds for about half of the cases). The purpose of this paper is to extend Gross's work to all cases. Specifically, we give an explicit construction of both \( g_- \) and \( g_+ \), regardless of the value of \( L(f, 1) \), together with the corresponding values of \( k_\pm \) in \((1.1) \). The proof of the validity of this construction will be given in a later publication and relies partly in the results of [B-M].

The construction gives \( g_\pm \) as a linear combination of (generalized) theta series associated to positive definite ternary quadratic forms. Computing the Fourier coefficients of these theta series up to \( x \) is tantamount to running over all lattice points in ellipsoid of volume proportional to \( x^{3/2} \). Doing this takes time roughly proportional to \( x^{3/2} \) which yields our claim above.

This approach to computing \( L(f, D, 1) \) has several other advantages over the standard method. First, the numbers \( c(|D|) \) are algebraic integers and are computed with exact arithmetic. Once \( c(|D|) \) is know it is trivial to compute \( L(f, D, 1) \) to any desired precision. Second, the \( c(D) \)'s have extra information; if \( f \) has coefficients in \( \mathbb{Z} \), for example, \((1.1) \) gives a specific square root of \( L(f, D, 1) \) (if non-zero), whose sign remains a mystery.

Moreover, the actual running time of our method vs. the standard method is, in practice, significantly better even for small \( x \).

2. Construction of \( g_\pm(z) \)

2.1. \( g_-(z) \): **when** \( L(f, 1) \neq 0 \). We recall Gross's construction of the map \( \theta_1 \).

Let \( B \) be the quaternion algebra over \( \mathbb{Q} \) ramified precisely at \( \infty \) and \( p \). Let \( R \) be a fixed maximal order in \( B \). A right ideal \( I \) of \( R \) is a lattice in \( B \) that is stable under right multiplication by \( R \). Two right ideals \( I \) and \( J \) are
in the same class if \( J = bI \) with \( b \in B^\times \). The set of right ideal class is finite; we denote its order by \( n \) and let \( \{ I_1, \ldots, I_n \} \) be the representatives.

Let \( R_i = \{ b \in B : bI_i \subset I_i \} \) be the left order of \( I_i \). Then \( R_i \) are also maximal orders in \( B \) and each conjugacy class of maximal orders has a representative \( R_i \) for some \( i \). Let \( 2w_i \) be the number of units in \( R_i^\times \), then Eichler’s mass formula states
\[
\sum_{i=1}^{n} \frac{1}{w_i} = \frac{p-1}{12}.
\]

For \( b \in B \), we use \( N^b \) to denote the reduced norm of \( b \). Let \( N I_i \) be the positive greatest common divisor of \( \{ N^b : b \in I_i \} \).

Let \( S_i := \mathbb{Z} + 2R_i \), a suborder of index 8 in \( R_i \). Let \( S_i^0 \) be the subset of \( S_i \) consisting of trace 0 elements. Define
\[
h_i(z) = \frac{1}{2} \sum_{b \in S_i^0} q^{N^b} = \frac{1}{2} \sum_{m \geq 0} c_i(m) q^m.
\]

Then \( h_i(z) \) is a weight \( \frac{3}{2} \) form with level \( 4p \) and satisfying \( c_i(m) = 0 \) whenever \( m \equiv 1, 2 \mod 4 \).

As mentioned before \( e_f \) is a function on the ideal classes \( I_i \). Let \( a_i = e_f(I_i) \), then
\[
(2.1) \quad g_-(z) = \theta_1(e_f) := \sum_i a_i h_i(z).
\]

### 2.2. \( g_-(z) \) and weight functions: \( L(f, 1) = 0 \) case.
When \( L(f, 1) = 0 \), we construct \( g_-(z) \) as follows:

1. Find a prime \( l \neq p \) such that \( l \equiv 1 \pmod{4} \) and \( L(f, l, 1) \neq 0 \); in particular, \( \left( \frac{1}{p} \right) \) has to be equal to the sign of the functional equation for \( L(f, s) \). From [B-F-H], there are infinitely many such \( l \).
2. Fix a normalized weight function \( \omega_l \) on \( R \), as defined below.
3. Transport \( \omega_l \) to weight functions \( \omega_l(I_i, \cdot) \) on \( R_i \), as explained below.
4. Define
\[
h_i(z) := \frac{1}{2} \sum_{b \in S_i^0} \omega_l(I_i, b) q^{N^b/l}.
\]
5. Let \( a_i = e_f(I_i) \), then
\[
g_-(z) = \theta_l(e_f) := \sum_i a_i h_i(z).
\]

**Definition 2.1.** Let \( R \) be a maximal order, and fix a prime \( l \neq p \). A weight function \( \omega_l \) on \( R \) is a nonzero function defined on \( R^0(\mathbb{Z}_l) \) (where \( R^0 \) is the subset of trace zero elements) satisfying the following equations:

\[
\omega_l(a^{-1}ba) = \left( \frac{N a}{l} \right) \omega_l(b), \quad a \in R^\times(\mathbb{Z}_l), \; b \in R^0(\mathbb{Z}_l);
\]

\[
\omega_l(kb) = \left( \frac{k}{l} \right) \omega_l(b), \quad k \in \mathbb{Z}_l^\times, \; b \in R^0(\mathbb{Z}_l);
\]

\[
\omega_l(y) = \sigma \int_{R^0(\mathbb{Z}_l)} \omega_l(x) \psi(\text{Tr}(xy)/l) \, dx.
\]

Here \( \psi(x) = q^{\iota(x)} \) where \( \iota(x) \in \mathbb{Q} \) satisfies \( x - \iota(x) \in \mathbb{Z}_l \); the measures are normalized so that \( \mathbb{Z}_l \) has volume 1, and
\[
\sigma = l^2 \int_{\mathbb{Z}_l^\times} \left( \frac{a}{l} \right) \psi(a/l) \, da.
\]

We say that \( \omega_l \) is normalized if \( \omega_l(b) \in \{0, \pm 1\} \) for all \( b \in R^0 \). A normalized weight function \( \omega_l \) exists and is unique up to sign.

Fix \( b_0 \in R^0 \) such that \( l \mid N b_0 \) and \( b_0 \not\in lR^0 \). Then \( \omega_l(b_0) \neq 0 \) for any weight function \( \omega_l \neq 0 \) on \( R \). We fix \( \omega_l \) to be the unique weight function on \( R \) such that \( \omega_l(b_0) = 1 \). Then \( \omega_l \) can be computed by using Algorithm 2.2 below, applied to \((R, b_0)\).

Let \( x_i \in I_i \) be a generator of \( I_i \otimes \mathbb{Z}_l \), so that \( x_i^{-1} R^0_i(\mathbb{Z}_l) x_i = R^0(\mathbb{Z}_l) \) and \( l \nmid n_i := N x / N I_i \in \mathbb{Z} \). If \( b \in R^0_i(\mathbb{Z}_l) \), we set
\[
\omega_l(I_i, b) := \left( \frac{n_i}{l} \right) \omega_l(x_i^{-1} b x_i).
\]

This determines a weight function \( \omega_l(I_i, \cdot) \) on \( R_i \). Note that we can always assume \( I_i \subseteq R \) and \( \left( \frac{N I_i}{l} \right) = 1 \), in which case we would have \( R^0_i(\mathbb{Z}_l) = R^0(\mathbb{Z}_l) \) and \( \omega_l(I_i, \cdot) = \omega_l \).
In any case, $b_{0,i} := n_i x_i b_0 x_i^{-1} \in R^0$ is such that $\omega_l(I_i, b_{0,i}) = \omega_l(b_0) = 1$; thus $\omega_l(I_i, \cdot)$ can also be computed by Algorithm 2.2 applied to $(R_i, b_{0,i})$.

**Algorithm 2.2.** Given a pair $(R, b_0)$, where $R$ is a maximal order and $b_0 \in R^0$ is such that $l | N b_0$, but $b_0 \not\in lR^0$, this algorithm computes the unique weight function $\omega_l$ on $R^0$ determined by $\omega_l(b_0) = 1$.

**Input:** $b \in R_0$.

**Output:** $\omega_l(b)$.

1. If $l \nmid N b$, return 0.
2. If $l \nmid N(b + b_0)$, return $\left(\frac{N(b + b_0)}{l}\right)$.
3. Otherwise, there is some $k \in \mathbb{Z}$ is such that $b - k b_0 \in lR^0$.

Find such a $k$, and return $\left(\frac{k}{l}\right)$.

2.3. $g_+(z)$ and weight function. The construction of $g_+(z)$ can be done as follows:

1. Identify a prime $l \neq p$ such that $l \equiv 3 \mod 4$ and $L(f, -l, 1) \neq 0$; in particular, $-\left(\frac{-l}{p}\right)$ has to be equal to the sign of the functional equation for $L(f, s)$. From [B-F-H], there are infinitely many such $l$.

2. Fix a normalized weight function $\omega_l$ on $R$ and transport it to weight functions $\omega_l(I_i, \cdot)$ on $R_i$ as in the previous section. Define another weight function $\omega_p$ on $B^0(\mathbb{Z}_p)$. As $S^0_i \mapsto S^0_i \otimes \mathbb{Q}_p \subset B^0(\mathbb{Z}_p)$ for all $i$, $\omega_p$ can be regarded as a function on $S^0_i$.

3. Define

$$h_i(z) = \frac{1}{2} \sum_{b \in S^0_i} q^{N b / l} \omega_l(b) \omega_p(b).$$

4. Let $a_i = e_f(I_i)$, then

$$g_+(z) = \theta_{-l}(e_f) := \sum_i a_i h_i(z).$$ (2.5)

The weight function $\omega_p(b)$ is a function satisfying:

1. $\omega_p$ is constant mod $p\mathbb{Z}_p$.
2. $\omega_p(a^{-1} b a) = [N a, l]_p \omega_p(b)$ for all $a \in B(\mathbb{Q}_p)$ and $b \in B^0(\mathbb{Z}_p)$. 

(3) \( \omega_p(kb) = \chi_p(k)\omega_p(b) \) for \( k \in \mathbb{Z}^\times \), and \( \chi_p \) is any fixed odd character of \((\mathbb{Z}/p)^\times \) considered as a character on \( \mathbb{Z}_p^\times \), ("odd" means \( \chi(-1) = -1 \)).

When \( \chi_p \) is fixed, there is a unique (up to scalar multiple) function satisfying the above conditions. Recall [P]

\[
B^0(\mathbb{Z}_p) = \{ b = \alpha I + \beta J + \gamma IJ : \alpha, \beta, \gamma \in \mathbb{Z}_p \}
\]

where \( I^2 = a \) and \( J^2 = b \), \( IJ = -JI \); \( a, b \) are negative integers satisfying \([a, b]_p = -1\) and \([a, b]_l = 1\) for all primes \( l \neq p \). We may assume \( a \) is a unit in \( \mathbb{Z}_p \) and \( b \) generates the prime ideal in \( \mathbb{Z}_p \). Then \( \omega_p \) can be defined as follows:

1. when \( \alpha \) is not a unit in \( \mathbb{Z}_p \), \( \omega_p(b) = 0 \).
2. when \( \alpha \) is a unit in \( \mathbb{Z}_p \), \( \omega_p(b) = \chi_p(\alpha) \).

3. An explicit formula

The construction of section 3 singles out, for each \( l \), an explicit form of weight \( 3/2 \) for the formula of Theorem 2.2. This follows the construction of Gross for \( l = 1 \), for which there is a formula with an explicit constant [Gr, Proposition 13.5]. We can extend it to the case of \( l \equiv 1 \pmod{4} \).

**Proposition 3.1.** Let \( l \neq p \) be a prime such that \( l \equiv 1 \pmod{4} \), and let \( -d < 0 \) be a fundamental discriminant. Then

\[
L(f, -d, 1) L(f, l, 1) = \ast \frac{\langle f, f \rangle |c_l(d)|^2}{\sqrt{dl} \langle e_f, e_f \rangle},
\]

where \( \theta_l(e_f) = \sum_{n=1}^{\infty} c_l(n)q^n \), and where \( \ast = 1 \) if \( p \nmid d \), \( \ast = 2 \) if \( p \mid d \).

**Proof.** We deal here with the case \( \left( -\frac{dl}{p} \right) = -1 \) and \( l \nmid d \). When \( \left( -\frac{dl}{p} \right) = +1 \), both sides are trivially 0. The remaining cases follow from similar methods, or from Theorem 2.2.

Consider the divisor

\[
c := \sum_i \frac{1}{2w_i} \sum_{b \in S_i^0 \atop N b = dl} \omega_l(I_i, b) [I_i].
\]
It is clear that \( c_l(d) = \langle c, e_f \rangle \), since \( \langle [I_i], [I_i] \rangle = w_i \) by definition of the height pairing. The pairs \((I_i, b)\) where \( b \in S^0_i \) is such that \( \mathcal{N} b = dl \), modulo conjugation by the units of \( R_i \), give a set of representatives for the special points of discriminant \(-dl\). Thus

\[
c_l = \frac{1}{2} \sum_x \omega_l(x) [x],
\]

where the sum is over the special points \( x = (I_i, b) \) of discriminant \(-dl\), and where \([x] := [I_i]\).

Let \( \mathcal{O} \) be the quadratic order of discriminant \(-d \cdot l\), and let \( \chi_l \) be the genus character of \( \text{Pic}(\mathcal{O}) \) corresponding to this discriminant factorization; namely, if \( A \in \text{Pic}(\mathcal{O}) \), then \( \chi_l(A) := (\frac{\mathcal{N} a}{l}) \), where \( a \in A \) is chosen so that \( l \nmid \mathcal{N} a \).

Recall that \( \text{Pic}(\mathcal{O}) \) acts freely on the special points of discriminant \(-dl\), and since \( \left( \frac{-dl}{p} \right) = -1 \) there are exactly two orbits that are permuted by complex conjugation (see [Gr,§3]). It follows from the definitions that, for \( A \in \text{Pic}(\mathcal{O}) \)

\[
\omega_l(A \cdot x) = \chi_l(A) \omega_l(x),
\]

and also that \( \omega_l(\overline{x}) = \omega_l(x) \) and \([\overline{x}] = [x]\). Consequently, we can write

\[
c = \omega_l(x_0) \sum_{A \in \text{Pic}(\mathcal{O})} \chi_l(A) [A \cdot x_0],
\]

where \( x_0 \) is a fixed special point of discriminant \(-dl\). Since \( l \nmid d \) and \( \omega_l \) is normalized, we have \( \omega_l(x_0) = \pm 1 \).

Apply now [Gr, Proposition 11.2] to \( \chi_l \), obtaining the formula

\[
L(f, \chi_l, 1) = \frac{\langle f, f \rangle \langle c, e_f \rangle^2}{\sqrt{dl} \langle e_f, e_f \rangle}.
\]

Since \( \chi_l \) is a genus character, we have a decomposition

\[
L(f, \chi_l, 1) = L(f, -d, 1) L(f, l, 1),
\]

and the result follows. \( \square \)
4. Examples

4.1. **37A, imaginary twists.** Let \( f = f_{37A} \), the modular form of level 37 and rank 1. Let \( B = B(-2, -37) \), the quaternion algebra ramified precisely at \( \infty \) and 37. A maximal order, and representatives for its right ideal classes, are given by

\[
R = I_1 = \left\langle 1, i, \frac{1 + i + j}{2}, \frac{2 + 3i + k}{4} \right\rangle \quad \text{with } N I_1 = 1,
\]

\[
I_2 = \left\langle 2, 2i, \frac{1 + 3i + j}{2}, \frac{6 + 3i + k}{4} \right\rangle \quad \text{with } N I_2 = 2,
\]

\[
I_3 = \left\langle 4, 2i, \frac{3 + 3i + j}{2}, \frac{6 + i + k}{2} \right\rangle \quad \text{with } N I_3 = 4.
\]

By computing the Brandt matrices, we find a vector

\[
e_f = \frac{[I_3] - [I_2]}{2}
\]

of height \( (e_f, e_f) = \frac{1}{2} \) corresponding to \( f \). Since \( L(f, 1) = 0 \) we know that \( 2\theta_1(e_f) = \theta_1([I_3]) - \theta_1([I_2]) = 0 \). Indeed, one checks that \( R_2 \) and \( R_3 \) are conjugate, which explains the identity \( \theta_1([I_2]) = \theta_1([I_3]) \).

Let now \( l = 5 \). One can compute \( L(f, 5, 1) \approx 5.3548616 \), and thus we expect \( \theta_5(e_f) \) to be nonzero. We note that, by the same reason that the orders are conjugate, we have \( \theta_5([I_3]) = -\theta_5([I_2]) \), except now there’s an extra sign, ultimately coming from the fact that \( (\frac{37}{5}) = -1 \). Thus, \( \theta_5(e_f) = \theta_5([I_3]) \). A basis for \( S_3^0 \) is given by

\[
S_3^0 = \left\langle b_1 = \frac{3i + 2j + k}{4}, \quad b_2 = \frac{7i - 2j + k}{4}, \quad b_3 = \frac{3i - k}{2} \right\rangle,
\]

with the norm in this basis (denoting \( x_1 b_1 + x_2 b_2 + x_3 b_3 \) by \( (x_1, x_2, x_3) \))

\[
N_3(x_1, x_2, x_3) = 15x_1^2 + 20x_2^2 + 23x_3^2 - 8x_2x_3 - 14x_1x_3 - 4x_1x_2.
\]
Choose $\omega_5$ so that $\omega_5(b_1) = +1$. Then $\omega_5(I_3, \cdot) = \omega_5$ can be computed as

$$
\omega_5(I_3, (x_1, x_2, x_3)) = \begin{cases} 
0 & \text{if } 5 \nmid N_3(x_1, x_2, x_3), \\
\left(\frac{x_2 + x_3}{5}\right) & \text{if } x_2 + x_3 \not\equiv 0 \pmod{5}, \\
\left(\frac{\cdot}{5}\right) & \text{otherwise}.
\end{cases}
$$

Table 1 shows the values of $c_5(d)$ and $L(f, -d, 1)$, where $0 < -d < 200$ is a fundamental discriminant such that $(\frac{-d}{37}) \neq -1$. The formula

$$
L(f, -d, 1) = k_5 \frac{c_5(d)^2}{\sqrt{d}} \cdot \begin{cases} 
1 & \text{if } (\frac{-d}{37}) = +1, \\
2 & \text{if } (\frac{-d}{37}) = 0, \\
0 & \text{if } (\frac{-d}{37}) = -1,
\end{cases}
$$

is satisfied, where

$$
k_5 = \frac{2(f, f)}{L(f, 5, 1)\sqrt{5}} \approx 4.902778763973580121708449663733...
$$

Note that in the case $(\frac{-d}{37}) = -1$ it is trivial that $c_5(d) = L(f, -d, 1) = 0$. 

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<th>$L(f, -d, 1)$</th>
<th>$d$</th>
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4.2. **43A, imaginary twists.** Let \( f = f_{43A} \), the modular form of level 43 and rank 1. Let \( B = B(-1, -43) \), the quaternion algebra ramified precisely at \( \infty \) and 43. A maximal order, and representatives for its right ideals classes, are given by

\[
R = I_1 = \left\langle 1, i, \frac{1 + j}{2}, \frac{i + k}{2} \right\rangle \quad \text{with} \quad \mathcal{N} I_1 = 1,
\]
\[
I_2 = \left\langle 2, 2i, \frac{1 + 2i + j}{2}, \frac{2 + 3i + k}{2} \right\rangle \quad \text{with} \quad \mathcal{N} I_2 = 2,
\]
\[
I_3 = \left\langle 3, 3i, \frac{1 + 2i + j}{2}, \frac{2 + 5i + k}{2} \right\rangle \quad \text{with} \quad \mathcal{N} I_3 = 3,
\]
\[
I_4 = \left\langle 3, 3i, \frac{1 + 4i + j}{2}, \frac{4 + 5i + k}{2} \right\rangle \quad \text{with} \quad \mathcal{N} I_4 = 3.
\]

By computing the Brandt matrices, we find a vector

\[
e_f = \frac{[I_4] - [I_3]}{2}
\]

of height \( \langle e_f, e_f \rangle = \frac{1}{2} \) corresponding to \( f \). We can use \( l = 5 \), since \( L(f, 5, 1) \approx 4.8913446 \) is nonzero. Again, we find \( \theta_5(e_f) = \theta_5([I_4]) \); in a convenient basis of \( S_0^0 \), the norm is

\[
\mathcal{N}_4(x_1, x_2, x_3) = 15x_1^2 + 23x_2^2 + 24x_3^2 + 12x_2x_3 + 8x_1x_3 + 2x_1x_2,
\]

and \( \omega_5(I_4, \cdot) = -\omega_5 \) can be computed by

\[
\omega_5(I_4, (x_1, x_2, x_3)) = \begin{cases} 
0 & \text{if } 5 \nmid \mathcal{N}_4(x_1, x_2, x_3), \\
\left( \frac{2x_2 + 3x_3}{5} \right) & \text{if } x_2 \not\equiv x_3 \pmod{5}, \\
\left( \frac{x_1}{5} \right) & \text{otherwise}.
\end{cases}
\]
**COMPUTATION OF CENTRAL VALUE**

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**Table 2.** Coefficients of $\theta_5(e_f)$ and central values for $f = f_{43A}$

Table 2 shows the values of $c_5(d)$ and $L(f, -d, 1)$, where $0 < -d < 200$ is a fundamental discriminant such that $\left(\frac{-d}{43}\right) \neq -1$. The formula

$$L(f, -d, 1) = k_5 \frac{c_5(d)^2}{\sqrt{d}} \cdot \begin{cases} 
1 & \text{if } \left(\frac{-d}{43}\right) = +1, \\
2 & \text{if } \left(\frac{-d}{43}\right) = 0, \\
0 & \text{if } \left(\frac{-d}{43}\right) = -1,
\end{cases}$$

is satisfied, where

$$k_5 = \frac{2(f, f)}{L(f, 5, 1)\sqrt{5}} \approx 5.452729672681734385570722785283...$$

Note that in the case $\left(\frac{-d}{43}\right) = -1$ it is trivial that $c_5(d) = L(f, -d, 1) = 0$.

### 4.3. 389A, Imaginary Twist

Let $f = f_{389A}$, the modular form of level 389 and rank 2. Let $B = B(-2, -389)$, the quaternion algebra ramified precisely at $\infty$ and 389. A maximal order, with 33 ideal classes, is given by

$$R = \left\langle 1, i, \frac{1 + i + j}{2}, \frac{2 + 3i + k}{4} \right\rangle.$$
There is a vector $e_f$ of height $\langle e_f, e_f \rangle = 5/2$ corresponding to $f$. We can use $l = 5$, since $L(f, 5, 1) \approx 8.9092552$.

We have omitted the 33 ideal classes; however, the computation of $\theta_l(e_f)$ involves only 14 distinct theta series. In table 3 we give the value of $e_f$ and the coefficients of the norm form $N_i$ and of $b_{0,i}$ on chosen bases of $S_0^i$.

Each row in the table allows one to compute an individual theta series

$$h_i(z) = \frac{1}{2} \sum_{b \in \mathbb{Z}^3} w_5(I_i, b)q^{N_i(b)/5}.$$ 

The ternary form corresponding to a sextuple $(A_1, A_2, A_3, A_{23}, A_{13}, A_{12})$ is

$$N_i(x_1, x_2, x_3) = A_1x_1^2 + A_2x_2^2 + A_3x_3^2 + A_{23}x_2x_3 + A_{13}x_1x_3 + A_{12}x_1x_2,$$
and the weight function $\omega_5(I_i, \cdot)$ is computed as per Algorithm 2.2 applied to $(\mathbb{Z}^3, b_{0,i})$. As an example, we show how to compute $h_1(z)$. First, we have

$$\mathcal{N}_1(x_1, x_2, x_3) = 15x_1^2 + 107x_2^2 + 416x_3^2 - 100x_2x_3 - 8x_1x_3 - 14x_1x_2.$$ 

A simple calculation shows that

$$\mathcal{N}_1(x_1 + 2, x_2 + 4, x_3 + 0) \equiv 4x_1 + 3x_2 + 4x_3 \pmod{5},$$

provided that $\mathcal{N}_1(x_1, x_2, x_3) \equiv 0 \pmod{5}$. Thus, $\omega_5$ can be computed as

$$\omega_5(I_1, (x_1, x_2, x_3)) = \begin{cases} 
0 & \text{if } 5 \nmid \mathcal{N}_1(x_1, x_2, x_3), \\
\left(\frac{4x_1 + 3x_2 + 4x_3}{5}\right) & \text{if } 4x_1 + 3x_2 + 4x_3 \not\equiv 0 \pmod{5}, \\
\left(\frac{x_3}{5}\right) & \text{otherwise},
\end{cases}$$

and we have

$$h_1(z) = q^3 - q^{12} - q^{27} + q^{39} + q^{40} + q^{48} - q^{83} - 2q^{92} + O(q^{100}).$$

Finally, we combine all of the theta series in

$$\theta_5(e_f) = \sum_{i=1}^{14} a_i h_i(z)$$

Table 4 shows the values of $c_5(d)$ and $L(f, -d, 1)$, where $0 < -d < 200$ is a fundamental discriminant such that $\left(\frac{-d}{389}\right) \neq +1$. The formula

$$L(f, -d, 1) = k_5 \frac{c_5(d)^2}{\sqrt{d}} \cdot \begin{cases} 
1 & \text{if } \left(\frac{-d}{389}\right) = -1, \\
2 & \text{if } \left(\frac{-d}{389}\right) = 0, \\
0 & \text{if } \left(\frac{-d}{389}\right) = +1,
\end{cases}$$

is satisfied, where

$$k_5 = \frac{2(f, f)}{5L(f, 5, 1)\sqrt{5}} \approx 7.886950806206592817689630792605...$$

Note that in the case $\left(\frac{-d}{389}\right) = +1$ it is trivial that $c_5(d) = L(f, -d, 1) = 0$. 
Table 4. Coefficients of $\theta_5(e_f)$ and central values for $f = f_{389A}$

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**References**


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