L-Functions of Hypergeometric Motives

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Hypergeometry I

Joint with Fernando Rodriguez-Villegas.

Hypergeometric data :

$$\alpha = (\alpha_1, \ldots, \alpha_r)\ , \quad \beta = (\beta_1, \ldots, \beta_r)\ ,$$

Pochammer symbol $(a)_m = a(a+1)\cdots(a+m-1)$, (generalized) hypergeometric function :

$$F(\alpha, \beta; t) = \sum_{m \geq 0} \frac{\prod_{1 \leq j \leq r}(\alpha_j)_m}{\prod_{1 \leq j \leq r}(\beta_j)_m} t^m$$

(classical case $\beta_r = 1$, so $(\beta_r)_m = m!$).

Hypergeometry II

Satisfies a linear differential equation with regular singular points at 0, 1, and ∞.
Monodromy at 1 is a complex reflection. Characteristic polynomials of the monodromy at 0 and ∞ :

$$P_0(T) = \prod_j (T - e^{2i\pi\beta_j}) \,, \quad P_\infty(T) = \prod_j (T - e^{2i\pi\alpha_j}) \,.$$

Hypergeometry III

Hypergeometric Assumption : we will assume that these characteristic polynomials are products of coprime and cyclotomic polynomials. Several equivalent formulations. The simplest is :

$$P_\infty(T)/P_0(T) = \prod_{\nu \geq 1} (T^\nu - 1)^{\gamma_\nu}$$

for some $\gamma_\nu \in \mathbb{Z}$ such that $\sum_\nu \nu\gamma_\nu = 0$. We set $\gamma(T) = \sum_\nu \gamma_\nu T^\nu$.

More complicated but more concrete equivalent statement : for any integer A, let $R(A)$ be the set of $\phi(A)$ rational numbers a/A such that $(a, A) = 1$ and $0 \leq a < A$. The assumption means that there exist integers A_i and B_i such that $\alpha = \bigcup_i R(A_i)$ and $\beta = \bigcup_i R(B_i)$ (concatenation and not union since the A_i or B_i are not necessarily distinct). It is clear that we may assume that the A_i are distinct from the B_i, and that $r = \sum_i \phi(A_i) = \sum_i \phi(B_i)$.

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Hypergeometry IV

Examples :
For $r = 1$, only possibility $\alpha = (1/2)$, $\beta = (0)$. For $r = 2$ exactly 13 possibilities (enumeration given below), for instance $\alpha = (1/3, 2/3)$, $\beta = (0, 0)$.
Examples for $r = 4$:

$$\alpha = (1/2, 1/2, 1/3, 2/3)\ , \quad \beta = (1/6, 1/6, 5/6, 5/6)\ ,$$

or

$$\alpha = (1/5, 2/5, 3/5, 4/5)\ , \quad \beta = (0, 0, 0, 0)\ .$$

In this example,

$$P_0(T) = (T - 1)^4\ , \quad P_\infty(T) = T^4 + T^3 + T^2 + T + 1\ ,$$

$P_\infty(T)/P_0(T) = (T^5 - 1)/(T - 1)^5$, hence $\gamma(T) = T^5 - 5T$.

Associated Motives I

Theorem (essentially N. Katz) : If (α, β) is hypergeometric data satisfying the fundamental hypergeometric assumption above, for each $t \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ there exists a motive $H(\alpha, \beta; t)$, which is defined over $\mathbb{Q}$, of rank r, and pure with a certain weight w, explicitly given in terms of data.

Conjecture : there must therefore exist a global L-function $\Lambda(s)$, holomorphic sur $\mathbb{C}$, with functional equation $\Lambda(w+1-s) = \pm\Lambda(s)$, and with an Euler product

$$\Lambda(s) = N^{s/2} L_\infty(s) L(s) \quad \text{with} \quad L(s) = \prod_p L_p(p^{-s}) = \sum_{n \geq 1} \frac{a(n)}{n^s} , \quad \text{where}$$

$$L_p(T) = \prod_{j=1}^{r} (1 - \xi_j T)^{-1} , \quad \text{with} \quad |\xi_j| = p^{w/2}$$

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Associated Motives II

Katz's theorem gives precise recipes for the weight w, the archimedean factor L_∞, and the factors L_p for the good primes p. On the other hand, not for the conductor N (of course divisible only by the bad primes), nor for the L_p for bad primes, nor for the sign $\pm$ of the f.e. Goal of our work : numerical check of the conjecture, and deduce conjectures for the unknown quantities (N, L_p for p bad, and $\pm$).

We have treated hundreds of examples, in degree $r = 2$, $r = 4$, and $r = 6$. One of the best-known is the case

$$\alpha = (1/5, 2/5, 3/5, 4/5)\ , \quad \beta = (0, 0, 0, 0)\ ,$$

which in fact corresponds to the theory of mirror symmetry on the Calabi–Yau quintic threefold

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5tx_1x_2x_3x_4x_5\ .$$

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Details on the Motive : The function $\mathcal{L}(x)$ I

Recall that if P_0 and P_∞ are the characteristic polynomials of the monodromies at 0 and ∞ of the hypergeometric diff. eq. we set

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for some $\gamma_\nu \in \mathbb{Z}$ such that $\sum_\nu \nu\gamma_\nu = 0$, and $\gamma(T) = \sum_\nu \gamma_\nu T^\nu$.

We define

$$\mathcal{L}(x) := \sum_{\nu\geq 1}\gamma_\nu(\{\nu x\} - 1/2)\ ,$$

where $\{z\}$ is the fractional part of z. Then $\mathcal{L}$ is locally constant, right continuous, periodic of period 1, and such that

$$\mathcal{L}^+(-x) = -\mathcal{L}^-(x) \quad \text{where} \quad \mathcal{L}^\pm(x) := \lim_{y\to x^\pm}\mathcal{L}(x)\ .$$

Note that by right continuity we have $\mathcal{L}^+(x) = \mathcal{L}(x)$.

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The function $\mathcal{L}(x)$ II

We define the weight w of the motive as

$$w = \max_{x\in[0,1[} \mathcal{L}(x) - \min_{x\in[0,1[} \mathcal{L}(x) - 1 \; .$$

It is a nonnegative integer,

$$\max_{x\in[0,1[} \mathcal{L}(x) = - \min_{x\in[0,1[} \mathcal{L}(x) = \frac{w+1}{2} \; ,$$

so that $\mathcal{L}^{\pm}(x) + (w+1)/2$ is integral-valued.

Since

$$\mathcal{L}^{-}(1) = \sum_{\nu\geq 1} \gamma_\nu(\nu - 1/2) = -(1/2)\sum_{\nu\geq 1}\gamma_\nu$$

and $\mathcal{L}^{-}(x) + (w+1)/2$ is integral valued, it follows that

$$w + 1 \equiv \sum_{\nu\geq 1} \gamma_\nu \pmod 2 \; .$$

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The Hodge Polynomial I

Set

$$\ell(x) := \mathcal{L}^+(x) - \mathcal{L}^-(x) ,$$

which measures the jumps at discontinuities. It is integral valued and in fact

$$\ell(x) = |\{i|\ \alpha_i = x\}| - |\{j|\ \beta_j = x\}| .$$

We also have $\ell(-x) = \ell(x)$.

We define the Hodge polynomial $h(T)$ as

$$h(T) := \sum_{\ell(x)>0} T^{\mathcal{L}^-(x)+(w+1)/2}[\ell(x)] ,$$

where

$$[\ell] := 1 + T + \cdots + T^{\ell-1} .$$

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The Hodge Polynomial II

Easy to show that :

1. h is reciprocal of degree w ($T^w h(1/T) = h(T)$) and has nonnegative integer coefficients.
2. $$h(T) = \sum_{\ell(x)<0} T^{\mathcal{L}^+(x)+(w+1)/2}[-\ell(x)]\ .$$

Conjecture (Corti–Golyshev) : The Hodge numbers of the motive $H(\alpha, \beta; t)$ are the coefficients of $h(T)$:

$$h(T) = \sum_{p+q=w} h^{p,q} T^p\ .$$

As already mentioned this gives the gamma factors of $\Lambda(s)$ at least when w is odd (otherwise must look at action of complex conjugation on $(w/2, w/2)$ piece) :

$$L_\infty(s) = \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h_{p,q}}\ .$$

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Recipe for the L_p for good primes p I

We will define below the trace of Frobenius $a_q(t) = \mathrm{Tr}(\mathrm{Frob}_q)$, and we will then set as usual

$$L_p(T) = \exp\left(\sum_{f\geq 1} \frac{a_{p^f}(t)}{f} T^f\right) .$$

The fundamental quantity which occurs in the definition of $a_q(t)$ is the following : for any multiplicative character χ of $\mathbb{F}_q^*$ we set

$$Q(\chi) = \prod_{\nu} \mathfrak{g}(\chi^{\nu})^{\gamma_{\nu}} ,$$

where γ_{ν} is as above and $\mathfrak{g}(\chi^{\nu})$ is the corresponding Gauss sum. Using Möbius inversion and elementary formulas for Gauss sums, for suitable easily computed integers a and b we also have

$$Q(\chi) = (-1)^a q^b \frac{\prod_i \prod_{d\mid A_i} \mathfrak{g}(\chi^{-d})^{\mu(A_i/d)}}{\prod_i \prod_{d\mid B_i} \mathfrak{g}(\chi^{-d})^{\mu(B_i/d)}} ,$$

where the A_i and B_i are as above.

Recipe for the L_p for good primes p II

The hypergeometric assumptions imply that $Q(\chi)$ can be expressed only in terms of Jacobi sums (which belong to a smaller number field), see below.

Finally, for $t \neq 0$, 1, or ∞ we set

$$a_q(t) = \frac{q^d}{1-q}\left(1 + \sum_{\chi \neq \chi_0} \chi(Mt)Q(\chi)\right) ,$$

where χ ranges over all nontrivial characters of $\mathbb{F}_q$, for certain constants d and M which I do not define here (of course M is only a normalizing factor, could change Mt into t, but less clean). By Galois theory this is in $\mathbb{Q}$, and it is easy to show that it is in fact in $\mathbb{Z}$, as it should if it is indeed a trace.

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Computation of the L_p for good primes p I

In the case $(\alpha, \beta) = (1/5, 2/5, 3/5, 4/5), (0, 0, 0, 0)$ mentioned above, we have the precise formula :

$$a_q(t) = \frac{1}{1-q}\left(1 + \sum_{\chi \neq \chi_0} \chi(5^5 t) J(\chi, \chi, \chi, \chi)\right) ,$$

where $q = p^f$ and J is the generalized Jacobi sum :

$$J(\chi_1, \dots, \chi_r) = \sum_{x_1 + \cdots + x_r = 1} \chi_1(x_1) \cdots \chi_r(x_r) .$$

Elementary idea 1 : the local factors L_p have degree 4 and are Weil-symmetrical. Therefore only need to compute a_p and a_{p^2}.
Elementary idea 2 : To obtain B terms of the Dirichlet series, need of course a_p for $p \leq B$, but a_{p^2} only for $p \leq B^{1/2}$.

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Computation of the L_p for good primes p II

Main computational task is the computation of the Jacobi sums.

(1) : Direct method.

By elementary properties of Jacobi sums, an order r Jacobi sum as above can be expressed as a product of $r-1$ usual Jacobi sums of order 2, of the form :

$$J(\chi_1,\chi_2)=\sum_{x\in\mathbb{F}_q}\chi_1(x)\chi_2(1-x)\ .$$

The direct computation of these sums requires essentially $O(q)$ operations, neglecting the computation of the χ_i values, and since we need $q-2$ Jacobi sums, the total cost of the computation of $a_q(t)$ is of the order of $O((r-1)q^2)$.

Computation of the L_p for good primes p III

(2). The use of Gauss sums.

Recall that

$$\mathfrak{g}(\chi)=\sum_{x\in\mathbb{F}_q^*}\chi(x)\exp(2i\pi\,\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x)/p)\;.$$

It is well-known that

$$J(\chi_1,\dots,\chi_r)=\frac{\prod_j\mathfrak{g}(\chi_j)}{\mathfrak{g}(\prod_j\chi_j)}$$

when all the characters are nontrivial (and simpler formulas otherwise). A priori, why use Gauss sums, more complicated than Jacobi sums, and belong to a larger number field than Jacobi sums ? A posteriori, they are in fact useful.

Computation of the L_p for good primes p IV

Let ω be a generator of the cyclic group $\widehat{\mathbb{F}_q^*}$. Taking our favorite example, we thus have

$$a_q(t) = \frac{1}{1-q}\left(1 + \sum_{1\le r\le q-2} \omega^r(5^5 t) J(\omega^r,\omega^r,\omega^r,\omega^r)\right)\ .$$

If we compute once and for all the $\mathfrak{g}(\omega^r)$ for $1 \le r \le q-2$, the above computation requires time $O(q)$, for a total time of $O(q^2)$, hence gain of a factor $r-1$.

(3). Use of Θ functions.

Idea of S. Louboutin : when $q = p$ is prime (by far the largest part of the computation), Gauss sums are direcly linked to the root numbers in the functional equation of L-functions, but even better, of Θ-functions, associated to the characters. These can be computed in time $O(q^{1/2+\varepsilon})$, so a considerable gain : combined with the preceding ideal, total cost $O(q^{3/2+\varepsilon})$.

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Computation of the L_p for good primes p V

(4). Use of p-adic methods.
The marvelous formula of Gross–Koblitz allows us to express Gauss sums (for all q, not only for $q = p$) in terms of the Morita p-adic gamma function Γ_p. There exist efficient methods to compute this function, time $O(p^{1+\varepsilon})$, so in principle no more efficient, even less than using theta functions.

The main interest of the method is that we only need to compute values modulo p or p^2, since we know that $a_q(t)$ is an integer and that we have Weil–Deligne bounds. Although this is a $O(q^2)$ method, it is the best available when $q = p^2$, and even when $q = p$, since we can work mod p and the implicit constant of $O()$ is very small, it is quite competitive in practice ($p \leq 10^4$ for instance).

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Verification of the Functional Equation I

(Also Implemented in several computer packages such as T. Dokschitser and `lcalc` of M. Rubinstein).

Even assuming that we know N and L_p for p bad, to check the functional equation $\Lambda(w+1-s) = \pm\Lambda(s)$ need to compute $\Lambda(s)$. For this, efficient general recipes :

Let $K(x)$ be the inverse Mellin transform of $L_\infty(s)$, i.e.,

$$\int_0^\infty t^{s-1}K(t)\,dt = L_\infty(s)\ ,$$

and set

$$\gamma(s,x) = \int_x^\infty t^{s-1}K(t)\,dt\ .$$

For any $t_0 > 0$ we have

$$L_\infty(s)L(s) = \sum_{n\geq 1}\frac{a(n)}{n^s}\gamma(s, nt_0/N^{1/2}) \pm \sum_{n\geq 1}\frac{a(n)}{n^{w+1-s}}\gamma(w+1-s, n/(t_0N^{1/2}))\ .$$

This series converges exponentially fast, but slows down if N is large.

Verification of the Functional Equation II

The “method” is thus as follows : we guess values for N and the bad Euler factors L_p (a lot of information is known about them), and we test if the above expression is independent of t_0, by choosing for instance $t_0 = 1$ and $t_0 = 1.1$.

Verification of the Functional Equation III

In the above computation, must do three costly computations. First, compute the inverse Mellin transform $K(x)$ (see below). Second, compute the partial Mellin transforms $\int_x^\infty t^{s-1}K(t)\,dt$. Finally, the third consists in computing the two series of the form. $\sum_{n\geq 1}(a(n)/n^s)\gamma(s,nt_0/N^{1/2})$.

Elementary idea 3 : to check the functional equation, there is no need to come back to the Dirichlet series by computing $\int_x^\infty t^{s-1}K(t)$. Indeed, we can write

$$\Lambda(s)=\int_0^\infty t^{s-1}F(N^{1/2}t)\,dt\ ,$$

so the f.eq. is in our case equivalent to

$$F(1/t)=\pm t^{w+1}F(t)\ .$$

Thus we only check this, and so only need to compute $K(x)$ and series sums.

Verification of the Functional Equation III

In the above computation, must do three costly computations. First, compute the inverse Mellin transform $K(x)$ (see below). Second, compute the partial Mellin transforms $\int_x^\infty t^{s-1}K(t)\,dt$. Finally, the third consists in computing the two series of the form. $\sum_{n\geq 1}(a(n)/n^s)\gamma(s, nt_0/N^{1/2})$.

Elementary idea 3 : to check the functional equation, there is no need to come back to the Dirichlet series by computing $\int_x^\infty t^{s-1}K(t)$. Indeed, we can write

$$\Lambda(s) = \int_0^\infty t^{s-1}F(N^{1/2}t)\,dt\ ,$$

so the f.eq. is in our case equivalent to

$$F(1/t) = \pm t^{w+1}F(t)\ .$$

Thus we only check this, and so only need to compute $K(x)$ and series sums.

Verification of the Functional Equation IV

The archimedean factor $L_\infty(s)$ is up to exponential factors Q^s, of the form $\Gamma(s)$, $\Gamma(s)\Gamma(s-1)$, $\Gamma(s)\Gamma(s-1)\Gamma(s-2)$ for instance (can of course also have $\Gamma(s/2)$, etc...) The precise recipe for this factor in a very general context is due to Serre around 1970. It depends on the Hodge numbers $h^{p,q}$ of the motive, which can easily be computed from the hypergeometric data, as can the weight w. More precisely : if as usual $\Gamma_{\mathbb{C}}(s)=(2\pi)^{-s}\Gamma(s)$ and $\Gamma_{\mathbb{R}}(s)=\pi^{-s/2}\Gamma(s/2)$ then

$$L_\infty(s)=N^{s/2}\prod_{\substack{p<q\\ p+q=w}}\Gamma_{\mathbb{C}}(s-p)^{h_{p,q}}\prod_{p=q=w/2}\Gamma_{\mathbb{R}}(s-p)^{h^+_{p,p}}\ ,$$

where the second product of course occurs only in even weight and $h^+_{p,p}$ is the dimension of the $+$-eigenspace of complex conjugation on $H^{p,p}$ of the motive.

Verification of the Functional Equation V

The computation of the inverse Mellin transform of the archimedean factor is a fundamental problem for which many solutions exist, and of course have been implemented.

Recall that the inverse Mellin transform is given by

$$K(x) = \frac{1}{2i\pi} \int_{\sigma - i\infty}^{\sigma + i\infty} \gamma(s) x^{-s} \, ds \;,$$

for σ sufficiently large. We move the line of integration to the left until $-\infty$, and we catch all the residues along the way. This allows us to express $K(x)$ as a power series, together with finitely many logarithmic terms or negative powers of x.

Verification of the Functional Equation VI

It is easy to show that these series converge like the series

$$\sum_{n\geq 0}(-1)^n\frac{x^n}{(n!)^{r/2}}\ ,$$

where r is the degree of the motive, in other words the number of α or of β, or equivalently the number of $\Gamma(s/2)$ factors, recalling that $\Gamma(s-p)$ counts for 2.
In the case $r=2$, the factor is $\Gamma(s)$ whose inverse Mellin transform is e^{-x}, nothing to add. For $r=4$, the inverse Mellin transform is essentially the well-known K-Bessel function $K_1(x)$. For $r=6$ the convergence is in $1/n!^3$.

Although some cancellation exists, it is no problem to use the above series even when x is large, for $r\geq 2$, and even better for $r\geq 4$. Otherwise, there exist asymptotic expansions for $x\to\infty$.

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Conjectures I

Three types of prime numbers :

(1) Wild : those which divide the denominator of one of the α_j or β_j.

(2) Tame : not wild, and such that either $v_p(t) > 0$, $v_p(1/t) > 0$, or $v_p(t-1) > 0$.

(3) Good : all the other primes.

We have given the recipe for $L_p(T)$ when p is good, and in that case $p \nmid N$, the conductor. When p is tame, we have been able to obtain precise (but complicated) conjectures, both for $L_p(T)$ and for $v_p(N)$. They should not be too difficult to prove.

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Conjectures II

On the other hand, when p is wild, we have a lot of experimental data and conjectures, but very little precise conjectures. Sample phenomenon, only partly understood : if p is wild and $v_p(t)$ tends to $-\infty$, then $v_p(N)$ has a behavior of the form (almost random numbers) :

$$12,\ 8,\ 9,\ 6,\ 5,\ 6,\ 4,\ 3,\ 0,\ 2,\ 2,\ 0,\ 2,\ 2,\ 0,\ 2,\ 2,\ 0$$

for $t = -1,\ -2,\ -3, \cdots$. In other words it starts to "decrease", with possible local increases, then it reaches 0 for a certain critical value of $v_p(t)$, and then it is periodic with small amplitued. We have a precise conjecture for the critical value.

Example in dimension 0 : the Belyi polynomials which occur for suitable hypergeometric data : let $a \geq 2$ and $b \geq 2$ be coprime (not absolutely necessary) and let

$$P_t(X) = X^a(1-X)^b - \frac{a^a b^b}{(a+b)^{a+b} t}\,.$$

Then the discriminant N of the number field generated by P_t has similar behavior for $p \mid ab(a+b)$ (i.e., wild).

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Examples

We give examples for $r = 2$. As mentioned, there are exactly 13 pairs (α, β) of possible hypergeometric data. Three have motivic weight 0 (hence functional equation $s \mapsto 1 - s$), corresponding to three families of number fields. The other ten have motivic weight 1 (func. eq. $s \mapsto 2 - s$), and correspond to families of elliptic curves, hence by Wiles to families of modular cusp forms of weight 2.

Even though very explicit and general, we do not know how to prove modularity of these ten families without Wiles, and a dream is that perhaps it could be possible, since the data is purely of a combinatorial (as opposed to geometric) nature.

Case (1)

$$\alpha = \{1/2, 1/2\}, \quad \beta = (0, 0) .$$

$$a_p(t) = \frac{1}{1-p}\left(1 + \sum_{1 \le r \le p-2} J(r,r)^2 \omega^r(16t)\right) .$$

If $v_p(t) = 0$:

$$a_p(t) = p - 3 - N_p(t) ,$$

$N_p(t)$ number of affine $\mathbb{F}_p$-points on $x(x-1)y(y-1) = a$, with $a = 1/(16t)$.

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If $v_p(t) = 0$:

$$pa_p(t) = p(p^2+6p-12) - \left(\left(\frac{-3t}{p}\right) + \left(\frac{t}{p}\right)_3 J(\rho_3, \rho_3)^2 + \left(\frac{t}{p}\right)_3^{-1} J(\rho_3^{-1}, \rho_3^{-1})^2\right) -$$

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$N_p(t)$ number of affine $\mathbb{F}_p$-points on the genus 2 curve :
$y^3(1-y)^3 = x(1-x)/a$, with $a = 16/t$.

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$$Y^2 = X^3 - 3aX + 4a + a^2/4 .$$

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$$\alpha = \{1/3, 2/3\}, \quad \beta = (1/6, 5/6)\ .$$

If $v_p(t) = 0$:

$$a_p(t) = p - \left(\frac{t}{p}\right) + \left(\frac{4t}{p}\right)_3 J(\rho_3, \rho_3) + \left(\frac{4t}{p}\right)_3^{-1} J(\rho_3^{-1}, \rho_3^{-1}) - N_p(t)\ ,$$

$N_p(t)$ number of affine $\mathbb{F}_p$-points on the genus 2 curve :
$y^3(1-y)^3 = x(1-x)/a$, with $a = 16/t$.

$$a_p(t) = p + 1 - M_p(t)\ ,$$

$M_p(t)$ number of projective points on

$$Y^2 = X^3 - 3aX + 4a + a^2/4\ .$$

Case (12)

$$\alpha = \{1/3, 2/3\}, \quad \beta = (1/6, 5/6) \ .$$

If $v_p(t) = 0$:

$$a_p(t) = p - \left(\frac{t}{p}\right) + \left(\frac{4t}{p}\right)_3 J(\rho_3, \rho_3) + \left(\frac{4t}{p}\right)_3^{-1} J(\rho_3^{-1}, \rho_3^{-1}) - N_p(t) \ ,$$

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Case (13)

$$\alpha = \{1/4, 3/4\}, \quad \beta = (1/6, 5/6)\ .$$

If $v_p(t) = 0$:

$$a_p(t) = p - 2\left(\frac{3t}{p}\right) - \left(\frac{4t}{p}\right)_3 - \left(\frac{4t}{p}\right)_3^{-1} - N_p(t)\ ,$$

$N_p(t)$ number of affine $\mathbb{F}_p$-points on $y^4(1-y)^2 = x^2(1-x)/a$, with $a = 27/(4t)$.

$$a_p(t) = p + 1 - M_p(t)\ ,$$

$M_p(t)$ number of projective points on

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