The L-function of the quintic

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1 Introduction

This is a report on work our group did at the workshop *Higher rank L-functions* in Benasque, July 2009. Our group consisted of: Sal Baig, Philip Candelas, Henri Cohen, Xenia de la Ossa, Fernando Rodriguez Villegas and Mark Watkins. The goal was to compute the full L-function of the principal piece of the middle cohomology of the quintic

$$X_\psi : \qquad x_1^5 + \cdots + x_5 - 5\psi x_1 \cdots x_5 = 0,$$

for arbitrary $\psi \in \mathbb{Q}$.

Concretely, fix $\psi \in \mathbb{Q}$ with $\psi^5 \neq 1$. Then X_ψ is a smooth projective Calabi-Yau threefold. Consider the abelian subgroup of automorphisms

$$A := \{(\zeta_1, \ldots, \zeta_5) \,|\, \zeta_i^5 = 1, \zeta_1 \cdots \zeta_5 = 1\},$$

acting by $x_i \mapsto \zeta_i x_i$ and let $V = V_\psi$ be the subspace of $H^3(X_\psi, \mathbb{C})$ fixed by A.

Our goal is to:

i) Compute the complete L-function $\Lambda(V, s)$ of V, i.e, compute all of its Euler factor including those for bad primes and at infinity.

ii) Check numerically the functional equation of $\Lambda(V, s)$ and determine the corresponding sign.

iii) Check, if possible, the modularity of $L(V, s)$.

2 The L-function

By the general recipe (described in Serre [?] for the total space $H^k(X)$ of a smooth projective variety X) the shape of the L-function is as follows

$$\Lambda(V,s) = N^{s/2} L_\infty(V,s) \prod_p L_p(V,p^{-s})^{-1},$$

where N is the conductor, a positive integer, L_∞ is a product of gamma factors and $L_p(V,T)$ is a polynomial, generically of degree equal to $\dim V$.

2.1 Gamma factors and numerical test of the functional equation

The gamma factors are determined by the Hodge numbers of V. It is known that $\dim V = 4$ and that in fact $h^{p,q}(V) = 1$ for $p = 0, 1, \dots, 3$ and $p + q = 3$. This yields the following value for the Euler factor at ∞.

$$L_\infty(V,s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-1).$$

Let

$$L(V,s) := \prod_p L_p(V,p^{-s})^{-1} = \sum_{n\geq 1} \frac{a_n}{n^s}.$$

By using the Mellin transform we can write

$$\Lambda(V,s) = \int_0^\infty \varphi(t) t^s \frac{dt}{t},$$

where

$$\varphi(t) := \sum_{n\geq 1} a_n\, k\left(\frac{nt}{\sqrt{N}}\right), \qquad k(t) := \frac{1}{\pi\sqrt{t}} K_1\left(4\pi\sqrt{t}\right) \tag{1}$$

and K_1 is the usual K-Bessel function. The point is that $k(t)$ is the inverse Mellin transform of $L_\infty(V,s)$.

It is know that

$$K_1(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x}, \qquad x \to \infty.$$

Since V is pure of weight 3, being a subspace of $H^3(X_\psi)$ where X_ψ is smooth and projective, we know that

$$a_n = O(n^{3/2+\epsilon}),$$

for any $\epsilon > 0$. Hence the definition (1) gives φ as a sum of exponentially decaying terms. To compute it to a given accuracy we will need, as a rule of thumb, a number of terms in the series proportional to $\sqrt{N}$. The size of N will therefore be crucial for the feasibility of the calculations.

Since V is a piece of H^3 the expected functional equation is

$$\Lambda(4-s) = \epsilon\Lambda(s), \qquad \epsilon = \pm 1.$$

By taking the inverse Mellin transform this is equivalent to

$$\varphi(t^{-1}) = \epsilon t^4 \varphi(t).$$

Our numerical test will be to compute an approximation to the ratio

$$\varphi(t^{-1})/t^4\varphi(t)$$

for $t \approx 1$. The result should be close to $\epsilon = \pm 1$.

2.2 Hypergeometric trace and Euler factors

Let S be the finite set of primes p consisting of $p = 5$ and those satisfying $\psi^5 \equiv 1 \bmod p$ or $\psi \equiv \infty \bmod p$ (i.e., such that p divides the denominator of ψ). Any prime p outside S is a good prime and the corresponding Euler factor has the form

$$L_p(T) = 1 + aT + bpT^2 + ap^3T^3 + p^6T^4, \qquad a, b \in \mathbb{Z}, \qquad p \notin S. \tag{2}$$

The coefficients a and b that determine the whole polynomial can be computed using the p-adic methods of Dwork. We will give the final expression obtained in [**?**] in terms of the hypergeometric trace of Katz, which we now define.

Let $\mathbb{Q}_{(p)}$ be the ring of rational numbers with denominator coprime to p. Fix $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\beta = (\beta_1, \ldots, \beta_r)$ vectors in $\mathbb{Q}_{(p)}^r$ with $0 \leq \alpha_j, \beta_j < 1$ and f a positive integer. For $m = 0, 1, \ldots, q-2$, with $q := p^f$, we define a p-adic analogue of the Pochammer symbol

$$(x)^*_{m,q} := \frac{\Gamma^*_q\left(x + \frac{m}{1-q}\right)}{\Gamma^*_q(x)}, \qquad x \in \mathbb{Q}_{(p)} \subseteq \mathbb{Z}_p. \tag{3}$$

where to simplify the notation we set

$$\Gamma^*_p(x) := \Gamma_p\left(\{x\}\right), \qquad \Gamma^*_q(x) := \prod_{\nu=0}^{f-1} \Gamma^*_p(p^\nu x), \qquad x \in \mathbb{Q}_{(p)} \subseteq \mathbb{Z}_p.$$

To alleviate the notation we will drop the dependence on q when there is no risk of confusion.

For $x \in \mathbb{Q}_{(p)}$ and $m = 0, 1, \ldots, q-2$ we let

$$\eta_m(x) := \sum_{\nu=0}^{f-1} \left\{ p^\nu \left(x + \frac{m}{1-q} \right) \right\} - \{p^\nu x\}$$

and extend the definition to $x = (x_1, \ldots, x_r) \in \mathbb{Q}_{(p)}^r$ by setting

$$\eta_m(x) := \sum_{j=1}^{r} \eta_m(x_j).$$

We assume now that for all $m = 0, 1, \dots, q-2$ we have

$$\eta_m(\alpha) - \eta_m(\beta) \in \mathbb{Z} \tag{4}$$

and define

$$H_q\left(\begin{array}{c} \alpha \\ \beta \end{array} \middle| z \right) := \frac{1}{1-q} \sum_{m=0}^{q-2} (-p)^{\eta_m(\alpha)-\eta_m(\beta)}\, q^{\xi_m(\beta)} \prod_{j=1}^{r} \frac{(\alpha_j)^*_m}{(\beta_j)^*_m}\ \mathrm{Teich}(z)^m. \tag{5}$$

where

$$\xi_m(\beta) := \#\left\{ j \,|\, \beta_j = 0 \right\} - \#\left\{ j \,|\, \beta_j + \frac{m}{1-q} = 0 \right\}.$$

Remark 2.0.1 We have normalized the hypergeometric trace H_q of Katz so that it resembles the classical hypergeometric series. We should point, however, that α gives the exponents of the local monodromy at ∞ and β those at 0, whereas classically the exponents at 0 would be given as $1-\beta_j$ instead of β_j.

We then have that the trace of the geometric Frobenius Frob_q on V is given as a number in $\mathbb{Q}_p$ by

$$\mathrm{Tr}\left(\mathrm{Frob}_q \big|_{V_\psi} \right) = H_q\left(\begin{array}{cccc} \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 0 \end{array} \middle| \psi^{-5} \right), \qquad \psi \not\equiv 0 \bmod p. \tag{6}$$

If we abbreviate the right hand side by H_q then we have

$$a = -H_p, \qquad\qquad b = \tfrac{1}{2p}(H_p^2 - H_{p^2}).$$

For a prime $p \neq 5$ such that $\psi^5 \equiv 1 \bmod p$ we may still compute the right hand side of (6). These are the traces of an operator with characteristic polynomial

$$L_p(T) = (1 - \left(\tfrac{5}{p}\right) pT)(1 - a_p T + p^3 T^2), \qquad p \neq 5, \quad \psi^5 \equiv 1 \bmod p, \tag{7}$$

where a_p is the p-th coefficient of the Hecke eigenform of weight 4 and level 25 discovered by Schoen, which gives the trace of Frobenius acting on H^3 of a resolutions of singularities of the conifold X_1. Again, with the above notation

$$a_p + \left(\tfrac{5}{p}\right) p = -H_p, \qquad \left(\tfrac{5}{p}\right) a_p + p^2 = \tfrac{1}{2p}(H_p^2 - H_{p^2}).$$

For a prime $p \neq 5$ such that $\psi \equiv \infty \bmod p$ the right hand side of (6) gives the constant value 1 for all f. Hence the associated characteristic polynomials is simply

$$L_p(T) = 1 - T, \qquad p \neq 5, \quad \psi \equiv \infty \bmod p. \tag{8}$$

This seems to be the right answer.

If $\psi \equiv 0 \bmod p$ the formula (6) breaks down. However, the variety X_0 is the Fermat hypersurface

$$x_1^5 + \cdots x_5^5 = 0,$$

whose L-series was calculated by Weil in terms of Hecke characters. It is not difficult to work out the Hecke character corresponding to V_0. Let $K := \mathbb{Q}(\zeta_5)$, where ζ_5 is a primitive fifth root of unity and let $\mathcal{F} = (1 - \zeta_5)^2$. A prime $p \neq 5$ factors in the ring of integers O_K of K into primes

$$(p) = \prod_{i=1}^{s} \mathcal{P}_i,$$

where $s \mid 4$. The class number of K is 1 and we can in fact choose generators α_i of $\mathcal{P}_i$ such that

$$\alpha_i \equiv 1 \bmod \mathcal{F}, \qquad i = 1, 2, \ldots, s. \tag{9}$$

Indeed, it is not hard to verify that a generator ϵ of $O_K^\star$ generates $(O_K/\mathcal{F})^\star$ and hence given any generator of $\mathcal{P}_i$ we can multiply it by an appropriate power of ϵ to obtain α_i.

Let σ be the generator of $\mathrm{Gal}(K/\mathbb{Q})$ that takes ζ_5 to ζ_5^2. We define the Hecke character ϕ by setting

$$\phi(\mathcal{P}_i) := \alpha_i^{1+2\sigma^2+3\sigma^3}.$$

It is a short calculation to verify that this is well defined independent of the choice of α_i satisfying (9). Then the Euler factor at p is

$$L_p(T) := \prod_{i=1}^{s} (1 - \phi(\alpha_i) T^{4/s}).$$

For $p = 5$ we have

$$L_5(T) = 1.$$

The L-function of this Hecke character has functional equation of the form

$$\Lambda(s) := N^{s/2} L_\infty(s) L(\phi, s) = \Lambda(4 - s),$$

where $L(\phi, s) := \prod_p L_p(p^{-s})$. It is known that

$$N = \mathrm{disc}(K/\mathbb{Q}) \cdot \mathbb{N}_{K/\mathbb{Q}}(\mathcal{F}) = 5^3 \cdot 5^2 = 5^5.$$

We can verify that $L_\infty(s)$ is our previously computed factor $(2\pi)^{-2s}\Gamma(s)\Gamma(s-1)$ directly. Indeed, the infinite type of ϕ is $\mu := 1 + 2\sigma^2 + 3\sigma^3$ and $1, \sigma^2$ and σ, σ^3 correspond to pairs of complex conjugate embeddings of K. For the first pair we have $(1, \sigma^2) + (0, \sigma^2)$ in μ. This contributes a factor of $\Gamma_{\mathbb{C}}(s-1)$. For the second pair we have $(0, 3\sigma^3)$ in μ, which contributes a factor of $\Gamma_{\mathbb{C}}$.

2.3 The conductor

The conductor N is defined as a product over primes $\prod_p p^{f_p}$, where $f_p = 0$ for all but finitely many primes. The exponent f_p itself is a sum of two terms: $r_p := \dim V - \dim V^I$, where I is the inertia group at p, and a wild contribution δ_p. Since

$$L_p(T) := \det\left(I - \mathrm{Frob}_q\big|_{V^I}\right)$$

we see that $r_p = \dim V - \deg L_p$.

2.4 Modularity

We would like to test whether $L(V, s)$ is modular. A natural choice of automorphic L-function to compare $L(V, s)$ with is the *spinor L-function* of a Siegel modular form. For $g = 2$ this L-function has an Euler factor for good primes p of the form

$$L_p(T) = 1 - \lambda_p T + (\lambda_p^2 - \lambda_{p^2} - p^{2k-4})T^2 - \lambda_p p^{2k-3} T^3 + p^{4k-6} T^4,$$

where k is the weight of the Siegel modular form and λ_p and λ_{p^2} are the eigenvalues of the Hecke operators T_p and T_{p^2}.

For a Siegel modular F form of level 1 Andrianov showed that the L-function

$$Z_F(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s-1)\prod_p L_p(p^{-s})^{-1},$$

has a meromorphic continuation to all $s \in \mathbb{C}$ and satisfies a functional equation

$$Z_F(k+1-s) = (-1)^k Z_F(s).$$

So if we expect $Z_F(s)$ to equal $L(V, s)$ we need the weight k to equal 3. However, one needs to be aware that if F is in the Maass space $L_p(T)$ is not pure of weight 3. Indeed in that case the Z_F factors as $\zeta(s-k+1)\zeta(s-k+2)L(f, s)$ for an eigenform of weight $2k-2$ on $\Gamma_0(N)$. It is a conjecture of Arthur that if F is not in the Maass space then there is an associated motive of V rank 4 and pure weight 3 such that $L(V, s) = Z_F(s)$. Hence it is natural to expect that our V is such a motive for some F.

Our situation is similar to the case of elliptic curves since a Siegel modular form F of $g = 2$ and weight 3 determines a holomorphic differential in the corresponding Siegel threefold

$$F(z)\, dz_{1,1} \wedge dz_{1,2} \wedge dz_{2,2}, \qquad z = (z_{i,j}) \in \mathcal{H}_2.$$

Other than quadratic twists of $Z_F(s)$ for F a Siegel eigenform of level 1 I could not find in the literature a description of the Euler factors for primes dividing the conductor.

3 Other analogous families

There turn out to be fourteen families X_ψ of Calabi-Yau threefolds analogue to the quintic; i.e., hypergeometric and with $\beta = (0, 0, 0, 0)$, (maximally unipotent monodromy at $\psi = \infty$). The values of α are given in the following table together with the level N_1 of the weight 4 modular form and the conductor D of the Dirichlet character associated with the singularity at $\psi = 1$.

α	N_1	D
$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	8	1
$(\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4})$	9	24
$(\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4})$	16	8
$(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5})$	25	5
$(\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{1}{3})$	27	1
$(\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4})$	32	1
$(\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{2}{3})$	36	12
$(\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{5}{6})$	72	1
$(\frac{1}{3},\frac{2}{3},\frac{1}{6},\frac{5}{6})$	108	12
$(\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8})$	128	8
$(\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6})$	144	8
$(\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10})$	200	1
$(\frac{1}{6},\frac{5}{6},\frac{1}{6},\frac{5}{6})$	216	1
$(\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12})$	864	1

For all of these cases we can write an explicit model for X_ψ (for the first thirteen cases as complete intersections in weigted projective spaces [**?**]; the fourteenth case is described in []) . The family carries a period satisfiying the corresponding hypergeometric differential equation with parameters α,β. We again obtain a motive V_ψ of rank 4 and pure weight 3 for good primes p coming from a piece of the middle cohomology of X_ψ. The trace of Frob_q on V_ψ is given by

$$\mathrm{Tr}\left(\mathrm{Frob}_q\big|_{V_\psi}\right) = H_q\left(\begin{array}{c}\alpha\\ \beta\end{array}\middle|\,\psi^{-m}\right), \qquad \psi \not\equiv 0 \bmod p$$

for some positive integer m. (For the quintic case $\alpha = (\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5})$ and $m = 5$.)

We posit that in fact for each choice of α from the above list there is a rank 4 motive $\mathcal{H}_t = \mathcal{H}_t(\alpha,\beta)$, of pure weight 3 for good primes, such that for $t \in \mathbb{Q}$ we have

$$\mathrm{Tr}\left(\mathrm{Frob}_q\big|_{\mathcal{H}_t}\right) = H_q\left(\begin{array}{c}\alpha\\ \beta\end{array}\middle|\,t\right), \qquad p \notin S,$$

where S is the finite set of primes dividing $t, t-1, t^{-1}, \mathrm{denom}(\alpha)$ or $\mathrm{denom}(\beta)$.

Let us take, for example, $\alpha = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $t = -1$. Here are the first few values of H_p and H_{p^2}.

p	H_p	H_{p^2}
3	0	−12
5	−4	276
7	0	−476
11	0	−4972
13	−84	−1420
17	36	7620
19	0	−21964
23	0	24932
29	140	−62412

From the traces we can compute the coefficients of the Euler factor

$$L_p(T) = 1 + aT + pbT^2 + ap^3 + p^6T^4, \qquad a = -H_p, \quad b = \tfrac{1}{2p}(H_p^2 - H_{p^2}).$$

p	a	b
3	0	2
5	4	−26
7	0	34
11	0	226
13	84	326
17	−36	−186
19	0	578
23	0	−542
29	−140	1414

If we compare these with the corresponding table 7.6 of van Geemen and van Straaten we see that except for the signs of a (and the value of b for $p = 11$) they agree with those in the column for the Siegel modular form F_7. (Note that their polynomial is normalized as $T^4 - a_pT^3 + a_{p^2}T^2 - a_pp^3T + p^6$.)

Furhermore, they notice that $L_p(T)$ seems to be the Euler factor of the L-function associated to $f_2 \otimes f_3$ where f_2 is a CM eigenform of weight 2, level 32 and trivial character and f_3 an eigenform of weight 3, level 32 and character $\left(\frac{-4}{\cdot}\right)$. With this information we can extend their calculation and check the agreement of this L-function with ours. Here is a table of the p-coefficients of f_2 and f_3 from Stein's database.

p	f_2	f_3
2	0	0
3	0	$4i$
5	−2	2
7	0	$-8i$
11	0	$4i$
13	6	−14
17	2	18
19	0	$12i$
23	0	$40i$
29	−10	−14
31	0	$32i$
37	−2	−30

We see, for example, that the product of the p coefficients of f_2 and f_3 matches the values of H_p.

3.1 Special cases

Inspired by this example we experimented with the motive $\mathcal{H}_{\pm 1}$ where $\alpha = (\frac{1}{2}, \ldots, \frac{1}{2})$ of length r. Let us denote the motive by $W_r^{\pm}$. Our previous example associated to $f_2 \otimes f_3$ is then W_4^-. Here is a table of the values of H_p.

W_r^+

$p \backslash r$	1	2	3	4	5	6
3	0	−1	0	−1	0	−1
5	0	1	−6	3	20	−59
7	0	−1	0	31	0	95
11	0	−1	0	−33	0	−481
13	0	1	10	35	−300	933
17	0	1	−30	67	−60	−59
19	0	−1	0	63	0	3519

W_r^-

$p\backslash r$	1	2	3	4	5	6
3	−1	0	−1	0	−1	0
5	−1	−2	5	−4	−21	58
7	1	0	−7	0	−79	0
11	−1	0	−25	0	79	0
13	−1	6	13	−84	−101	−1102
17	−1	2	19	36	−699	614
19	−1	0	15	0	−161	0

There is a clear pattern that emerges, $W_r^{\pm}$ has $H_p = 0$ unless $p \equiv 1 \bmod 4$ when $\pm 1 = (-1)^{r-1}$.

It would seem that the L-function of W_5^+ equals $L(f_2 \otimes f_4, s)$, where f_2 is our previous CM form of weight 2 and f_4 is a modular form of weight 4 and level 32 with eigenvalues

p	f_4
2	0
3	8
5	−10
7	16
11	−40
13	−50
17	−30
19	40
23	48
29	−34
31	320
37	310

3.2 Stirling and Dirichlet

It turns out that the discriminant D giving the Dirichlet character of the linear factor of $L_p(T)$ corresponding to $\psi = 1$ can be given directly in terms of α and β as follows.

The hypergeometric series $F\left(\begin{matrix}\alpha\\ \beta\end{matrix}\middle| t\right)$ has a power series around $t = 0$ of the form

$$\sum_{n\geq 0} u_n \left(\frac{t}{K}\right)^n,$$

where

$$u_n := \prod_{\nu\geq 1} (\nu n)!^{\gamma_\nu}, \qquad K := \prod_{\nu \geq 1} \nu^{\nu\gamma_\nu}, \tag{10}$$

for certain integers γ_ν, which are zero for almost all ν. The relation between the parameters α, β and γ is the following

$$\prod_{\nu \geq 1} (1 - T^\nu)^{\gamma_\nu} = \frac{q_\infty(T)}{q_0(T)}, \qquad q_\infty(T) := \prod_j (1 - e^{2\pi i \alpha_j} T), \qquad q_0(T) := \prod_j (1 - e^{2\pi i \beta_j} T). \quad (11)$$

(We are assuming that q_0 and q_∞ have coefficients in $\mathbb{Z}$.)

By Stirling, as $n \to \infty$

$$u_n \sim \frac{\sqrt{\delta}}{(2\pi n)^{d/2}} K^n,$$

where

$$\delta := \prod_{\nu \geq 1} \nu^{\gamma_\nu}, \qquad d := -\sum_{\nu \geq 1} \gamma_\nu.$$

Numerically, it seems that D is the discriminant of the quadratic extension of $\mathbb{Q}$ given by adjoining a square root of $(-1)^d \delta$.

4 Hypergeometric motives

4.1 Hodge numbers

We expect the above situation to be true in greater generality. Let γ be a non-zero sequence of integers $\gamma = (\gamma_\nu)$ for $\nu \geq 1$, only finitely many of which are non-zero, and satisfying

$$\sum_{\nu \geq 1} \gamma_\nu \nu = 0, \quad (12)$$

a condition we will call *regularity*. We associate to γ a family of motives V_t with $t \in \mathbb{P}^1$ defined over $\mathbb{Q}$. Our goal is to describe the L-function of V_t completely.

To this end, define

$$\mathcal{L}^+(x) := \sum_{\nu \geq 1} \gamma_\nu \left(\tfrac{1}{2} - \{\nu x\} \right), \qquad x \in \mathbb{R},$$

where $\{\cdot\}$ denotes the ordinary fractional part of a real number. It is easy to check that $\mathcal{L}^+$ is periodic of period 1, locally constant, right continuous and satisfies

$$\mathcal{L}^+(-x) = -\mathcal{L}^-(x), \quad (13)$$

where $\mathcal{L}^-(x) := \lim_{y \to x^-} \mathcal{L}^+(y)$. Also, $\mathcal{L}^+$ has only finitely many discontinuities. If we let $l(x) := \mathcal{L}^+(x) - \mathcal{L}^-(x)$ then l takes integer values and is zero away from these discontinuities. In other words, the functions $\mathcal{L}^\pm$ have only jump discontinuities and the jumps are integral. We have $\sum_{x \in [0,1)} l(x) = 0$ and by (13) the symmetry $l(-x) = l(x)$.

Define the *weight* of γ by

$$w := \max_{x\in[0,1)} \mathcal{L}^+(x) - \min_{x\in[0,1)} \mathcal{L}^+(x) - 1.$$

Clearly, w is an integer and is in fact non-negative since $\mathcal{L}$ is not identically zero (we have assumed γ is not zero). Note that $\max_{x\in[0,1)} \mathcal{L}^\pm(x) = -\min_{x\in[0,1)} \mathcal{L}^\pm(x) = (w+1)/2$ and hence $\mathcal{L}^\pm(x) + (w+1)/2$ takes values in $\mathbb{Z}_{\geq 0}$.

Define the *Hodge polynomial* of γ by

$$h(T) := \sum_{l(x)>0} T^{\mathcal{L}^-(x)+(w+1)/2}[l(x)] \in \mathbb{Z}[T],$$

where $[l] := 1 + T + \cdots + T^{l-1}$ and the sum is over the finitely many $x \in [0,1)$ with $l(x) > 0$.

Lemma 4.1. *The Hodge polynomial is reciprocal of degree w and has non-negative integer coefficients.*

$$h(T^{-1}) = T^{-w}h(T).$$

Proof. It is clear from the definition that the coefficients of $h(T)$ are non-negative integers. Let x be the left endpoint of an interval in $(0,1)$ where $\mathcal{L}^+(x)$ achieves its maximum. Then $l(x) > 0$ and the corresponding term in the sum defining $h(T)$ has degree $\mathcal{L}^-(x) + (w+1)/2 + l(x) - 1 = w$. Hence the degree of h is w.

We have

$$h(T^{-1}) = \sum_{l(x)>0} T^{-\mathcal{L}^-(x)-(w+1)/2}T^{1-l(x)}[l(x)] = \sum_{l(x)>0} T^{\mathcal{L}^-(-x)+l(-x)-(w+1)/2}T^{1-l(x)}[l(x)]$$

by (13) and the right hand side simplifies to give $T^{-w}h(T)$ finishing the proof. □

We refine the Hodge polynomial by definining for every $m \in \mathbb{Z}_{\geq 0}$

$$h_0^{(m)}(T) := \sum_{l(x)=m} T^{\mathcal{L}^-(x)+(w+1)/2} \in \mathbb{Z}[T]$$

so that $h(T) = \sum_{m\geq 0} h^{(m)}(T)[m]$. As in the proof of Lemma 4.1 we find

$$h_0^{(m)}(T^{-1}) = \sum_{l(x)=m} T^{-\mathcal{L}^-(x)-(w+1)/2} = \sum_{l(x)=m} T^{\mathcal{L}^+(-x)-(w+1)/2} = \sum_{l(x)=m} T^{\mathcal{L}^-(x)+m-(w+1)/2}$$

so that

$$h_0^{(m)}(T^{-1}) = T^{-w+m-1}h_0^{(m)}(T).$$

An alternative way to compute the Hodge polynomial of γ is as follows.

Proposition 4.2. *We have*

$$h(T) = \sum_{l(x)<0} = T^{\mathcal{L}^+(x)+(w+1)/2}[-l(x)]. \tag{14}$$

Proof. Slightly deform $\mathcal{L}^+$ to a continuous function L as follows. Replace a jump of $\mathcal{L}^+$ with $l(x) > 0$ by an increasing function in a small interval $(x-\epsilon, x+\epsilon)$ for some $\epsilon > 0$ going from $\mathcal{L}^-(x)$ to $\mathcal{L}^+(x)$. Similarly replace a jump with $l(x) < 0$ by a decreasing function going from $\mathcal{L}^+(x)$ to $\mathcal{L}^-(x)$.

Then

$$h(T) = \sum T^{L(x)+(w+1)/2},$$

where the sum is over $x \in [0, 1)$ such that $L(x) \in \mathbb{Z}$ and $L(x) < L(x')$ for $x < x'$. Call such an x a *point of increase*.

On the other hand the right hand side of (14) is a similar sum over $x \in [0, 1)$ such that $L(x) \in \mathbb{Z}$ and $L(x) > L(x')$ for $x' < x$. Call such an x a *point of decrease*. By periodicity, for a given $y \in \mathbb{Z}$ the number of points of increase with $x \in [0, 1)$ with $L(x) = y$ is the same as those of decrease. This proves our claim. □

We define a new refinement of the Hodge polynomial. For $m \in \mathbb{Z}_{\geq 0}$ let

$$h_\infty^{(m)}(T) := \sum_{l(x)=-m} T^{\mathcal{L}^+(x)+(w+1)/2} \in \mathbb{Z}[T].$$

Then

$$h(T) = \sum_{m\geq 0} h_\infty^{(m)}(T)[m].$$

Note that if we replace γ by $-\gamma$ then $\mathcal{L}$ also changes sign and $h_0^{(m)}$ turns into $h_\infty^{(m)}$. Hence

$$h_\infty^{(m)}(T^{-1}) = T^{-w+m-1} h_\infty^{(m)}(T).$$

4.2 Tame primes

We will say that a prime p is *tame* if it does not divide the denominators of α or β and one of $v_p(t), v_p(t^{-1})$ or $v_p(t-1)$ is positive. We would like to describe the Euler factor $L_p(T)$ and power f_p of p in the conductor of the L-function associated to the motive $H_t(\alpha,\beta)$.