HYPERGEOMETRIC MOTIVES

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The main goal of this workshop is to obtain concrete results (both theoretical and computational) on the motives of the title, particularly, on their associated L-functions.

A hypergeometric motive is determined by very simple data, which can be given in various formats. In one version, it consists of a polynomial $\gamma(T) \in \mathbb{Z}[T]$ with the property that $\gamma(0) = \gamma'(1) = 0$. Attached to this data there is a family of motives $\mathcal{F}(\gamma|t)$ defined over $\mathbb{Q}$ depending on a parameter $t \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$. For each choice of t we obtain a global L-function

$$\Lambda(s) = N^{s/2} L_\infty(s) \prod_p L_p(p^{-s})^{-1},$$

which is expected to satisfy a functional equation $\Lambda(w + 1 - s) = \varepsilon\, \Lambda(s)$ for some non-negative integer w and some $\epsilon = \pm 1$. A motivating example is the degree four L-function associated to the 4-dimensional piece $V \subseteq H^3$ of a quintic in the pencil

$$X_\psi : \qquad x_1^5 + \cdots x_5^5 - 5\psi x_1 \cdots x_5 = 0, \tag{1}$$

fixed by the automorphisms: $x_i \mapsto \zeta_5^{a_i} x_i$, where ζ_5 is a primitive 5-th root of unity and $\sum_i a_i \equiv 0 \bmod 5$. Here $w = 3$ and $\gamma(T) = T^5 - 5T$.

One of the fundamental questions we would like to address is this: how do we calculate all the ingredients that define $\Lambda(s)$? It is not hard to describe and compute the Euler factors L_p for the good primes (see below). As usual, it is more difficult to find the other Euler factors, including the gamma factors $L_\infty(s)$, and the conductor N.

In a workshop in Benasque, Spain in 2009 we succeded in computing the L-function $\Lambda(s)$ of $\mathcal{F}(\gamma|t)$ in the case of the quintic for a fairly large range of values of t. To compute the unknown Euler factors for the bad primes, conductor and root number ε we used a test based on the purported functional equation $\Lambda(4 - s) = \varepsilon \Lambda(s)$. (This is a very robust test and we feel confident that we indeed found the right L-function.) Here is a short list of values of ψ and the corresponding conductor N and root number for $\Lambda(s)$.

ψ	$\psi^5 - 1$	N	ε
-1	2	$2 \cdot 5^5$	$+$
2	31	$31 \cdot 5^5$	$-$
-2	$3 \cdot 11$	$3 \cdot 11 \cdot 5^5$	$-$
3	$2 \cdot 11^2$	$2 \cdot 11 \cdot 5^5$	$-$
-3	$2^2 \cdot 61$	$2 \cdot 61 \cdot 5^5$	$-$
4	$3 \cdot 11 \cdot 31$	$3 \cdot 11 \cdot 31 \cdot 5^5$	$+$
-4	$5^2 \cdot 41$	$41 \cdot 5^4$	$+$
6	$5^2 \cdot 311$	$311 \cdot 5^4$	$+$
-6	$7 \cdot 11 \cdot 101$	$7 \cdot 11 \cdot 101 \cdot 5^5$	$+$

The motives $\mathcal{F}(\gamma|t)$ are related to the classical hypergeometric series as follows. Consider, for example,

$$ {}_rF_{r-1}\left[\begin{matrix}\alpha_1 & \dots & \alpha_r \\ \beta_1 & \dots & \beta_{r-1}\end{matrix} \mid t\right] \tag{2}$$

with α_j and β_j in $\mathbb{Q}$. Then $\gamma(T)=\sum_{\nu\geq 1}\gamma_\nu\, T^\nu$ is such that

$$\prod_{\nu\geq 1}(T^\nu-1)^{-\gamma_\nu}=\frac{\prod_{j=1}^r(T-e^{2\pi i\alpha_j})}{\prod_{j=1}^r(T-e^{2\pi i(1-\beta_j)})},\qquad \beta_r:=1.$$

In one of its incarnations $\mathcal{F}(\gamma|t)$ is the space of local solutions to the (hypergeometric) linear differential equation satisfied by this series. The rank of $\mathcal{F}(\gamma|t)$ is r and hence generically the Euler factors $L_p(T)$ have degree r.

For the running example of the quintic family X_ψ the hypergeometric series is

$${}_4F_3\left[\begin{matrix}1/5 & 2/5 & 3/5 & 4/5\\ 1 & 1 & 1 & \end{matrix}\mid t\right]=\sum_{n\geq 0}\frac{(5n)!}{n!^5}\left(\frac{t}{5^5}\right)^n$$

and this period of the family can be explicitly computed in terms of

$$\frac{1}{(2\pi i)^5}\int_{|x_1|=\cdots=|x_5|=1}\frac{dx_1\cdots dx_5}{x_1^5+\cdots x_5^5-5\psi x_1\cdots x_5},\qquad t=\psi^{-5}. \tag{3}$$

The gamma factors for the corresponding degree four L-function can be deduced from the Hodge numbers of V, which are $h^{0,3}=h^{2,1}=h^{1,2}=h^{0,3}=1$. By the standard recipe this yields the gamma factor $L_\infty(s)=(2\pi)^{-2s}\Gamma(s)\Gamma(s+1)$. (Since the weight $w=3$ is odd there are no $h^{p,p}$ Hodge numbers; these would require more information given by the action of complex conjugation.)

Conjecturally at least, the Hodge numbers (and hence almost all data to describe $L_\infty(s)$) can be computed in a purely combinatorial way directly in terms of $\gamma(T)$. A convenient formulation is as follows. Let

$$\mathcal{L}^+(x):=\sum_{\nu\geq 1}\gamma_\nu\left(\{\nu x\}-\tfrac{1}{2}\right),\qquad x\in\mathbb{R}.$$

and $\mathcal{L}^-(x):=\lim_{y\to x^-}\mathcal{L}^+(y)=-\mathcal{L}^+(-x)$. Then $\mathcal{F}(\gamma|t)$ is pure of weight

$$w:=\max_{x\in[0,1)}\mathcal{L}^+(x)-\min_{x\in[0,1)}\mathcal{L}^+(x)-1$$

and its Hodge polynomial is

$$h(T):=\sum_{0\leq i\leq w}h^{i,w-i}\,T^i=\sum_{l(x)>0}T^{\mathcal{L}^-(x)+(w+1)/2}[l(x)]=\sum_{l(x)<0}T^{\mathcal{L}^+(x)+(w+1)/2}[-l(x)],$$

where $l(x):=\mathcal{L}^+(x)-\mathcal{L}^-(x)$ is the jump at $x\in[0,1)$, and $[l]:=1+T+\cdots+T^{l-1}$ (there are only finitely many $x\in[0,1)$ with $l(x)\neq 0$). For the quintic family this gives

$$h(T)=1+T+T^2+T^3=1[1]+T[1]+T^2[2]+T^3[3]=[4].$$

A prime p not dividing a denominator of α_j,β_j nor dividing $t,t^{-1},t-1$, is of good reduction for $\mathcal{F}(\gamma|t)$. We may compute characteristic polynomial $L_p(T)$ of Frobenius via calculating the trace of its powers. These traces are given by hypergeometric sums which are finite p-adic analogues of the hypergeometric series (2). This fact is a p-adic version of the famous observation of Igusa that for the Legendre family of elliptic curves the Hasse invariant is essentially the polynomial

$$\sum_{n=1}^{(p-1)/2}\binom{2n}{n}^2\lambda^n \bmod p,$$

which satisfies the same linear (hypergeometric) differential equation as its periods. (In our setup H^1 of the Legendre family is the motive $\mathcal{F}(\gamma|t)$ for $\gamma=T^2-2T$.)

To compute the traces of powers of Frobenius we use the p-adic gamma function (as discussed in *Calabi–Yau manifolds over finite fields I & II* by Candelas, de la Ossa and Rodriguez-Villegas). The only issue here is to improve the speed of computation, as this is the bulk of it, in order to be able to compute larger examples (larger values of t or larger rank).

On the analytic side, the further development of efficient methods for computing the inverse Mellin transform of $L_\infty(s)$ is interesting per se. We have already used several approaches (generalized power series expansions, doubly-exponential integration) and would like to try a few other promising ones. To improve considerably the search for the bad Euler factors, we plan to implement the mollified version of the approximate functional equation a la D. Farmer and M. Rubinstein,

There are 42 families of hypergeometric motives with the same characteristics as that of the quintic above. Finding corresponding explicit families of varieties such as (1) for them is not a simple matter. In a sense our main point is precisely that we do not actually need them. For each family, a choice of t leads to a computable degree four L-function $\Lambda(s)$. The procedure outlined above does not require any direct counting of points of varieties over finite fields or the computation of an automorphic form (the two standard ways of producing L-functions). However, one sees that by the Langlands philosophy $\Lambda(s)$ should be the spinor L-function of a Siegel modular forms of genus 2 and weight 3. It would be quite interesting to explore this relation further and try to confirm it numerically in several examples (in fact, modularity of many of these L-functions should follow from work of Harris and Taylor). We should point out, however, that the size of most of the conductors in the above list for example are significantly larger that those that can be handled computationally on the Siegel modular form side.

In summary, our approach yields a large supply of computable L-functions with Euler factors of degree larger than two. Having these, it is natural to test numerically, within the scale of feasibility, the standard conjectures (e.g., the location and distribution of their zeros) and this is ultimately the main goal of this project.

We would like this workshop to be a healthy mixture of theory and computations. One one hand, a great deal of theory is necessary in order to dig deep into the subtle features that determine, for example, the nature of the Euler factors for the bad primes. On the other, to be able to carry out concrete computations with the resulting L-functions (of degree at least four) requires highly developed computational skills.

Our proposal is to bring together experts from these often disjoint communities (Galois representations, automorphic forms, computational number theory) to plan a two-pronged approach. In our experience, bootstrapping by alternatively performing numerical experiments and discussing theory is a very efficient way to achieve significant mathematical progress. In this regard, a workshop at AIM seems to be a very natural venue for the style of work envisioned. In fact, the origin of this project was the workshop along these lines held in Benasque, Spain in 2009 already mentioned. We made significant progress then and expect a similar outcome for what we propose here.