NT seminar UT Sep 15, 2011

w/ H. Cohen

Hyperg. motives

Higher deg. L-functions.

Automorphic forms

algebraic varieties

typically you search in some finite dim'l space by eigenvectors of Hecke operators

count points
Weil conjectures

wt 2 forms on $\Gamma_0(N)$ $\longleftrightarrow$ elliptic curves

- modular symbols
- quat. algebras

E.g. elliptic curves

cubics in $\mathbb{P}^2$

	dim
H^0	1
H^1	2
H^2	1

threefolds quintic

H^3 dim 204

$$x_1^5 + \cdots + x_5^5 - 5\psi x_1 \cdots x_5$$

ψ parameter

$$H^3 \simeq \underset{\dim 4}{H_0} \oplus \cdots$$

<u>Different approach</u>

Hyperg. story.

E.g.

$$ {}_2F_1\left(\begin{matrix} 1/2 & 1/2 \\ & 1 \end{matrix} \middle| t\right) = \sum_{n \geq 0} \binom{2n}{n}^2 \left(\frac{t}{4}\right)^n$$

<u>Gauss</u>

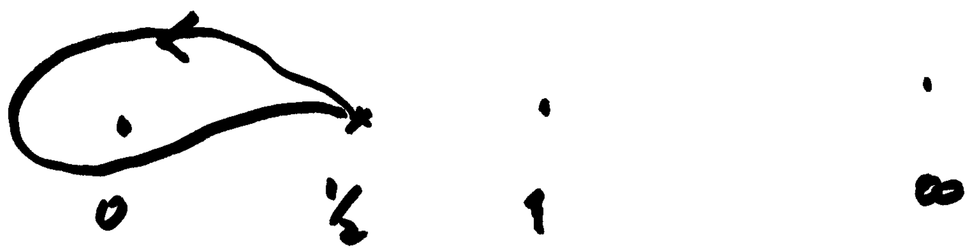
satisfies a linear diff eqn

$$L(u) = (1-t)\frac{d^2u}{dt^2} + (1-2t)\frac{du}{dt} - \frac{1}{4}u = 0$$

has singularities $t = 0, 1, \infty$

Take $V := \left\{ \begin{array}{c} \text{space of solutions} \\ \text{at } t = 1/2 \end{array} \right\}$ ③

$\dim V = 2$



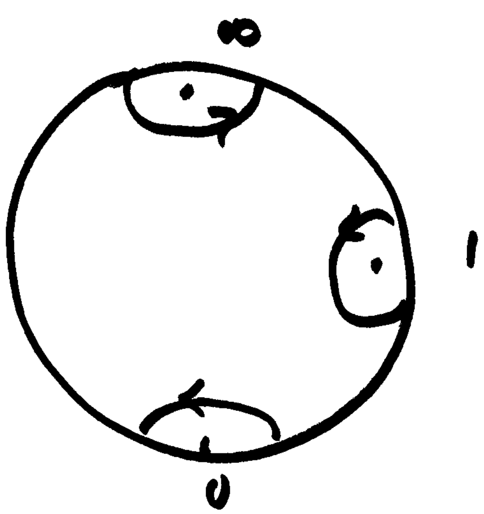
$$\rho : \pi_1(\mathbb{P}^1 \setminus S) \longrightarrow Gl(V)$$

$S = \{0, 1, \infty\}$

Let $h_s \in Gl(V)$

correspond to loops around s.

$$h_\infty h_1 h_0 = 1$$



$\text{char } h_\infty = (x+1)^2$

$\text{char } h_0 = (x-1)^2$

$h_1 \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$h_\infty = \left(x - e^{\frac{1}{2} 2\pi i}\right)\left(x - e^{\frac{1}{2} 2\pi i}\right) \quad ④$$

$1/2, 1/2$

are the parameters of ${}_2F_1$

Riemann.

This is rigid.

$$h_\infty \cdot h_1 \cdot h_0 = 1$$

with
$$\begin{cases} \mathrm{char}(h_\infty) = q_\infty \\ \mathrm{char}(h_0^{-1}) = q_0 \\ h_1 \text{ fixes a codim } 1 \\ \quad \text{space in } V \end{cases}$$

uniquely determine ρ up to conjugation (q_∞, q_0 are coprime)

Data needed to define ρ is simply q_∞ / q_0

Hyperg. data

q_∞, q_0 product of cyclotomic $\in \mathbb{Z}[X]$ same degree

relatively prime

$\rightsquigarrow \quad \rho : \pi_1(\mathbb{P}^1 \setminus S) \longrightarrow Gl(V)$

$\dim V = \deg q_\infty = \deg q_0$

Claim There exists a family of motives $\mathcal{H}_t$ pure of some weight w.

Igusa

Take $E_\lambda : \; y^2 = x(x-1)(x-\lambda)$

$\lambda \in \mathbb{F}_p \setminus \{0,1\}$

$$A_p(\lambda) := (-1)^{\frac{p-1}{2}} \sum_{m=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{m}^2 \lambda^m$$

p odd prime   Hasse invariant

$$\# E_\lambda(\mathbb{F}_p) \equiv p+1-A_p(\lambda) \mod p$$

Igusa noticed is that $A_p(\lambda)$
satisfies the <u>same</u> diff eqn
as ${}_2F_1\left(\begin{matrix}1/2 & 1/2\\ & 1\end{matrix}\,\middle|\,\lambda\right)$

over $\qquad$ is a period of E_λ

$\int \omega_\lambda$

There is a souped up version
of this, p-adic formula for
$\# E_\lambda(\mathbb{F}_p)$. It is a sort of
p-adic version of ${}_2F_1\left(\begin{matrix}1/2 & 1/2\\ & 1\end{matrix}\,\middle|\,\lambda\right)$.

We have hence a form to compute
the p^{th} euler factor of the
L-series of E_λ, for good p.

This works for any choice of hyperg. data!

I.e. we have a p-adic fmla for the Euler factor L_p of the L-series attached to $\mathcal{H}_t$. $t \in \mathbb{P}^1(\mathbb{Q}) \setminus S$.

for p of good reduction $p \nmid t, t^{-1}, t-1$ or disc q_∞, disc q_0.

E.g.

$$X_1^5 + \cdots + X_5^5 - 5\psi X_1 \cdots X_5$$

the 4-dim'l piece of H^3, $\mathcal{H}_0$ is the motive $\mathcal{H}_t$ with data

$$q_\infty = x^4 + \cdots + x + 1$$
$$q_0 = (x-1)^4$$

$${}_4F_3\left(\begin{matrix} 1/5 & 2/5 & 3/5 & 4/5 \\ & 1 & 1 & 1 \end{matrix} \,\middle|\, t\right)$$

<u>Concrete examples</u>

cases where $w = 0$

$$L_p(T) = \prod_{i=1}^{a} (1 - \alpha_i T)$$

$$|\alpha_i| = p^{w/2}$$

weight 0 $\longleftrightarrow$ Artin L-functions

$\longleftrightarrow$ algebraic solutions to diff. equ.

$\longleftrightarrow$ $\operatorname{Im}(\rho)$ is finite.

for ${}_2F_1$ Schwartz ~~the~~ classified all algebraic cases, deg 2.

Beukers - Heckman did the general case.

Example

$$\sum_{n \geq 0} \binom{2n}{n} \left(\frac{t}{4}\right)^n = \frac{1}{\sqrt{1-t}}$$

$$\begin{array}{ll} 1/2 & q_\infty = (x+1) \\ 1 & q_0 = (x-1) \end{array}$$

More generally.

Belyi polynomial

$$P_t(x) := x^a(1-x)^b - \frac{a^a b^b}{c^c}\, t$$

$c = a+b$, $\gcd(a,b)=1$

$$\frac{q_\infty}{q_0} = \frac{x^c - 1}{(x^a-1)(x^b-1)}$$

generically pick $t \in \mathbb{P}^1(\mathbb{Q}) \setminus S$

$$F_t := \mathbb{Q}(x)/(P_t)$$

$$\begin{tikzcd} F_t \arrow[d, dash, "\deg c"] \\ \mathbb{Q} \end{tikzcd}$$

$\zeta_{F_t}(s) / \zeta_{\mathbb{Q}}(s)$ = L-series of the motive $\mathcal{H}_t$

~~Stan~~ $S_c \longrightarrow GL(V)$

standard representation

Example

$$q_\infty = \Phi_{12} = x^4 - x^2 + 1$$

$$q_0 = \Phi_1 \Phi_2 \Phi_3 = (x^2+x+1)(x^2-1)$$

$\mathcal{H}_t$ motive of deg 4
weight 0.

Artin L-function associated (generically) to the reflection representation of the Weyl group of F_4. W has order 1152

$$u(t) = \sum_{n \geq 0} \frac{(12n)!\, n!}{(6n)!\,(4n)!\,(3n)!} (Kt)^n$$

is algebraic. There exists a $f(u,t)$ such that $f(u(t), t) = 0$.

$$\begin{array}{c} \mathbb{Q}(u)/f(u,t) \\ \Big| \; 96 \\ \mathbb{Q}(u) \end{array}$$

galois closure has galois group
$W(F_4)$.

We computed a number of examples
The easiest is say $t = -1$.

$t, t^{-1}, t-1$

- 2, 3, only bad primes. L_2, L_3 ?
- ∞, L_∞
- N conductor

for $t=-1$ $L_2 = L_3 = 1$
$N = 2^{13} \times 3^6$

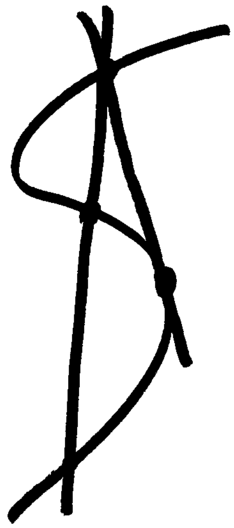
<u>Henri</u>: Jacobi sums
backwards engineer.

get
$\mathrm{Tr}(\mathrm{Frob}_q)$ in $\mathcal{H}_t$
is related to points on a
cubic surface

$$x^3 + x^2 - \frac{27}{4} t z^3 + z = y^2$$

missing  coplanar

lhs is a cubic curve



line at ∞

$$x^3 - \frac{27}{4} t z^3$$

$$\begin{array}{c} \mathbb{Q}(\beta) \\ 2 \,\big|\, \\ \mathbb{Q}(\alpha, \beta^2) \\ 4 \,\big|\, \\ \mathbb{Q}(\alpha) \\ \big|\, 3 \\ \mathbb{Q} \end{array}$$

Reduction of surface should lead to calculation of L_p for p of bad reduction.