p-adic Methods/L-functions

People

- Salman Butt
- Philip Candelas
- Henri Cohen
- Xenia de la Ossa
- Fernando Rodriguez Villegas
- Mark Watkins

Goals

Compute (conjectural) formulas for the conductor of the following Calabi-Yau manifold cut out by the equation

$$\sum_{i=1}^{d} x_i^d - d\psi \prod_{i=1}^{d} x_i = 0.$$

For example, for $d = 3$ we have the elliptic curve given in Weierstrass form by

$$y^2 - 3\psi xy + 9y = x^3 - 27(\psi^3 + 1)$$

with discriminant $\Delta = (\psi^3 - 1)^3 \cdot 3^9$. We write $\psi^3 - 1 = 3^a v$ with $(3, v) = 1$ and conjecture that the formula for the conductor is

$$N = \begin{cases} 3^3 \cdot sqf(v) & a \leq 2 \\ 3^2 \cdot sqf(v) & a \geq 3 \end{cases}$$

where $sqf(v)$ is the square-free part of v. This formula has been verified for all integral $\psi \in [-10000, 10000]$ not equal to 1 (though it was not investigated why a power of 3 is lost whenever $\psi^3 - 1 \equiv 0 \pmod{27}$).

The goal is to fill in the following table of local factors of the conductor (for $\psi \in \mathbf{Q}$):

d	$p \mid (\psi^d - 1)$	2	3	5
2				
3				

4				
5				

Such a formula will give us an idea (via the conductor) of the Siegel modular forms of interest to us.

Activities

Thursday, July 30

- Fernando, inspired by DNA, made a conjecture for the L-function associated to $\psi = \infty$ and, with some input from Mark, guessed the gamma factor was simply $\Gamma(s/2)$. This allowed us to guess that the local factor for a prime occurring in the denominator of $\psi^5 - 1$ is $1 - T$. Henri then ran some numerical computations to determine the appropriate conductor as well determining the correct local factor for when 5 was in the denominator of $\psi^5 - 1$, leading to the following numerically verified table:

ψ	$\psi^5 - 1$	N	ε
1/2	$2^{-5}.31$	$2^3.31.5^5$	+
1/3	$2.3^{-5}.11^2$	$2.3^3.11.5^5$	+
1/4	$2^{-10}.3.11.31$	2^3.3.11.31.5^5	-
-1/3	$2^2.3^{-5}.61$	$2.3^3.61.5^5$	+
2/3	$3^{-5}.211$	$3^3.211.5^5$	+
5	$2^2.11.71$	$2.11.71.5^5$	+
-5	2.3.521	$2.3.521.5^5$	-
1/5	$2^2.5^{-5}.11.71$	2.11.71	-
-3/5	$2^3.5^{-5}.421$	2.421	+

- Considering the rational ψ's, we conjecture the following formula for the sign of the functional equation:

$$\varepsilon = \varepsilon_\psi = \left(\frac{*}{5}\right) \times \prod_{v_p(\psi^5-1)<0, p\neq 5} \left(\frac{5}{p}\right) \times \prod_{v_p(\psi^5-1)>0, p\neq 5} \left(-\left(\frac{5}{p}\right)\right)$$

where $*$ is to be determined.

Wednesday, July 29

- Henri and Salman developed code to test out conjectural conductors for varying ψ, allowing them to fill in the following table using numerical experiments

ψ	$\psi^5 - 1$	N	ε
-1	2	2.5^5	+
2	31	$31.\ 5^5$	-
-2	3.11	$3.11.5^5$	-
3	2.11^2	$2.11.5^5$	-
-3	$2^2.61$	$2.61.5^5$	-
4	3.11.31	$3.11.31.5^5$	+
-4	$5^2.41$	41.5^4	+
6	$5^2.311$	311.5^4	+
-6	7.11.101	$7.11.101.5^5$	+
7	2.3.2801	$2.3.2801.5^5$	-
-7	$2^3.11.191$	$2.11.191.5^5$	+
8	3.7.151	$7.31.151.5^5$	+
-8	$3^2.11.331$	$3.11.331.5^5$	+
-15	$2^4.31.1531$	$2.31.1531.5^5$	+
1/3	$2.3^{-5}.11^2$	?	?
1/11	$2.5^2.11^{-5}.3221$	?	?

We conjecture the sign of the functional equation is simply

$$\prod_{p|N, p\neq 5} \left(-\left(\frac{5}{p}\right)\right).$$

For integral ψ, we conjecture N is just $5^{5-i} \cdot sqf(\psi)$ where $sqf(\psi)$ is just the squarefree part of ψ and $i = 1$ if 5 divides $\psi^5 - 1$ and $i = 0$ otherwise. The case of rational ψ is still left unresolved.

Tuesday, July 28

- Following a suggestion of David Farmer, Henri and Salman wrote gp code to compute only the a coefficient out to 10^5, which took 6.5 hours. At the same time, they optimized the code heavily by performing all the computations mod p^2 instead of p-adically, which reduced the computation of the a's from 6.5 hours to about 11 minutes.
- Mark and Henri numerically verified independently that the gamma factors and conjectural conductor was correct for $\psi = -1$ (or was it $\psi = 2$?).

Friday, July 24

- Henri and Fernando worked out the formulas for the approximal functional equation in the case of our L-function in the most naive way (setting the smoothing function g in Mike's notation to be 1). Let $W(s)$ encapsulate the gamma factors, conductor, and power of 2π for our L-function:

$$W(s) := \left(\frac{2\pi}{a}\right)^{-2s} \Gamma(s-1)\Gamma(s),$$

where our functional equation will send s to $4-s$. Then the Mellin transform of W is given by the K-Bessel function:

$$\mathcal{M}^{-1}(W) = Y(x) = \frac{a}{\pi} x^{-1/2} K_1\left(\frac{4\pi}{a} x^{1/2}\right).$$

The incomplete Mellin transform is then

$$F(x,s) = \int_x^\infty t^s Y(t) \frac{dt}{t} = W(s) - \int_0^x t^s Y(t) \frac{dt}{t}.$$

Setting $W(x,s) = \int_0^x t^s Y(t) \frac{dt}{t}$, we find

$$W(x,s) = 8\left(\frac{a}{4\pi}\right)^{2s} \left[-\left(\frac{4\pi}{a} x^{1/2}\right)^{2s-2} K_0\left(\frac{4\pi}{a} x^{1/2}\right) + (2s-2) I\left(\frac{4\pi}{a} x^{1/2}, 2s-2\right) \right],$$

where

$$I(x,s) = \int_0^x t^s K_0(t) \frac{dt}{t}.$$

A little work shows that

$$I(x,s) = (\log 2 - \gamma - \log x) \sum_{k \geq 0} \frac{x^{2k+s}}{(2k+s) 2^{2k} k!^2} + \sum_{k \geq 0} \left[\frac{x^{2k+s}}{(2k+s) 2^{2k} k!^2} \left(H_k + \frac{1}{2k+s} \right) \right],$$

where γ is Euler's constant and $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$ is the harmonic series truncated at k. Putting all this together allows us to compute $F(x, s)$, and using the coefficients of our L-function for the quintic surface, we aim to determine the unknown a (and hopefully the conductor) by varying x and s.

Thursday, July 23

- Fernando presented a p-adic hypergeometric function (following Katz): Fix p a prime and set

$$H\left(\begin{matrix}\alpha \\ \beta\end{matrix}\middle|\lambda\right) = \frac{1}{(1-p)c_0\left(\begin{matrix}\alpha \\ \beta\end{matrix}\right)} \sum_{m=0}^{p-2} c_m\left(\begin{matrix}\alpha \\ \beta\end{matrix}\right) Teich(\lambda)^{-m},$$

where α, β are are sequences of rational numbers, λ is a parameter, and the c_m's are given below.

$$c_m\left(\begin{matrix}\alpha \\ \beta\end{matrix}\right) = \prod_{j=1}^{r} \frac{\Gamma_p\left(\left\{\alpha_j + \frac{m}{p-1}\right\}\right)}{\Gamma_p\left(\left\{1-\beta_j + \frac{m}{p-1}\right\}\right)} \times (-p)^{\mathcal{L}_0(\alpha,\beta,m)} \times p^{-\mathcal{L}_1(\beta,m)} \times \Gamma_p(1/2)^d,$$

where r is the length of α and β, Γ_p is the p-adic gamma function, $\{a\}$ is the fractional part of a,

$$\mathcal{L}_0(\alpha, \beta, m) = \sum_{j=1}^{r} \left(\left\{\alpha_j + \frac{m}{p-1}\right\} - \left\{1-\beta_j + \frac{m}{p-1}\right\}\right) + \frac{d}{2},$$

$$\mathcal{L}_1(\beta, m) = \#\left\{j \mid \left\{1-\beta_j + \frac{m}{p-1}\right\} = 0\right\},$$

and

$$d = \#\{j \mid \beta_j = 1\}.$$

Using this, we were able to show computationally that for $w \neq 0, 1, 1/2$

$$H\left(\begin{matrix}1/4 & 3/4 \\ 1 & 1\end{matrix}\middle|w\right)^2 = p + H\left(\begin{matrix}1/4 & 1/2 & 3/4 \\ 1 & 1 & 1\end{matrix}\middle|4w(1-w)\right).$$

The left hand side of the above equation is the square of the trace of Frobenius on H^1 (of ?) while the right hand side is the trace of Frobenius on H^2 (of ?).

Wednesday, July 22

- We pushed to the $d = 4$ case (we switch from ψ to t):

$$S_t : x_1^4 + x_2^4 + x_3^4 + x_4^4 - 4tx_1x_2x_3x_4 = 0.$$

This surface has periods corresponding to the hypergeometric sum

$${}_3F_2\left(\begin{matrix} \frac{1}{4} \ \frac{3}{4} \\ 1 \end{matrix} \middle| 4^4\lambda \right)$$

where $\lambda = 1/(4t)^4$. We have

$${}_3F_2\left(\begin{matrix} \frac{1}{4} \ \frac{3}{4} \\ 1 \end{matrix} \middle| 4^4\lambda \right) = {}_2F_1\left(\begin{matrix} \frac{1}{8} \ \frac{3}{8} \\ 1 \end{matrix} \middle| 4^4\lambda \right)^2.$$

Using some relations on hypergeometric sums, we determined

$${}_2F_1\left(\begin{matrix} \frac{1}{8} \ \frac{3}{8} \\ 1 \end{matrix} \middle| 4z(z-1) \right) = {}_2F_1\left(\begin{matrix} \frac{1}{4} \ \frac{3}{4} \\ 1 \end{matrix} \middle| z \right).$$

Moreover

$${}_2F_1\left(\begin{matrix} \frac{1}{4} \ \frac{3}{4} \\ 1 \end{matrix} \middle| z \right) = \sum \frac{(4n)!}{(2n)!n!^2} \left(\frac{z}{4^3}\right)^n = \omega_E\left(\frac{z}{4^3}\right)$$

where ω_E is the period of the elliptic curve given by the equation

$$E : x^2y + y^2 + 1 = uxy,$$

and $z = 4^3/u^4$. In Weierstrass form,

$$E : y^2 - uxy = x^3 + x$$

with discriminant $u^4 - 4^3$. Moreover, we have the following relations on our parameters:

$$\lambda = \frac{u^4 - 4^3}{u^8}, \qquad t^4 - 1 = \frac{(u^4 - 128)^2}{4^4(u^4 - 4^3)}.$$

We can make a further change of variables by setting $u = 1/v$ to get

$$E_v : y^2 - xy = x^3 + vx.$$

In this form we have

$$\sum_{n\geq 0} \frac{(4n)!}{n!^4}(v(1-4^3v))^n = \left(\sum_{n\geq 0} \frac{(4n)!}{(2n)!n!^2} v^n \right)^2.$$

We know hope to compute the conductor of the symmetric square of E with Mark's help.

Tuesday, July 21

- Xenia (and Philip) introduced Calabi-Yau manifolds on a "baby-baby" level as a precursor/preview of the later workshop talk. They also discussed why Calabi-Yau manifolds are interesting to physicists and why they would be interested in the number of their $\mathbf{F}_p$-solutions (which are intimately related to the periods of the manifold, which are of prime importance to Calabi-Yau manifolds).
- Philip discussed the complex structure and Kahler class of the moduli spaces of Calabi-Yau manifolds. He also discussed mirror symmetry and its implications for the associated zeta functions, as well as motivating the interest in Siegel modular forms.
- The group set up a goal to determine the size of the conductor of the Siegel modular forms that may be related to Siegel modular forms. Henri ran some (very quick!) experiments in Pari/GP to see what happens in the elliptic curve case from which we settled on the formula above.

Monday, July 20

- Fernando discussed some (number-theoretic) motivation for why we want to look at p-adic methods to compute L-functions. In particular, he gave an overview of how to count the number of points on the Legendre family of elliptic curves

  $$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

  using the p-adic gamma function. He also suggested an overarching strategy of computing traces of Frobenius using p-adic methods and without knowing anything about the geometric object on which Frobenius is acting.
- Henri presented the p-adic gamma function, using a construction using the Hurwitz ζ-function and Volkenborn integral that avoids viewing the gamma function initially as an interpolation. One can recover interpolation however with a little bit of care.

Benasque09/p-adic (last edited 2009-07-30 17:28:16 by SalmanButt)