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Quantum Dynamical Entropies and Complexity in Dynamical Systems

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Introduction

Classical chaos is associated with motion on a compact phase–space with high sensitivity to initial conditions: trajectories diverge exponentially fast and nevertheless remain confined to bounded regions [1–7].

In discrete time, such a behaviour is characterized by a positive Lyapounov exponent $\log \lambda$, $\lambda > 1$, and by a consequent spreading of initial errors δ such that, after n time–steps, $\delta \mapsto \delta_n \simeq \delta\, \lambda^n$. Exponential amplification on a compact phase–space cannot grow indefinitely, therefore the Lyapounov exponent can only be obtained as:

$$\log \lambda := \lim_{t\to\infty} \frac{1}{n} \lim_{\delta\to 0} \log\left(\frac{\delta_n}{\delta}\right) \ , \tag{1}$$

that is by first letting $\delta \to 0$ and only afterwards $n \to \infty$.

In quantum mechanics non–commutativity entails absence of continuous trajectories or, semi–classically, an intrinsic coarse–graining of phase–space determined by Planck's constant $\hbar$: this forbids δ (the minimal error possible) to go to zero. Indeed, nature is fundamentally quantal and, according to the correspondence principle, classical behaviour emerges in the limit $\hbar \to 0$.

Thus, if chaotic behaviour is identified with $\log \lambda > 0$, then it is quantally suppressed, unless, performing the classical limit first, we let room for $\delta \to 0$ [6].

Another way to appreciate the regularity emerging from quantization, is to observe that quantization on compacts yields discrete energy spectra which in term entail quasi–periodic time–evolution [8].

In discrete classical systems, one deals with discretized versions of continuous classical systems, or with cellular automata [9–11] and neural networks [12] with finite number of states. In this case, roughly speaking, the minimal distance between two states or configurations is strictly larger than zero; therefore, the reason why $\log \lambda$ is trivially zero is very much similar to the one encountered in the field of quantum chaos, its origin being now not in non–commutativity but in the lack of a continuous structure. Alternative methods have thus to be developed in order to deal with the granularity of phase–space [9–11,13,14].

A signature of chaotic properties of quantized/discretized dynamical systems is the presence of a so called *breaking–time* τ_B, that is a time (depending on the quantization parameter $\hbar$) fixing the time–scale where quantum and classical mechanics are expected to almost coincide. Usually τ_B scales as $\hbar^{-\alpha}$ for some $\alpha > 0$ [7] for regular classical limits, that is for systems that are (classically) regular; conversely, for chaotic systems, the semi-classical regime typically scales as $-\log\hbar$ [5–7]. Both time scales diverge when $\hbar \to 0$, but the shortness of the latter means that classical mechanics has to be replaced by quantum mechanics much sooner for quantum systems with chaotic classical behaviour. The *logarithmic breaking time* $-\log\hbar$ has been considered by some as a violation of the correspondence principle [15,16], by others, see [6] and Chirikov in [5], as the evidence that time and classical limits do not commute.

This phenomenon has also been studied for quantized/discretized dynamical systems with finite number of states, possessing a well–defined classical/continuous limit. For instance, consider a discretized classical dynamical system: the breaking–time can be heuristically estimated as the time when the minimal error permitted, (δ: in the present case coincide with the $\hbar$–like parameter, that is the lattice spacing of the grid on which we discretize the system) , becomes of the order of the phase–space bound Δ. Therefore, when, in the continuum, a Lyapounov exponent $\log\lambda > 0$ is present, the breaking–time scales as $\tau_B = \dfrac{1}{\log\lambda}\log\dfrac{\Delta}{\delta}$.

In order to inquire how long the classical and quantum behaviour mimic each other, we need a witness of such "classicality", that be related (as we have seen) to the presence of positive Lyapounov exponent. By the theorems of Ruelle and Pesin [17], the positive Lyapounov exponents of smooth, classical dynamical systems are related to the dynamical entropy of Kolmogorov [3] (KS–entropy or *metric entropy*) which measures the information per time step provided by the dynamics. The phase–space is partitioned into cells by means of which any trajectory is encoded into a sequence of symbols. As times goes on, the richness in different symbolic trajectories reflects the irregularity of the motion and is associated with strictly positive dynamical entropy [18].

So, since the metric entropy is related to the positive Lyapounov exponent, and the latter are indicators of chaos in the semi–classical regime (for classically chaotic systems), it is evident how the KS–entropy could be profitably used to our purpose. However, the metric entropy can be defined only on measurable classical systems, and we need then to replace it with some tool more appropriate to finite, discrete context. In view of the similarities between quantization and discretization, our proposal is to use quantum extension of the metric entropy.

There are several candidates for non–commutative extensions of the latter [19–23]: in the following we shall use two of them [19, 20] and study their classical/continuous limits.

The most powerful tools in studying the semi–classical regime consist essentially in focusing, via coherent state (C.S.) techniques, on the phase space localization of specific time evolving quantum observables. For this reason we will make use of an Anti–Wick procedure of quantization, based on C.S. states, which can be applied also to algebraically discretize of classical continuous systems. Developing discretization–methods, mimicking quantization procedures, allow us to compute quantum dynamical entropies in both quantum and classical–discrete systems.

The entropies we will use are the CNT–entropy (Connes, Narnhofer and Thirring) and the ALF–entropy (Alicky, Lindblad and Fannes) generically which differ on quantum systems but coincide with the Kolmogorov metric entropy on classical ones. All these dynamical entropies are long–time entropy rates and therefore all vanish in systems with finite number of states. However, this does not mean that on finite–time scales, there might not be an entropy production, but only that sometimes it has to stop.

It is exactly the analytical/numerical study of this phenomenon of finite–time chaos that we will be concerned within this work [24, 25].

As particular examples of quantum dynamical systems with chaotic classical limit, we shall consider finite dimensional quantizations of hyperbolic automorphisms of the 2-torus, which are prototypes of chaotic behaviour; indeed, their trajectories separate exponentially fast with a Lyapounov exponent $\log\lambda > 0$ [26, 27]. Standard quantization, à la Berry, of hyperbolic automorphisms [28, 29] yields Hilbert spaces of a finite dimension N. This dimension plays the role of semi–classical parameter and sets the minimal size $1/N$ of quantum phase space cells.

On this family of quantum dynamical systems we will compute the two entropies mentioned above, showing that, from both of them, one recovers the Kolmogorov entropy by computing the average quantum entropy produced over a logarithmic time scale and then taking the classical limit [24]. This confirms the numerical results in [30], where the dynamical entropy [20] is applied to the study of the quantum kicked top. In this approach, the presence of logarithmic time scales indicates the typical scaling for a joint time–classical limit suited to preserve positive entropy production in quantized classically chaotic quantum systems.

For what concerns discrete systems, we will enlarge the set of classical systems from the hyperbolic automorphisms of the 2-torus to the larger class of Sawtooth Maps. For such systems, in general singular contrary to smooth hyperbolic ones, we will provide a rigorous discretization scheme with corresponding continuous limit in which we will study the behavior of the ALF–entropy.

The ALF–entropy is based on the algebraic properties of dynamical systems, that is on the fact that, independently on whether they are commutative or not, they are describable by suitable algebras of observables, their time evolution by linear maps on these algebras and their states by expectations over them.

Profiting from the powerful algebraic methods to inquire finite–time chaos, we will numerically compute the ALF–entropy in discrete systems, and the performed analysis [25] clearly show the consistency between the achieved results and our expectations of finding a logarithmic breaking–time.

Chapter 1

Quantization and Discretization on the Torus

1.1. Algebraic settings

1.1.1. Dynamical Systems

Usually, continuous classical motion is described by means of a measure space $\mathcal{X}$, the phase–space, endowed with the Borel σ–algebra and a normalized measure μ, $\mu(\mathcal{X}) = 1$. The "volumes"

$$\mu(E) = \int_E \mathrm{d}\mu(\boldsymbol{x})$$

of measurable subsets $E \subseteq \mathcal{X}$ represent the probabilities that a phase–point $\boldsymbol{x} \in \mathcal{X}$ belong to them. By specifying the statistical properties of the system, the measure μ defines a "state" of it.

In such a scheme, a reversible discrete time dynamics amounts to an invertible measurable map $T : \mathcal{X} \mapsto \mathcal{X}$ such that $\mu \circ T = \mu$ and to its iterates $\{T^k \mid k \in \mathbb{Z}\}$. Phase–trajectories passing through $\boldsymbol{x} \in \mathcal{X}$ at time 0 are then sequences $\left\{T^k \boldsymbol{x}\right\}_{k\in\mathbb{Z}}$ [3].

Classical dynamical systems are thus conveniently described by triplets $(\mathcal{X}, \mu, T)$.

In the present work we shall focus upon the following:

- $\mathcal{X}$ – a compact metric space:

the 2–dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 = \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 \pmod 1\}$. We will use the symbol $\mathcal{X}$ to refer to generic compact measure spaces, otherwise the specific symbol $\mathbb{T}^2$;

- μ – the Lebesgue measure $\mu(\mathrm{d}\boldsymbol{x}) = \mathrm{d}x_1\, \mathrm{d}x_2$ on $\mathbb{T}^2$;
- T – invertible measurable transformations from $\mathcal{X}$ to itself such that T^{-1} are also measurable.

For this kind of systems we also provide an algebraic description, consisting in associating to them algebraic triplets $(\mathcal{M}, \omega, \Theta)$, where:

- $\mathcal{M}$ – is a C* or a Von Neumann -algebra. Non–commutative algebras characterize quantum dynamical systems and the elements of $\mathcal{M}$ are nothing but the observables, usually acting as bounded operators on a suitably defined Hilbert space $\mathcal{H}$.

  Commutativity will be characteristic of algebras describing classical systems, as the ones that we are going to introduce in the next Section 1.1.2.
- ω – denotes a reference state on $\mathcal{M}$, that is a positive linear and normalized functional on it.
- $\{\Theta^k \mid k \in \mathbb{Z}\}$ – is the discrete group of *-automorphisms[1] of $\mathcal{M}$ implementing the dynamics that leave the state ω invariant, i.e. $\omega \circ \Theta = \omega$.

1.1.2. Two useful algebras on the torus

We introduce now two functional spaces, that will be profitably used for later purpose.

The first one is the Abelian C*–algebra $\mathcal{C}^0(\mathcal{X})$ of complex valued continuous functions with respect to the topology given by the *uniform norm*

$$\|f\|_0 = \sup_{\boldsymbol{x}\in\mathcal{X}} \Big| f(\boldsymbol{x}) \Big| \,. \tag{1.1}$$

The second functional space we are going to introduce is the Abelian (Von Neumann) algebra $L^\infty_\mu(\mathcal{X})$ of essentially bounded functions on $\mathcal{X}$. The meaning of “essentially” is

[1] A *-automorphism Θ of a C* algebra $\mathcal{M}$ is defined to be a *-isomorphism of $\mathcal{M}$ into itself, i.e., Θ is a *-morphism of $\mathcal{M}$ with range equal to $\mathcal{M}$ and kernel equal to zero. In order to be defined as a *-automorphism, a map Θ has to preserve the algebraic structure of $\mathcal{M}$, namely for all $m_1, m_2 \in \mathcal{M}$ it must hold: $\Theta(m_1 + m_2) = \Theta(m_1) + \Theta(m_2)$, $\Theta(m_1 m_2) = \Theta(m_1)\,\Theta(m_2)$ and $\Theta(m_1^*) = \Theta^*(m_1)$

that these function have to be bounded with respect to the so called "essential norm" $\|\cdot\|_\infty$, namely the essential supremum defined by [31]:

$$\|f\|_\infty := \operatorname*{ess\,sup}_{\boldsymbol{x}\in\mathcal{X}} |\, f\,| = \inf\left\{ \; a \in \mathbb{R} \;\middle|\; \mu\Big(\{\; \boldsymbol{x} \;:\; |f\,(\boldsymbol{x})\,| > a \;\}\Big) = 0 \;\right\} \cdot \tag{1.2}$$

This norm is slightly different from the one defined in (1.1), when taken on functions belonging to $L^\infty_\mu(\mathcal{X})$; for instance, two functions that differ only on a set of null measure (for instance on a single point), will have the same norm given by (1.2). Also, if $f \in \mathcal{C}^0(\mathcal{X})$, then $\|f\|_\infty = \|f\|_0$.

From now on we adopt the symbol $\mathcal{A}_\mathcal{X}$ to denote both algebras distinguishing them when necessary.

The Lebesgue measure μ defines a state ω_μ on $\mathcal{A}_\mathcal{X}$ via integration

$$\omega_\mu : \mathcal{A}_\mathcal{X} \ni f \longmapsto \omega_\mu(f) := \int_\mathcal{X} \mathrm{d}\mu(\boldsymbol{x})\; f(x) \in \mathbb{R}^+ \; ; \tag{1.3}$$

this will be our reference state for the algebras of $\mathcal{A}_\mathcal{X}$–types.

1.2. Quantization procedures

Once the algebraic triplet $\big(\mathcal{A}_\mathcal{X}, \omega_\mu, \Theta\big)$ has been fixed, the approach of Section 1.1.1 provide a general formalism that allows us to deal with generic dynamical systems.

Remarks 1.2.1

i) Of course we could provide different triplets describing systems that, in a suitable classical limit (argument of next Chapter 2), "correspond" to the same classical dynamical system $\big(\mathcal{X}, \mu, T\big)$. In particular, different quantum systems (mimicking each other in the semi classical limit) can be constructed by using different algebras $\mathcal{M}$, with the latter chosen among commutative or not, finite or infinite dimensional, and so on.

ii) In the future we will restrict ourselves to consider finite dimensional algebras $\mathcal{M}_N$, but even with this restriction, the set of possible choice is quite large. Intuitively we can think that a classical dynamical system is supposed to be described by using an abelian algebra (and this is the case), nevertheless it is not enough to say that a non–abelian algebra provide a "good" description of quantum systems.

Assigned a classical dynamical system $(\mathcal{A}_{\mathcal{X}}, \omega_\mu, \Theta)$, the aim of a quantization–dequantization procedure (specifically an N–dimensional quantization) is twofold:

- to find a couple of *-morphism, $\mathcal{J}_{N,\infty}$ mapping $\mathcal{A}_{\mathcal{X}}$ into a non abelian finite dimensional algebra $\mathcal{M}_N$ and $\mathcal{J}_{\infty,N}$ mapping backward $\mathcal{M}_N$ into $\mathcal{A}_{\mathcal{X}}$;
- to provide an automorphism Θ_N acting on $\mathcal{M}_N$ representing the quantized classical evolution Θ such that the two dynamics, the classical one on $\mathcal{A}_{\mathcal{X}}$ and the quantum one on $\mathcal{M}_N$, commute with the action of the two *-morphisms connecting the two algebras, that is
$$\mathcal{J}_{N,\infty} \circ \Theta^j \simeq \Theta_N^j \circ \mathcal{J}_{\infty,N} \tag{1.4}$$
The latter requirement can be seen as a modification of the so called Egorov's property (see [32]).

The difficulties in finding a convenient quantization procedure are due to two (equivalent) facts:

- as far as we know from quantum mechanics, once we assign in the algebra $\mathcal{M}_N$ the operators corresponding to classical observables, some relations have to be respected. These relation connected with the physics underlying our system. For instance, in our work, we will impose Canonical Commutation Relations (CCR for short);
- once a quantization parameter (something playing the role of $\hbar$, on which the two *-morphism $\mathcal{J}_{N,\infty}$ and $\mathcal{J}_{\infty,N}$ have to be dependent) is let to go to zero, the correspondence between classical and quantum observables has to be fixed in a way that allow us to speak of a "classical limit".

The latter observation will be discussed in Chapter 2, in which we will provide our quantization procedure and a suitable classical limit, whereas the CCR problem will be the core of the next Section.

1.2.1. Finite dimensional Quantization on the torus

We now consider the (non commutative) finite dimensional algebra $\mathcal{M}_N$ of $N \times N$ matrices acting on a N–dimensional Hilbert space $\mathcal{H}_N = \mathbb{C}^N$. Let us give a Definition of a state τ_N on matrix algebras, that will be used in the following.

Definition 1.2.1

We will denote by τ_N the state given by the following positive, linear and normalized functional over $\mathcal{M}_N$:

$$\tau_N : \mathcal{M}_N \ni M \longmapsto \tau_N(M) := \frac{1}{N}\,\mathrm{Tr}\,(M) \in \mathbb{R}^+ \cdot$$

Due to the finiteness dimension, it is impossible to find in $\mathcal{M}_N$ two operators $\hat{Q}$, $\hat{P}$ playing the role of position, respectively momentum, satisfying CCR [33]. Indeed, taking the trace of the basic equation

$$\left[\hat{Q}, \hat{P}\right] = i\,\hbar\,\mathbb{1} \ , \tag{1.5}$$

the ciclicity property of the trace gives us $0 = i\,\hbar\,N$.

Nevertheless, as in the Schrödinger representation, $\hat{P}$ is the generator of the (compact) Lie group of space translations, while $\hat{Q}$ acts as the generator of the group of momentum translations. The form and the action of the shift operator $\hat{U}$, $\hat{V}$, in position (q), respectively momentum (p), coordinates are given by:

$$\hat{U}\,(\mathrm{d}q)\,|\,q\,\rangle := e^{-\frac{i\,\hat{P}\,\mathrm{d}q}{\hbar}}\,|\,q\,\rangle = |\,q + \mathrm{d}q\,\rangle \quad , \quad \hat{U}\,(\mathrm{d}q)\,|\,p\,\rangle := e^{-\frac{i\,\hat{P}\,\mathrm{d}q}{\hbar}}\,|\,p\,\rangle = e^{-\frac{i\,p\,\mathrm{d}q}{\hbar}}\,|\,p\,\rangle \ , \tag{1.6}$$

$$\hat{V}\,(\mathrm{d}p)\,|\,p\,\rangle := e^{\ \frac{i\,\hat{Q}\,\mathrm{d}p}{\hbar}}\,|\,p\,\rangle = |\,p + \mathrm{d}p\,\rangle \quad , \quad \hat{V}\,(\mathrm{d}p)\,|\,q\,\rangle := e^{\ \frac{i\,\hat{Q}\,\mathrm{d}p}{\hbar}}\,|\,q\,\rangle = e^{\ \frac{i\,q\,\mathrm{d}p}{\hbar}}\,|\,q\,\rangle \ . \tag{1.7}$$

Using (1.5) and the Baker–Hausdorff's Lemma, we get from (1.6–1.7):

$$\hat{U}\,(\mathrm{d}q)\,\hat{V}\,(\mathrm{d}p) = \hat{V}\,(\mathrm{d}p)\,\hat{U}\,(\mathrm{d}q)\,e^{-\frac{i\,\mathrm{d}q\,\mathrm{d}p}{\hbar}} \ . \tag{1.8}$$

Of course (1.8) is unchanged if we define $\hat{U}$ and $\hat{V}$ up to phases.

The latter relation can be a good starting point for a quantization procedure [34, 35]. Given a N–dimensional Hilbert space $\mathcal{H}_N = \mathbb{C}^N$, its basis can be labeled by $\{|\,q_\ell\,\rangle\}_{\ell=0\,\ldots,N-1}$. If we want to interpreter this basis as a "$\mathbb{T}^2$: q–coordinates" basis, we have to respect the toral topology and to add the folding condition, namely $|\,q_{\ell+N}\,\rangle = |\,q_\ell\,\rangle$ for all ℓ belonging to $(\mathbb{Z}/N\mathbb{Z})$, the residual class (mod N). In a similar way we could choose a "$\mathbb{T}^2$: p–coordinates" representation, by choosing a basis $\{|\,p_m\,\rangle\}_{m=0\,\ldots,N-1}$ endowed with the same folding condition $|\,p_{m+N}\,\rangle = |\,p_m\,\rangle\,,\ \forall m \in (\mathbb{Z}/N\mathbb{Z})$.

The coordinates $(q_\ell, p_m) = \left(\frac{\ell}{N}, \frac{m}{N}\right)$ will label the points of a square grid of lattice spacing $\frac{1}{N}$ lying on the torus $\mathbb{T}^2$.

On this grid, we can construct two unitary shift operators U_N and V_N mimicking equations (1.6–1.7); we will explicitly indicate the dependence of the representation on two arbitrary phases (α, β):

$$U_N\,(\mathrm{d}q)\,|\,q_j\,\rangle := e^{i\,\alpha}\,|\,q_j + \mathrm{d}q\rangle \quad , \quad U_N\,(\mathrm{d}q)\,|\,p_j\,\rangle := e^{i\,\alpha}e^{-i\frac{j\,\mathrm{d}q}{N\hbar}}\,|\,p_j\,\rangle \ , \tag{1.9}$$

$$V_N\,(\mathrm{d}p)\,|\,p_j\,\rangle := e^{i\,\beta}\,|\,p_j + \mathrm{d}p\rangle \quad , \quad V_N\,(\mathrm{d}p)\,|\,q_j\,\rangle := e^{i\,\beta}e^{\ i\frac{j\,\mathrm{d}p}{N\hbar}}\,|\,q_j\,\rangle \ . \tag{1.10}$$

If we want that these operator act "infinitesimally", we have to tune them according to the minimal distance (in q and p) coordinates permitted by the granularity of the phase–space, that is we have to fix $U_N := \hat{U}\left(\mathrm{d}q = q_1 = \frac{1}{N}\right)$ and $V_N := \hat{V}\left(\mathrm{d}p = p_1 = \frac{1}{N}\right)$. Thus the action of U_N and V_N on the q–basis can be rewritten as

$$U_N\,|\,q_j\,\rangle := e^{i\,\alpha}\,|\,q_{j+1}\,\rangle \quad , \quad V_N\,|\,q_j\,\rangle := e^{i\,\beta}e^{\ \frac{i}{N}\frac{1}{N\hbar}j}\,|\,q_j\,\rangle \ . \tag{1.11}$$

Now it remains to impose the folding condition on the operators U_N and V_N, that is

$$U_N^N = e^{2i\pi u}\,\mathbb{1}_N, \quad V_N^N = e^{2i\pi v}\,\mathbb{1}_N \tag{1.12}$$

where $u = \alpha\,\frac{N}{2\pi}$ and $v = \beta\,\frac{N}{2\pi}$ can be chosen to belong to $[0,1)$ and are parameters labeling the representations.

If we want $V_N^N = e^{2i\pi v}\,\mathbb{1}_N$ to hold we have o fix $\frac{1}{N\hbar} = \frac{2\pi}{Nh} = 2\pi s \in \mathbb{Z}$; without loss of generality [29], we choose $s = -1$.
Then, from identity $h = -\frac{1}{N}$, it turns out that our quantization parameter is given by N, the dimension of Hilbert space, and we expect to recover the classical behaviour (namely commutativity) when $N \to \infty$. This is evident from (1.8), that now reads

$$U_N V_N = e^{\frac{2\pi i}{N}}\,V_N U_N \ \cdot \tag{1.13}$$

Upon changing the labels of the o.n.b.[2] of the $\mathcal{H}_N$ by letting $|\,q_j\,\rangle \longmapsto |\,j\,\rangle$, equation (1.11) can more conveniently be written as

$$U_N\,|\,j\,\rangle := e^{\frac{2\pi i}{N}u}\,|\,j+1\,\rangle \quad \text{and} \quad V_N\,|\,j\,\rangle := e^{\frac{2\pi i}{N}(v-j)}\,|\,j\,\rangle \ , \tag{1.14}$$

By mimicking the usual algebraic approach to CCR in the continuous case, we introduce Weyl operators labeled by $\boldsymbol{n} = (n_1, n_2) \in \mathbb{Z}^2$

$$W_N(\boldsymbol{n}) := e^{\frac{i\pi}{N}n_1 n_2}\,V_N^{n_2} U_N^{n_1} \ , \tag{1.15}$$

$$W_N^*(\boldsymbol{n}) = W_N(-\boldsymbol{n}) \ \cdot \tag{1.16}$$

[2]orthonormal basis

Their explicit action on the o.n.b. $\{|\,j\,\rangle\}_{j=0,1,\cdots,N-1}$ is given by

$$W_N(\boldsymbol{n})\;|\,j\,\rangle = \exp\left(\frac{i\pi}{N}(-n_1n_2+2n_1u+2n_2v)\right)\,\exp\left(-\frac{2i\pi}{N}jn_2\right)\;|\,j+n_1\,\rangle\;, \tag{1.17}$$

whence

$$W_N(N\boldsymbol{n}) = e^{i\pi(Nn_1n_2+2n_1u+2n_2v)}\;, \tag{1.18}$$

$$W_N(\boldsymbol{n})W_N(\boldsymbol{m}) = e^{\frac{i\pi}{N}\sigma(\boldsymbol{n},\boldsymbol{m})}\,W_N(\boldsymbol{n}+\boldsymbol{m}), \tag{1.19}$$

where $\sigma(\boldsymbol{n},\boldsymbol{m}) = n_1m_2 - n_2m_1$ is the so–called simplectic form. From equation (1.19) one derives

$$[W_N(\boldsymbol{n}), W_N(\boldsymbol{m})] = 2i\sin\left(\frac{\pi}{N}\,\sigma(\boldsymbol{n},\boldsymbol{m})\right)\,W_N(\boldsymbol{n}+\boldsymbol{m}),$$

which shows once more how recovering Abelianness is related to $N \longrightarrow \infty$.

Definition 1.2.2

The *Weyl Algebra* is the C*-algebra over $\mathbb{C}$ generated by the (discrete) group of Weyl operators

$$\{W_N(\boldsymbol{n})\}_{\boldsymbol{n}\in\mathbb{Z}^2}\;.$$

Remarks 1.2.2 (The Weyl group)

i) Let us comment now on the role played by the two parameter (u, v) introduced in (1.12): until now they are arbitrary parameters and we will fix them by inserting the dynamics into our scheme of quantization. Actually, although the Weyl group introduced in Definition 1.2.2 is just supposed to fulfill relations (1.16) and (1.19), choosing a couple of parameters (u, v) we choose a definite representation $\pi_{(u,v)}$ of the (abstract) Weyl group $\{W_N(\boldsymbol{n})\}_{\boldsymbol{n}\in\mathbb{Z}^2}$. In order to classify all possible representations of the Weyl group, we cite now [29] a useful

Theorem 1 :

a) $\pi_{(u,v)}$ is an irreducible *-representation of $\{W_N(\boldsymbol{n})\}_{\boldsymbol{n}\in\mathbb{Z}^2}$

b) $\pi_{(u,v)}$ is unitarily equivalent[3] to $\pi_{(\widetilde{u},\widetilde{v})}$ iff $(u, v) = (\widetilde{u}, \widetilde{v})$

[3] It means that exists an unitary operator U such that $U\pi_{(u,v)}U^\dagger = \pi_{(\widetilde{u},\widetilde{v})}$

ii) Once the generators of the group[4] $w_i := W_N(\hat{\boldsymbol{e}}_i)$ are assigned, a representation π is chosen; the whole Weyl group $\{W_N(\boldsymbol{n})\}_{\boldsymbol{n}\in\mathbb{Z}^2}$ can be constructed just by using relations (1.16) and (1.19).

Finally, by manipulating the matrix element of $W_N(\boldsymbol{n})$ given in (1.17), one easily derives the following

Properties 1.2.1

Let τ_N of Definition 1.2.1 be our quantum reference state; then it holds

$$\tau_N(W_N(\boldsymbol{n})) = e^{\frac{i\pi}{N}(-n_1 n_2 + 2n_1 u + 2n_2 v)}\,\delta^{(N)}_{\boldsymbol{n},\boldsymbol{0}}\ , \tag{1.20}$$

$$\frac{1}{N}\sum_{p_1,p_2=0}^{N-1} W_N(-\boldsymbol{p})\,W_N(\boldsymbol{n})\,W_N(\boldsymbol{p}) = \mathrm{Tr}\left(W_N(\boldsymbol{n})\right)\,\mathbb{1}_N\ , \tag{1.21}$$

$$\mathcal{M}_N \ni X = \sum_{p_1,p_2=0}^{N-1} \tau_N\Big(X\,W_N(-\boldsymbol{p})\Big)\,W_N(\boldsymbol{p})\ , \tag{1.22}$$

where in (1.20) we have introduced the periodic Kronecker delta, that is $\delta^{(N)}_{\boldsymbol{n},\boldsymbol{0}} = 1$ if and only if $\boldsymbol{n} = \boldsymbol{0} \pmod N$.

Notice that, according to (1.22), the Weyl algebra coincides with the $N\times N$ matrix algebra $\mathcal{M}_N$.

1.2.2. Weyl Quantization on the torus

Weyl operators have a nice interpretation in terms of the group of translations generated by $\hat{Q}$ and $\hat{P}$. The two operators U_N and V_N are given in (1.9–1.10) by mimicking the action of $\hat{U}(\mathrm{d}q)$ and $\hat{V}(\mathrm{d}p)$ in (1.6–1.7) (up to two phases u and v), with $\mathrm{d}q = \mathrm{d}p = -h = \frac{1}{N}$. Explicitly, they are formally related to $\hat{Q}$ and $\hat{P}$ by

$$U_N = e^{2\pi i\,\hat{P}} \quad \text{and} \quad V_N = e^{-2\pi i\,\hat{Q}} \tag{1.23}$$

Using Baker–Hausdorff's Lemma, together with (1.15), we obtain

$$W_N(\boldsymbol{n}) = e^{2\pi i\left(n_1\hat{P} - n_2\hat{Q}\right)}\ . \tag{1.24}$$

[4] Here $\hat{\boldsymbol{e}}_1 := \binom{1}{0}$ and $\hat{\boldsymbol{e}}_2 := \binom{0}{1}$, the two basis vector of $\mathbb{R}^2$.

If now we restrict to a subalgebra $\mathcal{A}_{\mathcal{X}}$ consisting of functions sufficiently smooth and regular to be Fourier decomposed, denoting with $\boldsymbol{x} = (x_1, x_2)$ the canonical coordinates (q, p), then the exponential functions $\left\{e^{2\pi i\,\sigma(\boldsymbol{n},\boldsymbol{x})}\right\}_{\boldsymbol{n}\in\mathbb{Z}^2}$ generate $\mathcal{A}_{\mathcal{X}}$, in the sense that

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \hat{f}_{\boldsymbol{n}}\, e^{2\pi i\,\sigma(\boldsymbol{n},\boldsymbol{x})} \tag{1.25}$$

$$\text{where} \quad \hat{f}_{\boldsymbol{n}} = \iint_{\mathbb{T}^2} \mathrm{d}\mu(\boldsymbol{x})\; f(\boldsymbol{x})\; e^{-2\pi i\,\sigma(\boldsymbol{n},\boldsymbol{x})} \tag{1.26}$$

The Weyl Quantization procedure associates functions f to operators $W_{N,\infty}(f) \in \mathcal{M}_N$ via the following *-morphism [34, 35]

$$W_{N,\infty} : \mathcal{A}_{\mathbb{T}^2} \ni f \longmapsto W_{N,\infty}(f) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \hat{f}_{\boldsymbol{n}}\, W_N(\boldsymbol{n}) \in \mathcal{M}_N \ . \tag{1.27}$$

We will postpone the construction of the de–quantizing *-morphism $W_{\infty,N}$, because it involves coherent states that will be introduced in Section (1.4), moreover this construction is completely analogue to the Anti–Wick way of de–quantizing, presented in Section (1.5). In Section 1.3.1 we will construct a concrete example of a Weyl "quantization" procedure, and in Section 1.6 we will invert such a scheme.

1.3. Discretization of the torus over a $N \times N$ square grid

In the following we proceed to a discretization of classical dynamical systems on the torus $\mathbb{T}^2$ that, according to Section 1.1, will be identified with $(\mathcal{A}_{\mathbb{T}^2}, \omega_\mu, \Theta)$. As in the introduction to the quantization methods, we postpone the role played by the dynamics to Chapter 2 and we start with phase–space discretization.

Roughly speaking, given an integer N, we shall force the continuous classical systems $(\mathcal{A}_{\mathbb{T}^2}, \omega_\mu, \Theta)$ to live on a lattice $L_N \subset \mathbb{T}^2$, of lattice–spacing $\frac{1}{N}$

$$L_N := \left\{ \frac{\boldsymbol{p}}{N} \;\middle|\; \boldsymbol{p} \in (\mathbb{Z}/N\mathbb{Z})^2 \right\} \ , \tag{1.28}$$

where $(\mathbb{Z}/N\mathbb{Z})$ denotes the residual class $(\mathrm{mod}\ N)$. In order to set an algebraic structure for the discretization scheme, we give now some

Definitions 1.3.1

i. $\mathcal{H}_N^D$ will denote an N^2–dimensional Hilbert space;

ii. $\mathcal{D}_N$ will denote the abelian algebra $D_{N^2}(\mathbb{C})$ of $N^2 \times N^2$ matrices ($\mathcal{D}$ standing for diagonal with respect to a chosen o.n.b. $\{|\boldsymbol{\ell}\rangle\}_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} \in \mathcal{H}_N^D$);

iii. To avoid difficulties due to the fact that the "quantum" algebra $\mathcal{M}_N$ and the "discretized" algebra $\mathcal{D}_N$ are indexed by the same "N" but their dimension is different ($N \times N$, $N^2 \times N^2$ respectively), when it will be important to refer to the dimensionality of Hilbert spaces ($\mathcal{H}_N$, $\mathcal{H}_N^D$ respectively) we will use the symbol: $N_\hbar := \dim(\mathcal{H})$.

We can compare discretization of classical continuous systems with quantization; to this aim, we define in the next Section a discretization procedure resembling the Weyl quantization of Sections 1.2.1 and 1.2.2; in practice, we will construct a *-morphism $\mathcal{J}_{N,\infty}$ from $\mathcal{A}_{\mathbb{T}^2}$ into the abelian algebra $\mathcal{D}_N$. The basis vectors will be labeled by the points of L_N, defined in (1.28)

The main point is that, although $\mathcal{J}_{N,\infty}$ maps $\mathcal{A}_{\mathbb{T}^2}$ into a finite dimensional algebra $\mathcal{D}_N$ (and this will be very useful for our purpose), $\mathcal{D}_N$ is abelian, and so endowed with very nice properties. In this scheme discretization can be considered a very useful "toy model" for testing the similarities with quantization and quantum systems as source of granular description, leaving inside non–commutativity.

1.3.1. Weyl Discretization: from $\mathcal{C}^0(\mathbb{T}^2)$ to $\mathcal{D}_N$

In order to define the discretization morphism $\mathcal{J}_{N,\infty}$, we use Fourier analysis and restrict ourselves to the *-subalgebra $\mathcal{W}_{\exp} \in \mathcal{A}_{\mathbb{T}^2}$ generated by the exponential functions

$$W(\boldsymbol{n})(\boldsymbol{x}) = \exp(2\pi i\, \boldsymbol{n}\cdot\boldsymbol{x}) \;, \tag{1.29}$$

where $\boldsymbol{n}\cdot\boldsymbol{x} = n_1 x_1 + n_2 x_2$. The generic element of $\mathcal{W}_{\exp}$ is:

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \hat{f}_{\boldsymbol{n}} W(\boldsymbol{n})(\boldsymbol{x}) \tag{1.30}$$

with finitely many coefficients $\hat{f}_{\boldsymbol{n}} = \iint_{\mathbb{T}^2} \mathrm{d}\boldsymbol{x}\; f(\boldsymbol{x})\; e^{-2\pi i \boldsymbol{n}\boldsymbol{x}}$ different from zero.
On $\mathcal{W}_{\exp}$, formula (1.3) defines a state such that

$$\omega_\mu\left(W(\boldsymbol{n})\right) = \delta_{\boldsymbol{n},\boldsymbol{0}}. \tag{1.31}$$

Following Weyl quantization, we get elements of $\mathcal{D}_N$ out of elements of $\mathcal{W}_{\rm exp}$ by replacing, in (1.30), exponentials with diagonal matrices:

$$W(\boldsymbol{n}) \longmapsto \widetilde{W}(\boldsymbol{n}) := \sum_{\boldsymbol{\ell} \in (\mathbb{Z}/N\mathbb{Z})^2} e^{\frac{2\pi i \boldsymbol{n} \boldsymbol{\ell}}{N}} \left| \boldsymbol{\ell} \right\rangle \left\langle \boldsymbol{\ell} \right| \ , \qquad \boldsymbol{\ell} = (\ell_1, \ell_2) \cdot \tag{1.32}$$

We will denote by $\mathcal{J}^{\mathcal{W}}_{N,\infty}$, the *-morphism from the *-algebra $\mathcal{W}_{\rm exp}$ into the diagonal matrix algebra $\mathcal{D}_N$, given by:

$$\mathcal{W}_{\rm exp} \ni f \longmapsto \mathcal{J}^{\mathcal{W}}_{N,\infty}(f) := \sum_{\boldsymbol{n} \in \mathbb{Z}^2} \hat{f}_{\boldsymbol{n}} \, \widetilde{W}(\boldsymbol{n}) \tag{1.33}$$

$$= \sum_{\boldsymbol{\ell} \in (\mathbb{Z}/N\mathbb{Z})^2} f\left(\frac{\boldsymbol{\ell}}{N}\right) \left| \boldsymbol{\ell} \right\rangle \left\langle \boldsymbol{\ell} \right| \cdot \tag{1.34}$$

Remarks 1.3.1

i) The completion of the subalgebra $\mathcal{W}_{\rm exp}$ with respect to the uniform norm given in equation (1.1) is the C*-algebra $\mathcal{C}^0\left(\mathbb{T}^2\right)$ [36].

ii) With the usual *operator norm* $\|\cdot\|_{N^2}$ of $\mathcal{B}\left(\mathcal{H}_N^D\right)$, $(\mathcal{D}_N, \|\cdot\|_{N^2})$ is the C* algebra of $N^2 \times N^2$ diagonal matrices.

iii) The *-morphism $\mathcal{J}^{\mathcal{W}}_{N,\infty} : (\mathcal{W}_{\rm exp}, \|\cdot\|_0) \longmapsto (\mathcal{D}_N, \|\cdot\|_{N^2})$ is bounded by $||\mathcal{J}^{\mathcal{W}}_{N,\infty}|| = 1$. Using the Bounded Limit Theorem [36], $\mathcal{J}^{\mathcal{W}}_{N,\infty}$ can be uniquely extended to a bounded linear transformation (with the same bound) $\mathcal{J}_{N,\infty} : \left(\mathcal{C}^0\left(\mathbb{T}^2\right), \|\cdot\|_0\right) \longmapsto (\mathcal{D}_N, \|\cdot\|_{N^2})$.

iv) Using Remark iii, equation (1.34) can be taken as a definition of $\mathcal{J}_{N,\infty}$, as in the following

Definition 1.3.2

We will denote by $\mathcal{J}_{N,\infty}$, the *-morphism from the C*–algebra $\mathcal{C}^0\left(\mathbb{T}^2\right)$ into the diagonal matrix algebra $\mathcal{D}_N$, given by:

$$\mathcal{J}_{N,\infty} : \mathcal{C}^0\left(\mathbb{T}^2\right) \ni f \longmapsto \mathcal{J}_{N,\infty}(f) = \sum_{\boldsymbol{\ell} \in (\mathbb{Z}/N\mathbb{Z})^2} f\left(\frac{\boldsymbol{\ell}}{N}\right) \left| \boldsymbol{\ell} \right\rangle \left\langle \boldsymbol{\ell} \right| \in \mathcal{D}_N \ \cdot$$

Remark 1.3.2

i. The expectation $\tau_N\left(\mathcal{J}_{N,\infty}\left(f\right)\right)$ (τ_N given in Definition 1.2.1) corresponds to the numerical calculation of the integral of f realized on the grid L_N of (1.28).

1.4. Coherent States

The next Quantization procedure we are going to consider, the Anti–Wick quantization, makes use of coherent states (CS, for short). Moreover, when we use Weyl quantization, the dequantizing operator is constructed by means of CS; actually the most successful semi-classical tools used to study the classical limit, are based on the use of CS. For this reason, in this section we will give a suitable definitions of CS, in the abelian case $\big(\mathcal{A}_{\mathcal{X}}, \omega_\mu, \Theta\big)$ and in the non–abelian one $\big(\mathcal{M}_N, \tau_N, \Theta_N\big)$, that will be of use in quantization schemes.

We remind the reader that in the following, in particular in Definition 1.4.1, $N_\hbar$ introduced in Definition 1.3.1 (iii.), denotes the dimension of the Hilbert space $\mathcal{H}_N = \mathbb{C}^{N_\hbar}$ associated to the algebra $\mathcal{M}_N$ (of $N_\hbar \times N_\hbar$ matrices). As it has already been seen in Section 1.2.1, $N_\hbar$ play also the role of quantization parameter, i.e. we use $1/N_\hbar$ as an h-like parameter. The quantum reference state is τ_N of Definition 1.2.1 and the dynamics is given in terms of a unitary operator U on $\mathcal{H}_N$ in the standard way: $\Theta_N(X) := U^* X\, U$.

In full generality, coherent states will be identified as follows.

Definition 1.4.1

A family $\{|C_N(\boldsymbol{x})\rangle \mid \boldsymbol{x} \in \mathcal{X}\} \in \mathcal{H}_N$ of vectors, indexed by points $\boldsymbol{x} \in \mathcal{X}$, constitutes a set of coherent states if it satisfies the following requirements:

1. Measurability: $\boldsymbol{x} \mapsto |C_N(\boldsymbol{x})\rangle$ is measurable on $\mathcal{X}$;

2. Normalization: $\|C_N(\boldsymbol{x})\|^2 = 1$, $\boldsymbol{x} \in \mathcal{X}$;

3. Overcompleteness: $N_\hbar \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, |C_N(\boldsymbol{x})\rangle\langle C_N(\boldsymbol{x})| = \mathbb{1}$;

4. Localization: given $\varepsilon > 0$ and $d_0 > 0$, there exists $N_0(\epsilon, d_0)$ such that for $N \geq N_0$ and $d_{\mathcal{X}}(\boldsymbol{x}, \boldsymbol{y}) \geq d_0$ one has

$$N_\hbar |\langle C_N(\boldsymbol{x}), C_N(\boldsymbol{y})\rangle|^2 \leq \varepsilon.$$

The symbol $d_{\mathcal{X}}(\boldsymbol{x}, \boldsymbol{y})$ used in the localization property stands for the length of the shorter segment connecting the two points on $\mathcal{X}$. Of course the latter quantity does depend on the topological properties of $\mathcal{X}$ so, with the aim of using it when $\mathcal{X} = \mathbb{T}^2$, we give now the following

Definition 1.4.2

We shall denote by

$$d_{\mathbb{T}^2}(\boldsymbol{x},\boldsymbol{y}) := \min_{\boldsymbol{n}\in\mathbb{Z}^2} \|\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{n}\|_{\mathbb{R}^2} \tag{1.35}$$

the distance on $\mathbb{T}^2$.

The overcompleteness condition may be written in dual form as

$$N_\hbar \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, \langle C_N(\boldsymbol{x}), X\, C_N(\boldsymbol{x})\rangle = \mathrm{Tr}\, X, \quad X \in \mathcal{M}_N.$$

Indeed,

$$N_\hbar \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, \langle C_N(\boldsymbol{x}), X\, C_N(\boldsymbol{x})\rangle = N_\hbar\, \mathrm{Tr}\left(\int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, |C_N(\boldsymbol{x})\rangle\langle C_N(\boldsymbol{x})|\, X\right) = \mathrm{Tr}\, X.$$

In the next three Sections, we define three different sets of states, two of them satisfying Properties in Definition 1.4.1 and then rightly named Coherent States (CS). These sets belong to the two different Hilbert spaces $\mathcal{H}_N$ and $\mathcal{H}_N^{\mathcal{D}}$ defined up to now, and are indexed by points of the torus $\mathbb{T}^2$.

1.4.1. First set of C.S.: $\{|C_N^1(\boldsymbol{x})\rangle \mid \boldsymbol{x} \in \mathbb{T}^2\} \in \mathcal{H}_N$

We shall construct a family $\{|C_N^1(\boldsymbol{x})\rangle \mid \boldsymbol{x} \in \mathbb{T}^2\}$ of coherent states on the 2-torus $\mathbb{T}^2$ by means of the discrete Weyl group introduced in Definition 1.2.2. We define

$$|C_N^1(\boldsymbol{x})\rangle := W_N(\lfloor N\boldsymbol{x}\rfloor)\, |C_N\rangle, \tag{1.36}$$

where $\lfloor N\boldsymbol{x}\rfloor = (\lfloor Nx_1\rfloor, \lfloor Nx_1\rfloor)$, $0 \leqslant \lfloor Nx_i\rfloor \leqslant N-1$ is the largest integer smaller than Nx_i and the fundamental vector $|C_N\rangle$ is chosen to be

$$|C_N\rangle = \sum_{j=0}^{N-1} C_N(j)|j\rangle, \qquad C_N(j) := \frac{1}{2^{(N-1)/2}}\sqrt{\binom{N-1}{j}}. \tag{1.37}$$

Measurability and normalization are immediate, overcompleteness comes as follows. Let Y be the operator in Definition 1.4.1 on the left hand side of property 3 . If $\tau_N(Y\, W_N(\boldsymbol{n})) = \tau_N(W_N(\boldsymbol{n}))$ for all $\boldsymbol{n} = (n_1, n_2)$ with $0 \leqslant n_i \leqslant N-1$, then according to (1.22) applied to

Y it follows that $Y = \mathbb{1}$. This is indeed the case as, using (1.18), (1.20) and N-periodicity,

$$\begin{aligned}\tau_N(Y\,W_N(\boldsymbol{n})) &= \int_{\mathbb{T}} d\boldsymbol{x}\,\langle C_N^1(\boldsymbol{x}), W_N(\boldsymbol{n})\,C_N^1(\boldsymbol{x})\rangle \\ &= \int_{\mathbb{T}} d\boldsymbol{x}\,\exp\Big(\frac{2\pi i}{N}\sigma(\boldsymbol{n},\lfloor N\boldsymbol{x}\rfloor)\Big)\,\langle C_N, W_N(\boldsymbol{n})\,C_N\rangle \\ &= \frac{1}{N^2}\sum_{p_1,p_2=0}^{N-1}\exp\Big(\frac{2\pi i}{N}\sigma(\boldsymbol{n},\boldsymbol{p})\Big)\,\langle C_N, W_N(\boldsymbol{n})\,C_N\rangle \\ &= \tau_N(W_N(\boldsymbol{n})). \end{aligned} \tag{1.38}$$

In the last line we used that when $\boldsymbol{x}$ runs over $[0,1)$, $\lfloor Nx_i\rfloor$, $i=1,2$ runs over the set of integers $\{0,1,\cdots,N-1\}$.

The proof the localization property in Definition 1.4.1 requires several steps. First, we observe that, due to (1.12),

$$\begin{aligned} E(n) &:= \Big|\langle C_N, W_N(\boldsymbol{n})\,C_N\rangle\Big| \\ &= \frac{1}{2^{N-1}}\Bigg|\sum_{\ell=0}^{N-n_1-1}\exp\Big(-\frac{2\pi i}{N}\ell n_2\Big)\sqrt{\binom{N-1}{\ell}\binom{N-1}{\ell+n_1}} \\ &\quad + \sum_{\ell=N-n_1}^{N-1}\exp\Big(-\frac{2\pi i}{N}\ell n_2\Big)\sqrt{\binom{N-1}{\ell}\binom{N-1}{\ell+n_1-N}}\Bigg| \end{aligned} \tag{1.39}$$

$$\begin{aligned} &\leqslant \frac{1}{2^{N-1}}\Bigg[\sum_{\ell=0}^{N-n_1-1}\sqrt{\binom{N-1}{\ell}\binom{N-1}{\ell+n_1}} \\ &\quad + \sum_{\ell=N-n_1}^{N-1}\sqrt{\binom{N-1}{\ell}\binom{N-1}{\ell+n_1-N}}\Bigg]. \end{aligned} \tag{1.40}$$

Second, using the entropic bound of the binomial coefficients

$$\binom{N-1}{\ell} \leqslant 2^{(N-1)\,\eta(\frac{\ell}{N-1})}\quad , \tag{1.41}$$

where

$$\eta(t) := \begin{cases} -t\log_2 t-(1-t)\log_2(1-t) & \text{if } 0<t\leqslant 1 \\ 0 & \text{if } t=0 \end{cases}\quad , \tag{1.42}$$

we estimate

$$E(n) \leqslant \frac{1}{2^{N-1}} \left[\sum_{\ell=0}^{N-1-n_1} 2^{\frac{N-1}{2}\left[\eta\left(\frac{\ell}{N-1}\right)+\eta\left(\frac{\ell+n_1}{N-1}\right)\right]} + \sum_{\ell=N-n_1}^{N-1} 2^{\frac{N-1}{2}\left[\eta\left(\frac{\ell}{N-1}\right)+\eta\left(\frac{\ell+n_1-N}{N-1}\right)\right]} \right] . \tag{1.43}$$

The exponents in the two sums are bounded by their maxima

$$\eta\left(\frac{\ell}{N-1}\right) + \eta\left(\frac{\ell+n_1}{N-1}\right) \leqslant 2\,\eta_1(n_1), \quad (0 \leqslant \ell \leqslant N-n_1-1) \tag{1.44}$$

$$\eta\left(\frac{\ell}{N-1}\right) + \eta\left(\frac{\ell+n_1-N}{N-1}\right) \leqslant 2\,\eta_2(n_1), \quad (N-n_1 \leqslant \ell \leqslant N-1) \tag{1.45}$$

where

$$\eta_1(n_1) := \eta\left(\frac{1}{2} - \frac{n_1}{2(N-1)}\right) \leqslant 1 \tag{1.46}$$

$$\eta_2(n_1) := \eta\left(\frac{1}{2} + \frac{N-n_1}{2(N-1)}\right) \leqslant \eta_2 < 1. \tag{1.47}$$

Notice that η_2 is automatically < 1, while $\eta_1(n_1) < 1$ if $\lim_{N\to\infty} \frac{n_1}{N} \neq 0$. If so, the upper bound

$$E(n) \leqslant N\Big(2^{-(N-1)(1-\eta_1(n_1))} + 2^{-(N-1)(1-\eta_2)}\Big) \tag{1.48}$$

implies $N\big|\langle C_N, W_N(\boldsymbol{n})\, C_N\rangle\big|^2 \longmapsto 0$ exponentially with $N \to \infty$.

The condition for which $\eta_1(n_1) < 1$ is fulfilled when $|x_1 - y_1| > \delta$; in fact, $\boldsymbol{n} = \lfloor N\boldsymbol{y}\rfloor - \lfloor N\boldsymbol{x}\rfloor$ and

$$\lim_{N\to\infty} \frac{\lfloor Nx_1\rfloor - \lfloor Ny_1\rfloor}{N} = x_1 - y_1 \cdot$$

On the other hand, if $x_1 = y_1$ and $n_2 = \lfloor Nx_2\rfloor - \lfloor Ny_2\rfloor \neq 0$, one explicitly computes

$$N\big|\langle C_N, W_N((0,n_2))\, C_N\rangle\big|^2 = N\left(\cos^2\left(\frac{\pi n_2}{N}\right)\right)^{N-1}. \tag{1.49}$$

Again, the above expression goes exponentially fast to zero, if $\lim_{N\to\infty} \frac{n_2}{N} \neq 0$ which is the case if $x_2 \neq y_2$.

1.4.2. A second set of states, not overcomplete: $\{|\beta_N(\boldsymbol{x})\rangle \mid \boldsymbol{x} \in \mathbb{T}^2\} \in \mathcal{H}_N^D$

As in the previous Section 1.4.2 $\lfloor\cdot\rfloor$ denotes the integer part of a real number; moreover we introduce the notation $\langle\cdot\rangle$ to denote fractional parts, namely $\langle x\rangle := x - \lfloor x\rfloor$, so that we can express each $\boldsymbol{x} \in \mathbb{T}^2$ as

$$\boldsymbol{x} = \left(\frac{\lfloor Nx_1\rfloor}{N}, \frac{\lfloor Nx_2\rfloor}{N}\right) + \left(\frac{\langle Nx_1\rangle}{N}, \frac{\langle Nx_2\rangle}{N}\right) .$$

Then we associate $\boldsymbol{x} \in \mathbb{T}^2$ with vectors of $|\beta_N(\boldsymbol{x})\rangle \in \mathcal{H}_N^D$ as follows:

$$\begin{aligned}\mathbb{T}^2 \ni \boldsymbol{x} \mapsto |\beta_N(\boldsymbol{x})\rangle = \lambda_{00}(\boldsymbol{x})\,|\,\lfloor Nx_1\rfloor, \lfloor Nx_2\rfloor\,\rangle +\\ + \lambda_{01}(\boldsymbol{x})\,|\,\lfloor Nx_1\rfloor, \lfloor Nx_2\rfloor + 1\,\rangle + \lambda_{10}(\boldsymbol{x})\,|\,\lfloor Nx_1\rfloor + 1, \lfloor Nx_2\rfloor\,\rangle +\\ + \lambda_{11}(\boldsymbol{x})\,|\,\lfloor Nx_1\rfloor + 1, \lfloor Nx_2\rfloor + 1\,\rangle \in \mathcal{H}_N^D. \end{aligned} \quad (1.50)$$

We choose the coefficients $\lambda_{\mu\nu}$ in order to have Measurability, normalization, and invertibility of the mapping in (1.50):

$$\begin{cases} \lambda_{00}(\boldsymbol{x}) = \cos\left(\frac{\pi}{2}\langle Nx_1\rangle\right)\cos\left(\frac{\pi}{2}\langle Nx_2\rangle\right) \\ \lambda_{01}(\boldsymbol{x}) = \cos\left(\frac{\pi}{2}\langle Nx_1\rangle\right)\sin\left(\frac{\pi}{2}\langle Nx_2\rangle\right) \\ \lambda_{10}(\boldsymbol{x}) = \sin\left(\frac{\pi}{2}\langle Nx_1\rangle\right)\cos\left(\frac{\pi}{2}\langle Nx_2\rangle\right) \\ \lambda_{11}(\boldsymbol{x}) = \sin\left(\frac{\pi}{2}\langle Nx_1\rangle\right)\sin\left(\frac{\pi}{2}\langle Nx_2\rangle\right) \end{cases} \quad (1.51)$$

Before going in the proof of other properties, let us remind to the reader that in this case $N_\hbar$ in Definition 1.4.1 stands for N^2, the dimension of the Hilbert space.

Overcompleteness fails and we refer to Appendix A for a proof. Since we will not use them in the Anti–Wick quantization, in which that property is required, this is no trouble. On the other hand, the states $|\beta_N(\boldsymbol{x})\rangle$ are useful to invert the Weyl discretization developed in Section 1.3.1, as we shall see in Section 1.6. Here we simply note that, although overcompleteness is not satisfied by this family of states, nevertheless it is "not too far from being true", in the sense that they provide via Definition 1.4.1 property 3 an operator $I_{\boldsymbol{\ell},\boldsymbol{m}}$ actually very near to the identity operator $\mathbb{1}$.

We now prove localization.

The states $|\beta_N(\boldsymbol{x})\rangle$ are constructed by choosing, among the elements of the basis of $\mathcal{H}_N^D$, the four ones labeled by elements of L_N that are neighbors of $\boldsymbol{x}$; it follows that $|\beta_N(\boldsymbol{x})\rangle$ is orthogonal to every basis element labeled by a point of L_N whose toral distance

$d_{\mathbb{T}^2}$ (see Definition (1.4.2)) from $\boldsymbol{x}$ is greater than $\sqrt{2}/N$.

As a consequence, the quantity $N^2\langle\beta_N(\boldsymbol{x}),\beta_N(\boldsymbol{y})\rangle = 0$ if the distance on the torus between $\boldsymbol{x}$ and $\boldsymbol{y}$ is greater than $2\sqrt{2}/N$.

Thus, given $d_0 > 0$, it is sufficient that $d_0 > 2\sqrt{2}/N$, that is $N_0(\epsilon, d_0) > 2\sqrt{2}/d_0$, to have

$$N > N_0(\epsilon, d_0) \Longrightarrow N^2\langle\beta_N(\boldsymbol{x}),\beta_N(\boldsymbol{y})\rangle = 0$$

1.4.3. A third set of C.S.: $\{|C^3_N(\boldsymbol{x})\rangle \mid \boldsymbol{x} \in \mathbb{T}^2\} \in \mathcal{H}^D_N$

The new family of CS we are going to introduce in this Section, is not too different from the one introduced in the previous Section, as it will also consist of states in the same Hilbert space and constructed by grouping a small cluster of nearest neighbors in the basis of $\mathcal{H}^D_N$.

Nevertheless there is one big difference between the two example: in the present case, the mapping from $\mathbb{T}^2$ into $\mathcal{H}^D_N$ defining the family of coherent states is as follows:

$$\mathbb{T}^2 \ni \boldsymbol{x} \mapsto |C^3_N(\boldsymbol{x})\rangle = |\,\lfloor Nx_1 + \tfrac{1}{2}\rfloor\,,\lfloor Nx_2 + \tfrac{1}{2}\rfloor\,\rangle \in \mathcal{H}^D_N \tag{1.52}$$

and is not invertible. Measurability and normalization are clearly satisfied and localization can be proved in the same way as in the previous Section. Now we shall give a direct proof of overcompleteness.

Overcompleteness property of Definition 1.4.1 can be expressed as[5]

$$N^2 \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\,\langle \boldsymbol{\ell}\;|C^3_N(\boldsymbol{x})\rangle\langle C^3_N(\boldsymbol{x})|\;\boldsymbol{m}\rangle = \delta^{(N)}_{\boldsymbol{\ell},\boldsymbol{m}}, \qquad \forall \boldsymbol{\ell},\boldsymbol{m} \in (\mathbb{Z}/N\mathbb{Z})^2 \tag{1.53}$$

and this is exactly what we are going to prove; let us take the quantity

$$\begin{aligned}
I_{\boldsymbol{\ell},\boldsymbol{m}} &:= N^2 \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\,\langle \boldsymbol{\ell}\;|C^3_N(\boldsymbol{x})\rangle\langle C^3_N(\boldsymbol{x})|\boldsymbol{m}\rangle \\
&= N^2 \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \left\langle \ell_1,\ell_2 \;\middle|\; \lfloor Nx_1 + \tfrac{1}{2}\rfloor\,,\lfloor Nx_2 + \tfrac{1}{2}\rfloor \right\rangle \times \\
&\quad \times \left\langle \lfloor Nx_1 + \tfrac{1}{2}\rfloor\,,\lfloor Nx_2 + \tfrac{1}{2}\rfloor \;\middle|\; m_1, m_2 \right\rangle =
\end{aligned}$$

[5] For the definition of $\delta^{(N)}_{\boldsymbol{\ell},\boldsymbol{m}}$, see in Properties 1.2.1.

$$= N^2 \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \;\; \delta^{(N)}_{\ell_1\,,\,\lfloor Nx_1+\frac{1}{2}\rfloor} \, \delta^{(N)}_{\ell_2\,,\,\lfloor Nx_2+\frac{1}{2}\rfloor} \, \delta^{(N)}_{m_1\,,\,\lfloor Nx_1+\frac{1}{2}\rfloor} \, \delta^{(N)}_{m_2\,,\,\lfloor Nx_2+\frac{1}{2}\rfloor} =$$

$$= N^2 \, \delta^{(N)}_{\ell_1\,,\,m_1} \, \delta^{(N)}_{\ell_2\,,\,m_2} \left[\int_0^1 \mathrm{d}x_1 \, \delta^{(N)}_{\ell_1\,,\,\lfloor Nx_1+\frac{1}{2}\rfloor}\right] \left[\int_0^1 \mathrm{d}x_2 \;\, \delta^{(N)}_{\ell_2\,,\,\lfloor Nx_2+\frac{1}{2}\rfloor}\right] \quad . \tag{1.54}$$

Note that, in order to have the integrand of (1.54) different from zero we must have $\ell_i \leqslant Nx_i + \frac{1}{2} < \ell_i + 1$ for $i = 1, 2$, that is $\frac{\ell_i - \frac{1}{2}}{N} \leqslant x_i < \frac{\ell_i + \frac{1}{2}}{N}$. Then (1.54) reads:

$$\begin{aligned} I_{\boldsymbol{\ell},\boldsymbol{m}} &= N^2 \;\left(\delta^{(N)}_{\ell_1\,,\,m_1} \, \delta^{(N)}_{\ell_2\,,\,m_2}\right) \left[\int_{\frac{\ell_1-\frac{1}{2}}{N}}^{\frac{\ell_1+\frac{1}{2}}{N}} \mathrm{d}x_1\right] \left[\int_{\frac{\ell_2-\frac{1}{2}}{N}}^{\frac{\ell_2+\frac{1}{2}}{N}} \mathrm{d}x_2\right] \quad . \\ &= N^2 \, \delta^{(N)}_{\boldsymbol{\ell},\boldsymbol{m}} \times \frac{1}{N} \times \frac{1}{N} = \delta^{(N)}_{\boldsymbol{\ell},\boldsymbol{m}} \end{aligned} \tag{1.55}$$

and hence overcompleteness is proved.

1.5. Anti–Wick Quantization

In order to study the classical limit and, more generally, the semi–classical behaviour of $(\mathcal{M}_N, \Theta_N, \tau_N)$ when $N \to \infty$, we introduce two linear maps. The first, $\mathcal{J}_{N\infty}$, (anti-Wick quantization) associates $N \times N$ matrices to functions in $\mathcal{A}_{\mathcal{X}}$, the second one, $\mathcal{J}_{\infty N}$, maps $N \times N$ matrices into functions in $\mathcal{A}_{\mathcal{X}}$.

Definitions 1.5.1

Given a family $\{|C_N(\boldsymbol{x})\rangle \mid \boldsymbol{x} \in \mathcal{X}\}$ of coherent states in $\mathcal{H}_N$, the Hilbert space of dimension $N_\hbar$, the anti-Wick quantization scheme will be described by a (completely) positive unital map $\mathcal{J}_{N\infty} : \mathcal{A}_{\mathcal{X}} \to \mathcal{M}_N$

$$\mathcal{A}_{\mathcal{X}} \ni f \mapsto N_\hbar \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \, f(\boldsymbol{x}) \, |C_N(\boldsymbol{x})\rangle\langle C_N(\boldsymbol{x})| =: \mathcal{J}_{N\infty}(f) \in \mathcal{M}_N \quad .$$

The corresponding dequantizing map $\mathcal{J}_{\infty N} : \mathcal{M}_N \to \mathcal{A}_{\mathcal{X}}$ will correspond to the (completely) positive unital map

$$\mathcal{M}_N \ni X \mapsto \langle C_N(\boldsymbol{x}), X \, C_N(\boldsymbol{x})\rangle =: \mathcal{J}_{\infty N}(X)(\boldsymbol{x}) \in \mathcal{A}_{\mathcal{X}} \quad .$$

Both maps are identity preserving because of the conditions imposed on the family of coherent states and are also completely positive since the domain of $\mathcal{J}_{N\infty}$ is a commutative algebra as well as the range of $\mathcal{J}_{\infty N}$. Moreover,

$$\|\mathcal{J}_{\infty N} \circ \mathcal{J}_{N\infty}(g)\| \leq \|g\|, \quad g \in \mathcal{A}_{\mathcal{X}}, \tag{1.56}$$

where $\|\cdot\|$ denotes the norm with respect to which the C*-algebra $\mathcal{A}_{\mathcal{X}}$ is complete ($\|\cdot\|_0$ for $\mathcal{C}^0(\mathcal{X})$ and $\|\cdot\|_\infty$ for $L^\infty_\mu(\mathcal{X})$).

1.5.1. Classical limit in the anti–Wick quantization scheme

Performing the classical limit or a semi-classical analysis consists in studying how a family of algebraic triples $(\mathcal{M}_N, \tau_N, \Theta_N)$ depending on a quantization $\hbar$-like parameter is mapped onto $(\mathcal{A}_{\mathcal{X}}, \omega_\mu, \Theta)$ when the parameter goes to zero.

We shall give now two equivalent properties that can be taken as requests on any well–defined quantization–dequantization scheme for observables. In the sequel, we shall need the notion of quantum dynamical systems $(\mathcal{M}_N, \tau_N, \Theta_N)$ tending to the classical limit $(\mathcal{X}, \mu, T)$. Indeed, a request upon any sensible quantization procedure is to recover the classical description in the limit $\hbar \to 0$; in a similar way, our quantization (or discretization) should recover the classical (or continuous) system in the $\frac{1}{N} \to 0$ limit. Moreover we not only need convergence of observables but also of the dynamics: this aspect will be considered in Section 2.4.

Here $\mathcal{M}_N$ will denote a general $N_\hbar \times N_\hbar$ matrix algebra and the following two proposition will be proved for both $\mathcal{A}_{\mathcal{X}} = \mathcal{C}^0(\mathcal{X})$ and $\mathcal{A}_{\mathcal{X}} = L^\infty_\mu(\mathcal{X})$, the two functional spaces introduced in Section 1.1.2.

Proposition 1.5.1

For all $f \in \mathcal{A}_{\mathcal{X}}$

$$\lim_{N\to\infty} \mathcal{J}_{\infty N} \circ \mathcal{J}_{N\infty}(f) = f \quad \mu - \text{a.e.}$$

Proposition 1.5.2

For all $f, g \in \mathcal{A}_{\mathcal{X}}$

$$\lim_{N\to\infty} \tau_N\big(\mathcal{J}_{N\infty}(f)^* \mathcal{J}_{N\infty}(g)\big) = \omega_\mu(\overline{f} g) = \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, \overline{f(\boldsymbol{x})} g(\boldsymbol{x}).$$

Proof of Proposition 1.5.1:

We first prove the assertion when $\mathcal{A}_{\mathcal{X}} = \mathcal{C}^0(\mathcal{X})$ and then we extend to $\mathcal{A}_{\mathcal{X}} = L^\infty_\mu(\mathcal{X})$. We show that the quantity

$$\begin{aligned} F_N(\boldsymbol{x}) &:= \left| f(\boldsymbol{x}) - \mathcal{J}_{\infty N} \circ \mathcal{J}_{N\infty}(f)(\boldsymbol{x}) \right| \\ &= \left| f(\boldsymbol{x}) - N \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\, f(\boldsymbol{y})\, |\langle C_N(\boldsymbol{x}), C_N(\boldsymbol{y})\rangle|^2 \right| \\ &= N \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\, (f(\boldsymbol{y}) - f(\boldsymbol{x}))\, |\langle C_N(\boldsymbol{x}), C_N(\boldsymbol{y})\rangle|^2 \right| \end{aligned}$$

becomes arbitrarily small for N large enough, uniformly in $\boldsymbol{x}$. Selecting a ball $B(\boldsymbol{x}, d_0)$ of radius d_0, using the mean-value theorem and property (1.4.1.3), we derive the upper bound

$$\begin{aligned} F_N(\boldsymbol{x}) &\le N \left| \int_{B(\boldsymbol{x},d_0)} \mu(\mathrm{d}\boldsymbol{y})\, (f(\boldsymbol{y}) - f(\boldsymbol{x}))\, |\langle C_N(\boldsymbol{x}), C_N(\boldsymbol{y})\rangle|^2 \right| \\ &\quad + N \left| \int_{\mathcal{X}\setminus B(\boldsymbol{x},d_0)} \mu(\mathrm{d}\boldsymbol{y})\, (f(\boldsymbol{y}) - f(\boldsymbol{x}))\, |\langle C_N(\boldsymbol{x}), C_N(\boldsymbol{y})\rangle^2 \right| \qquad (1.57) \\ &\le |f(\boldsymbol{c}) - f(\boldsymbol{x})| + \int_{\mathcal{X}\setminus B(\boldsymbol{x},d_0)} \mu(\mathrm{d}\boldsymbol{y})\, |f(\boldsymbol{y}) - f(\boldsymbol{x})|\, N\, |\langle C_N(\boldsymbol{x}), C_N(\boldsymbol{y})\rangle|^2, \qquad (1.58) \end{aligned}$$

where $c \in B(\boldsymbol{x}, d_0)$.

Because $\mathcal{X}$ is compact, f is uniformly continuous. Therefore, we can choose d_0 in such a way that $|f(\boldsymbol{c}) - f(\boldsymbol{x})| < \varepsilon$ uniformly in $\boldsymbol{x} \in \mathcal{X}$. On the other hand, from the localization property (1.4.1.4), given $\varepsilon' > 0$, there exists an integer $N_0(\varepsilon', d_0)$ such that $N|\langle C_N(\boldsymbol{x}), C_N(\boldsymbol{y})\rangle|^2 < \varepsilon'$ whenever $N > N_0(\varepsilon', d_0)$. This choice leads to the upper bound

$$\begin{aligned} F_N(\boldsymbol{x}) &\le \varepsilon + \varepsilon' \int_{\mathcal{X}\setminus B(\boldsymbol{x},d_0)} \mu(\mathrm{d}\boldsymbol{y})\, |f(\boldsymbol{y}) - f(\boldsymbol{x})| \\ &\le \varepsilon + \varepsilon' \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\, |f(\boldsymbol{y}) - f(\boldsymbol{x})| \le \varepsilon + 2\varepsilon' \|f\|_\infty. \qquad (1.59) \end{aligned}$$

To get rid of the continuity of f, that is when $f \in L^\infty_\mu(\mathcal{X})$, we use a corollary 1 of Lusin's theorem [31, 37, 38]. For later use, we write down both statements, in a form slightly adapted to our case (for instance our compact space $\mathcal{X} = \mathbb{T}^2$ is a "locally compact Hausdorff space", but we do not need these generic settings, and the same is true for the class of f we will refer to):

Theorem 2 (Lusin's) : Every measurable function $f(\boldsymbol{x})$ on a measurable set $\mathcal{X}$ can be made continuous by removing from $\mathcal{X}$ the points contained in suitably chosen open intervals whose total measure is arbitrarily small.

Corollary 1 (of Lusin's Theorem) : Given $f \in L^\infty_\mu(\mathcal{X})$, with $\mathcal{X}$ compact, there exists a sequence $\{f_n\}$ of continuous functions on $\mathcal{X}$ such that $|f_n| \leq \|f\|_\infty$ and converging to f μ – almost everywhere.

Thus, for $f \in L^\infty_\mu(\mathcal{X})$, we pick such a sequence and estimate

$$\begin{aligned} F_N(\boldsymbol{x}) \leq &\Big|f(\boldsymbol{x}) - f_n(\boldsymbol{x})\Big| + \Big|f_n(\boldsymbol{x}) - \mathcal{J}_{\infty N} \circ \mathcal{J}_{N\infty}(f_n)(\boldsymbol{x})\Big| \\ &+ \Big|\mathcal{J}_{\infty N} \circ \mathcal{J}_{N\infty}(f_n - f)(\boldsymbol{x})\Big|. \end{aligned}$$

The first term can be made arbitrarily small (μ – a.e) by choosing n large enough because of Lusin's theorem, while the second one goes to 0 when $N \to \infty$ since f_n is continuous. Finally, the third term becomes as well vanishingly small with $n \to \infty$ as one can deduce from

$$\begin{aligned} &\int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \Big|\mathcal{J}_{\infty N} \circ \mathcal{J}_{N\infty}(f - f_n)(\boldsymbol{x})\Big| \\ &\quad = \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \left|\int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\, (f(\boldsymbol{y}) - f_n(\boldsymbol{y}))\, N\, |\langle C_N(\boldsymbol{x}), C_N(\boldsymbol{y})\rangle|^2\right| \\ &\quad \leq \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\, |f(\boldsymbol{y}) - f_n(\boldsymbol{y})| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, N\, |\langle C_N(\boldsymbol{x}), C_N(\boldsymbol{y})\rangle|^2 \\ &\quad = \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\, |f(\boldsymbol{y}) - f_n(\boldsymbol{y})|, \end{aligned}$$

where exchange of integration order is harmless because of the existence of the integral (1.56). The last integral goes to zero with n by dominated convergence and thus the result follows. ∎

Proof of Proposition 1.5.2:

Consider

$$\begin{aligned} \Omega_N &:= \Big|\tau_N\big(\mathcal{J}_{N\infty}(f)^* \mathcal{J}_{N\infty}(g)\big) - \omega_\mu(\overline{f} g)\Big| \\ &= N \left|\int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, \overline{f(\boldsymbol{x})} \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\, (g(\boldsymbol{y}) - g(\boldsymbol{x}))\, |\langle C_N(\boldsymbol{x}), C_N(\boldsymbol{y})\rangle|^2\right| \\ &\leq \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, |f(\boldsymbol{x})| \left|\int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\, (g(\boldsymbol{y}) - g(\boldsymbol{x}))\, N\, |\langle C_N(\boldsymbol{x}), C_N(\boldsymbol{y})\rangle|^2\right|. \end{aligned}$$

By choosing a sequence of continuous g_n approximating $g \in L^\infty_\mu(\mathcal{X})$, and arguing as in the previous proof, we get the following upper bound:

$$\begin{aligned}
\Omega_N \le N \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\,|f(\boldsymbol{x})| \left|\int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\,(g(\boldsymbol{y}) - g_n(\boldsymbol{y}))\,|\langle C_N(\boldsymbol{x}), C_N(\boldsymbol{y})\rangle|^2\right| \\
+ N \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\,|f(\boldsymbol{x})| \left|\int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\,(g_n(\boldsymbol{y}) - g_n(\boldsymbol{x}))\,|\langle C_N(\boldsymbol{x}), C_N(\boldsymbol{y})\rangle|^2\right| \\
+ N \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\,|f(\boldsymbol{x})| \left|\int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\,(g(\boldsymbol{x}) - g_n(\boldsymbol{x}))\,|\langle C_N(\boldsymbol{x}), C_N(\boldsymbol{y})\rangle|^2\right|.
\end{aligned}$$

The integrals in the first and third lines go to zero by dominated convergence and Lusin's theorem. As regards the middle line, one can apply the argument used for the quantity $F_N(\boldsymbol{x})$ in the proof of Proposition 1.5.1. ■

1.5.2. Discretization/Dediscretization of $L^\infty_\mu(\mathcal{X})$ by means of $\{|C^3_N(\boldsymbol{x})\rangle\}_{\boldsymbol{x}\in\mathbb{T}^2}$

Now that we have proved the so called classical limit for the anti–Wick quantization in the general case, we have all ingredient to build a concrete example of such a quantization procedure. In particular we will apply Definitions 1.5.1 and discretize $L^\infty_\mu(\mathcal{X})$ by means of the CS set $\{|C^3_N(\boldsymbol{x})\rangle \mid \boldsymbol{x} \in \mathbb{T}^2\} \in \mathcal{H}^D_N$ introduced in Section 1.4.3.

In this framework, the discretizing/dediscretizing operators of Definitions 1.5.1 now reads:

$$L^\infty_\mu(\mathbb{T}^2) \ni f \mapsto N^2 \int_{\mathbb{T}^2} \mu(\mathrm{d}\boldsymbol{x})\, f(\boldsymbol{x})\, |C^3_N(\boldsymbol{x})\rangle\langle C^3_N(\boldsymbol{x})| =: \mathcal{J}_{N\infty}(f) \in \mathcal{D}_N \quad \cdot \tag{1.60}$$

$$\mathcal{D}_N \ni X \mapsto \langle C^3_N(\boldsymbol{x}),\, X\, C^3_N(\boldsymbol{x})\rangle =: \mathcal{J}_{\infty N}(X)(\boldsymbol{x}) \in \mathcal{S}(\mathbb{T}^2)^{6} \subset L^\infty_\mu(\mathbb{T}^2) \quad \cdot \tag{1.61}$$

In this Section we will give an interpretation of these two operators that will be useful in the following. Let us start by computing the matrix elements of $\mathcal{J}_{N\infty}(f)$ in (1.60).

$$M^{(f)}_{\boldsymbol{\ell},\boldsymbol{m}} := \langle \boldsymbol{\ell}\,,\, \mathcal{J}_{N\infty}(f)\, \boldsymbol{m}\rangle \tag{1.62}$$

$$= N^2 \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, f(\boldsymbol{x})\, \langle \boldsymbol{\ell}\, |C^3_N(\boldsymbol{x})\rangle\langle C^3_N(\boldsymbol{x})|\boldsymbol{m}\rangle \tag{1.63}$$

$$\begin{aligned}
&= N^2 \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2\; f(\boldsymbol{x}) \left\langle \ell_1, \ell_2 \,\middle|\, \lfloor N x_1 + \tfrac{1}{2}\rfloor\,,\, \lfloor N x_2 + \tfrac{1}{2}\rfloor \right\rangle \times \\
&\quad \times \left\langle \lfloor N x_1 + \tfrac{1}{2}\rfloor\,,\, \lfloor N x_2 + \tfrac{1}{2}\rfloor \,\middle|\, m_1, m_2 \right\rangle
\end{aligned} \tag{1.64}$$

[6] The symbol $\mathcal{S}(\mathbb{T}^2)$ denotes the set of *simple functions* on the torus. A function f belong to that set if it holds that f assumes on $\mathbb{T}^2$ only a finite number of values [37]. The relation $\mathrm{Ran}(\mathcal{J}_{\infty N}) = \mathcal{S}(\mathbb{T}^2)$ will be shown in (1.72).

$$= N^2 \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2\, f(\boldsymbol{x})\, \delta^{(N)}_{\ell_1\,,\,\lfloor Nx_1+\frac{1}{2}\rfloor}\, \delta^{(N)}_{\ell_2\,,\,\lfloor Nx_2+\frac{1}{2}\rfloor}\, \delta^{(N)}_{m_1\,,\,\lfloor Nx_1+\frac{1}{2}\rfloor}\, \delta^{(N)}_{m_2\,,\,\lfloor Nx_2+\frac{1}{2}\rfloor}$$

$$= N^2\, \delta^{(N)}_{\ell_1\,,\,m_1}\, \delta^{(N)}_{\ell_2\,,\,m_2} \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2\, f(\boldsymbol{x})\, \delta^{(N)}_{\ell_1\,,\,\lfloor Nx_1+\frac{1}{2}\rfloor}\, \delta^{(N)}_{\ell_2\,,\,\lfloor Nx_2+\frac{1}{2}\rfloor} \quad\cdot \tag{1.65}$$

As already observed (between eqs. (1.54) and (1.55)), in order to have the integrand of (1.65) different from zero we must have $\ell_i \leqslant Nx_i + \frac{1}{2} < \ell_i + 1$ for $i = 1, 2$, that is $\frac{\ell_i - \frac{1}{2}}{N} \leqslant x_i < \frac{\ell_i + \frac{1}{2}}{N}$ and (1.65) reads:

$$M^{(f)}_{\boldsymbol{\ell},\boldsymbol{m}} = N^2\, \delta^{(N)}_{\boldsymbol{\ell},\boldsymbol{m}} \int_{\frac{\ell_1-\frac{1}{2}}{N}}^{\frac{\ell_1+\frac{1}{2}}{N}} \mathrm{d}x_1 \int_{\frac{\ell_2-\frac{1}{2}}{N}}^{\frac{\ell_2+\frac{1}{2}}{N}} \mathrm{d}x_2\; f(\boldsymbol{x}) \quad\cdot \tag{1.66}$$

From the latter equation we see that $\mathrm{Ran}\left(\mathcal{J}_{N\infty}\right) = \mathcal{D}_N$. We will reduce (1.66) to a nicer expression, but to this aim we introduce now a new

Definition 1.5.2 (Running Average Operator (RAO))

- We will denote by $Q_N\left(\boldsymbol{x}\right)$ the small square of side $1/N$, oriented with sides parallel to the axis of the torus, centered on $\boldsymbol{x}$.
- By means of the latter, we introduce now the Running Average Operator $\Gamma_N : L^\infty_\mu\left(\mathcal{X}\right) \longmapsto \mathcal{C}^0\left(\mathbb{T}^2\right)$ defined by:

$$L^\infty_\mu\left(\mathcal{X}\right) \ni f(\boldsymbol{x}) \longmapsto \Gamma_N\left(f\right)\left(\boldsymbol{x}\right) =: N^2 \int_{Q_N(\boldsymbol{x})} \mu(\mathrm{d}\boldsymbol{x})\; f(\boldsymbol{y}) \;\in \mathcal{C}^0\left(\mathbb{T}^2\right) \quad\cdot$$

Propositions 1.5.3

1) Given $f \in L^\infty_\mu\left(\mathbb{T}^2\right)$, the function $f^{(Q)}_N := \Gamma_N\left(f\right)$ is uniformly continuous on $\mathbb{T}^2$
2) Denoting by

$$\left\|\Gamma_N\right\|_{\mathcal{B}} := \sup_{f\in L^\infty_\mu(\mathbb{T}^2)} \frac{\left\|\Gamma_N\left(f\right)\right\|_0}{\left\|f\right\|_\infty} \tag{1.67}$$

we have that $\left\|\Gamma_N\right\|_{\mathcal{B}} = 1$.

Proof of Propositions 1.5.3:

Given a two functions, $f \in L^\infty_\mu\left(\mathbb{T}^2\right)$ and $g \in L^1_\mu\left(\mathbb{T}^2\right)$, and denoting by $\left\|\cdot\right\|_1$ the $L^1_\mu\left(\mathbb{T}^2\right)$-norm, that is

$$\left\|g\right\|_1 := \int_{\mathbb{T}^2} \mu(\mathrm{d}\boldsymbol{x})\; \left|g(\boldsymbol{x})\right| \;, \tag{1.68}$$

one has:

$$\|fg\|_1 \leqslant \|f\|_\infty \|g\|_1 \tag{1.69}$$

Equation (1.69) can be seen as an extension of the Hölder's inequality, but its proof is more easily deduced by integrating the obvious relation $|fg| \leqslant \|f\|_\infty |g|$; using (1.69) we are going to prove the two statement of Proposition 1.5.3.

1) Let's take two points $\boldsymbol{x}_0 \in \mathbb{T}^2$ and $\boldsymbol{x} \in Q_N(\boldsymbol{x}_0)$, and let $\mathcal{X}_E$ denote the characteristic function of $E \subset \mathbb{T}^2$. By Definition (1.5.2):

$$\left| f_N^{(Q)}(\boldsymbol{x}_0) - f_N^{(Q)}(\boldsymbol{x}) \right| = N^2 \left| \int_{\mathbb{T}^2} \mu(\mathrm{d}\boldsymbol{y})\, f(\boldsymbol{y}) \,\left(\mathcal{X}_{Q_N(\boldsymbol{x}_0)}(\boldsymbol{y}) - \mathcal{X}_{Q_N(\boldsymbol{x})}(\boldsymbol{y})\right) \right|$$

therefore, carrying $|\cdot|$ inside the integral and using (1.69)

$$\begin{aligned} &\leqslant N^2 \,\|f\|_\infty \int_{\mathbb{T}^2} \mu(\mathrm{d}\boldsymbol{y}) \,\left|\mathcal{X}_{Q_N(\boldsymbol{x}_0)}(\boldsymbol{y}) - \mathcal{X}_{Q_N(\boldsymbol{x})}(\boldsymbol{y})\right| \\ &\leqslant N^2 \,\|f\|_\infty \Big[\mu\Big(Q_N(\boldsymbol{x}_0) \cup Q_N(\boldsymbol{x})\Big) \;-\; \mu\Big(Q_N(\boldsymbol{x}_0) \cap Q_N(\boldsymbol{x})\Big)\Big] \,. \end{aligned}$$

According to our hypothesis, $\boldsymbol{x} \in Q_N(\boldsymbol{x}_0)$, thus geometrical considerations lead to:

$$\begin{aligned} \mu\Big(Q_N(\boldsymbol{x}_0) \cup Q_N(\boldsymbol{x})\Big) &< \Big(\frac{1}{N} + |x_1 - x_{01}|\Big)\Big(\frac{1}{N} + |x_2 - x_{02}|\Big) \\ \mu\Big(Q_N(\boldsymbol{x}_0) \cap Q_N(\boldsymbol{x})\Big) &= \Big(\frac{1}{N} - |x_1 - x_{01}|\Big)\Big(\frac{1}{N} - |x_2 - x_{02}|\Big) \\ \mu\Big(Q_N(\boldsymbol{x}_0) \cup Q_N(\boldsymbol{x})\Big) \;-\; \mu\Big(Q_N(\boldsymbol{x}_0) \cap Q_N(\boldsymbol{x})\Big) &< \frac{2}{N}\Big(|x_1 - x_{01}| + |x_2 - x_{02}|\Big) \\ &< \frac{2\sqrt{2}}{N}\,\|\boldsymbol{x}_0 - \boldsymbol{x}\| \end{aligned}$$

Finally we can write

$$\left| f_N^{(Q)}(\boldsymbol{x}_0) - f_N^{(Q)}(\boldsymbol{x}) \right| \leqslant 2\sqrt{2}\, N \,\|f\|_\infty \,\|\boldsymbol{x}_0 - \boldsymbol{x}\|$$

and this prove continuity of $f_N^{(Q)}$ (and so uniform continuity too, being $\mathbb{T}^2$ a compact space). ■

2) A straightforward application of (1.69) give us that $\|\Gamma_N\|_{\mathcal{B}} \leqslant 1$. We reach the maximum by noting that it is attained when we choose f constant. ■

We can now rewrite equation (1.60) by using RAO together with (1.66); as a result:

$$\mathcal{J}_{N\infty}(f) = \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} \Gamma_N(f)\left(\frac{\boldsymbol{\ell}}{N}\right)|\boldsymbol{\ell}\rangle\langle\boldsymbol{\ell}| = \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} f_N^{(Q)}\left(\frac{\boldsymbol{\ell}}{N}\right)|\boldsymbol{\ell}\rangle\langle\boldsymbol{\ell}| \ \cdot \tag{1.70}$$

We now compute explicitly (1.61), considering the matrix elements $\{X_{\boldsymbol{\ell},\boldsymbol{\ell}}\}_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2}$ of X; namely

$$X = \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} X_{\boldsymbol{\ell},\boldsymbol{\ell}}\,|\boldsymbol{\ell}\rangle\langle\boldsymbol{\ell}| \ \cdot \tag{1.71}$$

By putting (1.72) in (1.61) we get:

$$\begin{aligned}
\mathcal{J}_{\infty N}(X)(\boldsymbol{x}) &= \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} X_{\boldsymbol{\ell},\boldsymbol{\ell}}\,\langle C_N^3(\boldsymbol{x})\,|\boldsymbol{\ell}\rangle\langle\boldsymbol{\ell}|\,C_N^3(\boldsymbol{x})\rangle \\
&= \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} X_{\boldsymbol{\ell},\boldsymbol{\ell}}\left\langle \lfloor Nx_1+\tfrac{1}{2}\rfloor, \lfloor Nx_2+\tfrac{1}{2}\rfloor \,\middle|\, \ell_1,\ell_2\right\rangle \times \\
&\quad \times \left\langle \ell_1,\ell_2 \,\middle|\, \lfloor Nx_1+\tfrac{1}{2}\rfloor, \lfloor Nx_2+\tfrac{1}{2}\rfloor\right\rangle = \\
&= \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} X_{\boldsymbol{\ell},\boldsymbol{\ell}}\left(\delta^{(N)}_{\ell_1,\lfloor Nx_1+\frac{1}{2}\rfloor}\right)^2\left(\delta^{(N)}_{\ell_2,\lfloor Nx_2+\frac{1}{2}\rfloor}\right)^2 \\
&= \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} X_{\boldsymbol{\ell},\boldsymbol{\ell}}\,\mathcal{X}_{Q_N(\frac{\boldsymbol{\ell}}{N})}(\boldsymbol{x}) \ ,
\end{aligned} \tag{1.72}$$

and it proves that $\mathrm{Ran}\,(\mathcal{J}_{\infty N}) = \mathcal{S}\left(\mathbb{T}^2\right)$.

Moreover, equations (1.70) and (1.72) can be combined and we get the form of the (simple) function that arises from one in $L^\infty_\mu(\mathcal{X})$, by performing the anti–Wick quantization/dequantization:

$$\left(\mathcal{J}_{\infty N}\circ\mathcal{J}_{N\infty}\right)(f)(\boldsymbol{x}) = \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} \Gamma_N(f)\left(\frac{\boldsymbol{\ell}}{N}\right)\ \mathcal{X}_{Q_N(\frac{\boldsymbol{\ell}}{N})}(\boldsymbol{x}) \ \cdot \tag{1.73}$$

1.6. Inverting the Weyl discretization by means of $\{|\beta_N(\boldsymbol{x})\rangle\}_{\boldsymbol{x}\in\mathbb{T}^2}$ states

In Sections 1.2.2 and 1.3.1 we developed the procedures of Weyl quantization, Weyl discretization respectively. In this Section, we will refer to the second scheme, the Weyl discretization of $\mathcal{C}^0\left(\mathbb{T}^2\right)$ into $\mathcal{D}_N$ described in Section 1.3.1.

In the algebraic discretization, our goals was to find the operator $\mathcal{J}_{N,\infty}$ in Definition 1.3.2; actually this is not complete. As in the anti–Wick scheme of quantization/de–quantization (equivalently discretization/de–discretization) developed in Section 1.5, described in Definitions 1.5.1 and based on a couple of *-automorphisms $\mathcal{J}_{N\infty}$ and $\mathcal{J}_{\infty N}$, also here we have to go back from $\mathcal{D}_N$ to $\mathcal{C}^0\left(\mathbb{T}^2\right)$ by defining a *-morphism $\mathcal{J}_{\infty,N}$ that "inverts" the $\mathcal{J}_{N\infty}$ introduced in Definition 1.3.2, at least in the $N \to \infty$ limit. We construct this operator by means of the family of states $\{|\beta_N(\boldsymbol{x})\rangle \mid \boldsymbol{x} \in \mathbb{T}^2\} \in \mathcal{H}_N^D$ introduced in Section 1.4.2, as follows

Definition 1.6.1

We will denote by $\mathcal{J}_{\infty,N} : \mathcal{D}_N \longmapsto$ the *-morphism defined by:

$$\mathcal{J}_{\infty,N} : \mathcal{D}_N \ni M \longmapsto \mathcal{J}_{\infty,N}(M)\,(\boldsymbol{x}) := \langle\,\beta_N\,(\boldsymbol{x})\,|\,M\,|\,\beta_N\,(\boldsymbol{x})\,\rangle \in \mathcal{C}^0\left(\mathbb{T}^2\right)\;\cdot$$

Therefore, from definitions (1.3.2) and (1.6.1) it follows that, when mapping $\mathcal{C}^0\left(\mathbb{T}^2\right)$ onto $\mathcal{D}_N$ and the latter back into $\mathcal{C}^0\left(\mathbb{T}^2\right)$, we get[7]:

$$\begin{aligned}\widetilde{f}_N\,(\boldsymbol{x}) &:= \left(\mathcal{J}_{\infty,N}\circ\mathcal{J}_{N,\infty}\right)(f)\,(\boldsymbol{x}) = \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} f\left(\frac{\boldsymbol{\ell}}{N}\right)\left|\,\langle\beta_N\,(\boldsymbol{x})\,|\boldsymbol{\ell}\rangle\,\right|^2 = \\ &= \frac{1}{4}\sum_{(\mu,\nu,\rho,\sigma)\in\{0,1\}^4} \cos\left(\pi\mu\,\langle Nx_1\rangle\right)\cos\left(\pi\nu\,\langle Nx_2\rangle\right)(-1)^{\mu\rho+\nu\sigma}\times \\ &\qquad\times f\left(\frac{\lfloor Nx_1\rfloor+\rho}{N},\frac{\lfloor Nx_2\rfloor+\sigma}{N}\right) \qquad (1.74)\end{aligned}$$

Remarks 1.6.1

i) As it was pointed out in Section 1.4.2, the family of states $\{|\beta_N(\boldsymbol{x})\rangle \mid \boldsymbol{x} \in \mathbb{T}^2\}$ cannot be considered as a set of C.S., for they does not fulfill overcompleteness. The latter property is a necessary condition in order to prove the classical limit in the anti–Wick scheme of quantization/dequantization, and to this aim it has been profitably used in the proofs of property (1.5.1–1.5.2). Nevertheless, in the Weyl scheme, we will provide in Theorem 3 an equivalent proof of the classical limit that does not depend on overcompleteness; conversely we can use other nice properties of our family of states, like invertibility.

[7] we omit here the details of the cumbersome calculation: equation (1.74) is derived by using the same technique as the one showed in Appendix A.

ii) From (1.74), $f = \widetilde{f}_N$ on lattice points. Moreover, although the first derivative of (1.74) is not defined on the lattice, its limit exists there and it is zero; thus, we can extend by continuity $\widetilde{f}_N$ to a function in $\mathcal{C}^1\left(\mathbb{T}^2\right)$ that we will denote again as $\widetilde{f}_N$.

iii) We note that $\mathrm{Ran}\left(\mathcal{J}_{\infty,N}\right)$ is a subalgebra strictly contained in $\mathcal{A}_{\mathcal{X}}$; this is not surprising and comes as consequence of Weyl quantization, where this phenomenon is quite typical [34, 35].

We show below that $\mathcal{J}_{\infty,N}\circ\mathcal{J}_{N,\infty}$ approaches $\mathbb{1}_{\mathcal{C}^0(\mathbb{T}^2)}$ (the identity function in $\mathcal{C}^0\left(\mathbb{T}^2\right)$) when $N\to\infty$. Indeed, a request upon any sensible quantization procedure is to recover the classical description in the limit $\hbar\to 0$; in a similar way, our discretization should recover the continuous system in the $N\to\infty$ limit.

Theorem 3 : Given $f\in\mathcal{C}^0\left(\mathbb{T}^2\right)$, $\displaystyle\lim_{N\to\infty}\left\|\left(\mathcal{J}_{\infty,N}\circ\mathcal{J}_{N,\infty}-\mathbb{1}_{\mathcal{C}^0(\mathbb{T}^2)}\right)(f)\right\|_0 = 0\cdot$

Proof of Theorem 3:

i) Since $\mathcal{X}=\mathbb{T}^2$ is compact, f is uniformly continuous on it, that is[8]

$$\begin{aligned}&\forall\,\varepsilon>0,\ \exists\,\delta_{f,\varepsilon}>0\ \text{ s.t. }\left|\boldsymbol{x}-\frac{\lfloor N\boldsymbol{x}\rfloor}{N}\right|<\delta_{f,\varepsilon}\Longrightarrow\\&\Longrightarrow\left|f\left(\boldsymbol{x}\right)-f\left(\frac{\lfloor N\boldsymbol{x}\rfloor}{N}\right)\right|<\frac{\varepsilon}{2}\,,\ \forall\boldsymbol{x}\in\mathbb{T}^2,\ \forall N\in\mathbb{N}^+\end{aligned}\tag{1.75}$$

Further, we can choose an $\bar{N}_{f,\varepsilon}=\bar{N}_{f,\varepsilon}(\delta_{f,\varepsilon})$ s.t. $\left|\boldsymbol{x}-\dfrac{\lfloor\bar{N}_{f,\varepsilon}\boldsymbol{x}\rfloor}{\bar{N}_{f,\varepsilon}}\right|<\delta_{f,\varepsilon}$, that is[9] $\bar{N}_{f,\varepsilon}>\dfrac{\sqrt{2}}{\delta_{f,\varepsilon}}$; therefore

$$\forall\varepsilon>0,\ \exists\bar{N}_{f,\varepsilon}\in\mathbb{N}^+\ \text{ s.t. }\ N>\bar{N}_{f,\varepsilon}\Longrightarrow\left|f\left(\boldsymbol{x}\right)-f\left(\frac{\lfloor N\boldsymbol{x}\rfloor}{N}\right)\right|<\frac{\varepsilon}{2}\,,\ \forall\boldsymbol{x}\in\mathbb{T}^2\cdot$$

ii) $\widetilde{f}_N\in\mathcal{C}^1\left(\mathbb{T}^2\right)\subset\mathcal{C}^0\left(\mathbb{T}^2\right)$ and the previous point **i)** let us write

$$\forall\varepsilon>0,\ \exists\bar{N}'_{f,\varepsilon}\in\mathbb{N}^+\ \text{ s.t. }\ N>\bar{N}'_{f,\varepsilon}\Longrightarrow\left|\widetilde{f}_N\left(\boldsymbol{x}\right)-\widetilde{f}_N\left(\frac{\lfloor N\boldsymbol{x}\rfloor}{N}\right)\right|<\frac{\varepsilon}{2}\,,\ \forall\boldsymbol{x}\in\mathbb{T}^2\cdot$$

[8] Notation: we use $\dfrac{\lfloor N\boldsymbol{x}\rfloor}{N}$ to denote $\left(\dfrac{\lfloor Nx_1\rfloor}{N},\dfrac{\lfloor Nx_2\rfloor}{N}\right)\cdot$

[9] $\forall\,i\in\{1,2\}\,,\ \left|Nx_i-\lfloor Nx_i\rfloor\right|<1\Longrightarrow\left|N\boldsymbol{x}-\lfloor N\boldsymbol{x}\rfloor\right|<\sqrt{2}\Longrightarrow\left|\boldsymbol{x}-\dfrac{\lfloor N\boldsymbol{x}\rfloor}{N}\right|<\dfrac{\sqrt{2}}{N}\Longrightarrow$
$\Longrightarrow\left|\boldsymbol{x}-\dfrac{\lfloor N\boldsymbol{x}\rfloor}{N}\right|<\dfrac{\sqrt{2}}{\bar{N}_{f,\varepsilon}},\ \forall\,N>\bar{N}_{f,\varepsilon}\cdot$

iii) $\widetilde{f}_N\left(\frac{\lfloor N\boldsymbol{x}\rfloor}{N}\right) = f\left(\frac{\lfloor N\boldsymbol{x}\rfloor}{N}\right) \quad \forall \boldsymbol{x} \in \mathbb{T}^2,\ \forall N \in \mathbb{N}^+.$ (see Remark 1.6.1.ii)

Then, using the triangle inequality, with $\bar{N}''_{f,\varepsilon} = \max\left\{\bar{N}'_{f,\varepsilon}, \bar{N}_{f,\varepsilon}\right\}$, we get

$$\begin{aligned}&\forall\varepsilon > 0,\ \exists \bar{N}''_{f,\varepsilon} \in \mathbb{N}^+ \ \text{ s.t. } N > \bar{N}''_{f,\varepsilon} \Longrightarrow \\ &\Longrightarrow \left| f(\boldsymbol{x}) - \widetilde{f}_N(\boldsymbol{x}) \right| \leqslant \left| f(\boldsymbol{x}) - f\left(\frac{\lfloor N\boldsymbol{x}\rfloor}{N}\right)\right| + \left|\widetilde{f}_N\left(\frac{\lfloor N\boldsymbol{x}\rfloor}{N}\right) - \widetilde{f}_N(\boldsymbol{x})\right| < \varepsilon\ ,\ \forall \boldsymbol{x} \in \mathbb{T}^2\end{aligned}$$

that is:

$$\lim_{N\to\infty} \left\| \widetilde{f}_N - f \right\|_0 = 0 \cdot$$

■

Chapter 2

Quantization of the Dynamics and its classical limit

2.1. Classical Automorphisms on the Torus

2.1.1. Classical description of Sawtooth Maps and Cat Maps

The special kind of automorphisms of the torus that we are going to consider in this Section, namely the Sawtooth Maps [39,40], are a big family including the well known Cat Maps as a subset. From a classical point of view, in the spirit of Section 1.1.1, we describe these systems by means of triples $(\mathcal{X}, \mu, S_\alpha)$ where

$$\mathcal{X} = \mathbb{T}^2 \tag{2.1a}$$

$$S_\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1+\alpha & 1 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \langle x_1 \rangle \\ x_2 \end{pmatrix} \pmod 1 \;, \quad \alpha \in \mathbb{R} \tag{2.1b}$$

$$\mu(\mathrm{d}\boldsymbol{x}) = \mathrm{d}x_1 \, \mathrm{d}x_2 \;, \tag{2.1c}$$

where $\langle\cdot\rangle$ denotes the fractional part of a real number. Without $\langle\cdot\rangle$, (2.1b) is not well defined on $\mathbb{T}^2$ for not–integer α; in fact, without taking the fractional part, the same point $\boldsymbol{x} = \boldsymbol{x} + \boldsymbol{n} \in \mathbb{T}^2, \boldsymbol{n} \in \mathbb{Z}^2$, would have (in general) $S_\alpha(\boldsymbol{x}) \neq S_\alpha(\boldsymbol{x}+\boldsymbol{n})$. Of course, $\langle\cdot\rangle$ is not necessary when $\alpha \in \mathbb{Z}$.
The Lebesgue measure defined in (2.1c) is S_α–*invariant* for all $\alpha \in \mathbb{R}$.

After identifying $\boldsymbol{x}$ with canonical coordinates (q, p) and imposing the (mod 1) condition on both of them, the above dynamics can be rewritten as:

$$\begin{cases} q' &= q + p' \\ p' &= p + \alpha \langle q \rangle \end{cases} \quad \text{(mod 1)}, \tag{2.2}$$

This is nothing but the Chirikov Standard Map [6] in which $-\frac{1}{2\pi}\sin(2\pi q)$ is replaced by $\langle q \rangle$. The dynamics in (2.2) can also be thought of as generated by the (singular) Hamiltonian

$$H(q, p, t) = \frac{p^2}{2} - \alpha \frac{\langle q \rangle^2}{2} \delta_p(t), \tag{2.3}$$

where $\delta_p(t)$ is the periodic Dirac delta which makes the potential act through periodic kicks with period 1 [15].

Sawtooth Maps are invertible and the inverse is given by the expression

$$S_\alpha^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \left\langle \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle \quad \text{(mod 1)} \tag{2.4}$$

or, in other words,

$$\begin{cases} q &= \quad q' - p' \\ p &= -\alpha\, q \ + p' \end{cases} \quad \text{(mod 1)} . \tag{2.5}$$

It can indeed be checked that $S_\alpha\left(S_\alpha^{-1}(\boldsymbol{x})\right) = S_\alpha^{-1}\left(S_\alpha(\boldsymbol{x})\right) = \boldsymbol{x}$[1], $\forall \boldsymbol{x} \in \mathbb{T}^2$.

Another dynamics we are going to consider is the one described by (2.1), but with (2.1b) replaced by

$$T(\boldsymbol{x}) = T \cdot \boldsymbol{x} \quad \text{(mod 1)} , \tag{2.1b$'$}$$

where $T \in \mathrm{SL}_2\left(\mathbb{T}^2\right)$, namely the 2×2 matrices with integer entries and determinant equal to one. The Lebesgue measure defined in (2.1c) is *T–invariant* for all $T \in \mathrm{SL}_2\left(\mathbb{T}^2\right)$.

Definition 2.1.1

When $\alpha \in \mathbb{Z}$, we shall write T_α instead of S_α. In particular:

$\left(\mathcal{X}, \mu, T_\alpha\right)$ will be the classical dynamical system representing a generic T_α automorphism;

[1] of course $\boldsymbol{x}$ has to be intended as an element of the torus, that is an equivalent class of $\mathbb{R}^2$ points whose coordinates differ for integer value, indeed $\mathbb{T}^2 = \mathbb{R}^2 \setminus \mathbb{Z}^2$.

$(\mathcal{X}, \mu, S_\alpha)$ will represent Sawtooth Maps;

$(\mathcal{X}, \mu, T)$ with $T \in \mathrm{SL}_2(\mathbb{T}^2)$, will represent the so–called *Unitary Modular Group* [3] (UMG for short).

$T_1 = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ is the Arnold Cat Map [3].

Then, $T_1 \in \{T_\alpha\}_{\alpha\in\mathbb{Z}} \subset \mathrm{SL}_2(\mathbb{T}^2) \subset \mathrm{GL}_2(\mathbb{T}^2) \subset \mathrm{ML}_2(\mathbb{T}^2)$ where $\mathrm{ML}_2(\mathbb{T}^2)$ is the set of 2×2 matrices with integer entries and $\mathrm{GL}_2(\mathbb{T}^2)$ is the subset of invertible matrices.

Remarks 2.1.1

i. Sawtooth Maps $\{S_\alpha\}$ are *discontinuous* on the subset $\gamma_0 := \{\boldsymbol{x} = (0, p),\ p \in \mathbb{T}\} \in \mathbb{T}^2$: two points close to this border, $A := (\varepsilon, p)$ and $B := (1 - \varepsilon, p)$, have images that differ, in the $\varepsilon \to 0$ limit, by a vector $d^{(1)}_{S_\alpha}(A, B) = (\alpha, \alpha) \pmod 1$.

ii. Inverse Sawtooth Maps $\{S_\alpha^{-1}\}$ are *discontinuous* on the subset $\gamma_{-1} := S_\alpha(\gamma_0) = \{\boldsymbol{x} = (p, p),\ p \in \mathbb{T}\} \in \mathbb{T}^2$: two points close to this border, $A := (p + \varepsilon, p - \varepsilon)$ and $B := (p - \varepsilon, p + \varepsilon)$, have images that differ, in the $\varepsilon \to 0$ limit, by a vector $d^{(1)}_{S_\alpha^{-1}}(A, B) = (0, \alpha) \pmod 1$.

iii. The maps T_α and T_α^{-1} are *continuous*:
$\alpha \in \mathbb{Z} \Longrightarrow d^{(1)}_{T_\alpha}(A, B) = d^{(1)}_{T_\alpha^{-1}}(A, B) = (0, 0) \pmod 1$.
Also, all $T \in \mathrm{SL}_2(\mathbb{T}^2)$ are continuous.

iv. The eigenvalues of $\left(\begin{smallmatrix} 1+\alpha & 1 \\ \alpha & 1 \end{smallmatrix}\right)$ are $\left(\alpha + 2 \pm \sqrt{(\alpha+2)^2 - 4}\right) \Big/ 2$. They are complex conjugates if $\alpha \in [-4, 0]$, while one eigenvalue $\lambda > 1$ and the other $\lambda^{-1} < 1$ if $\alpha \notin [-4, 0]$.

v. For a generic matrix $T \in \mathrm{SL}_2(\mathbb{T}^2)$, denoting $t = \mathrm{Tr}(T)/2$, the eigenvalues are $(t \pm \sqrt{t^2 - 1})$. They are conjugate complex numbers if $|t| < 1$, while one eigenvalue $\lambda > 1$ if $|t| > 1$. The latter is our case of interest, indeed we will use only *hyperbolic* $T \in \mathrm{SL}_2(\mathbb{T}^2)$, that is $|t| > 1$.

vi. When a positive eigenvalue is present, that is $\lambda > 1$, distances are stretched along the direction of the eigenvector $|\boldsymbol{e}_+\rangle$, $S_\alpha|\boldsymbol{e}_+\rangle = \lambda|\boldsymbol{e}_+\rangle$, contracted along that of $|\boldsymbol{e}_-\rangle$, $S_\alpha|\boldsymbol{e}_-\rangle = \lambda^{-1}|\boldsymbol{e}_-\rangle$. In this case, we can see in $\log\lambda$, the (positive) Lyapounov exponent (compare (1)).

vii. The Lebesgue measure in (2.1c) is S_α^{-1}–invariant.

Notation 2.1.1

Let S_α be the matrix $\left(\begin{smallmatrix} 1+\alpha & 1 \\ \alpha & 1 \end{smallmatrix}\right)$. Then the expression $S_\alpha(\boldsymbol{x})$ will denote the action represented by (2.1b), whereas $S_\alpha \cdot \boldsymbol{x}$ will denote the matrix action of S_α on the vector $\boldsymbol{x}$.

When the dynamics arises from the action of a UMG map (so, in particular, when $\{T_\alpha\}_{\alpha\in\mathbb{Z}}$ is the family of toral automorphisms), the equation (2.1b) assumes the simpler form $T_\alpha(\boldsymbol{x}) = T_\alpha \cdot \boldsymbol{x} \pmod 1$.

Analogously, expression like $T_\alpha \cdot \boldsymbol{x}$, $T_\alpha^{\mathrm{tr}} \cdot \boldsymbol{x}$, $T_\alpha^{-1} \cdot \boldsymbol{x}$ and $\left(T_\alpha^{\mathrm{tr}}\right)^{-1} \cdot \boldsymbol{x}$, will denote action by T_α itself, its transposed, its inverse and the inverse of the transposed, respectively.

2.1.2. Algebraic description for the classical dynamical systems of toral automorphisms

In this Section we make use of the two commutative algebras introduced in Section 1.1.2 in order to describe the two family of toral automorphisms defined up to now.

For the (continuous) automorphisms in the $\{T_\alpha\}$ family, a convenient algebra [34, 35] of observables is the C*–algebra $\mathcal{C}^0(\mathcal{X})$, equipped with the uniform norm given in (1.1).

The discrete–time dynamics generates automorphisms Θ_α and its iterates $\Theta_\alpha^j : \mathcal{C}^0(\mathcal{X}) \mapsto \mathcal{C}^0(\mathcal{X})$ as follows:

$$\Theta_\alpha^j(f)(\boldsymbol{x}) := f(T_\alpha^j \cdot \boldsymbol{x}) \ , \quad j \in \mathbb{Z} \ , \ \alpha \in \mathbb{Z} \ \cdot \tag{2.6}$$

They preserve the state $\omega_\mu \circ \Theta_\alpha^j = \omega_\mu$.

Due to the discontinuity of Sawtooth Maps, the maps Θ_α^j in equation (2.6), with $\alpha \in \mathbb{R}\backslash\mathbb{Z}$, are no longer automorphisms of $\mathcal{C}^0(\mathcal{X})$.

For this reason, in order to deal with Sawtooth maps, we will make use of the (Von Neumann) algebra $L^\infty_\mu(\mathcal{X})$ of essentially bounded functions defined in Section 1.1.2.

We define $\Theta_\alpha^j : L^\infty_\mu(\mathcal{X}) \mapsto L^\infty_\mu(\mathcal{X})$ by

$$\Theta_\alpha^j(f)(\boldsymbol{x}) := f(S_\alpha^j(\boldsymbol{x})) \ , \quad j \in \mathbb{Z} \ , \ \alpha \in \mathbb{R} \ , \tag{2.7}$$

These maps are now automorphisms of $L^\infty_\mu(\mathcal{X})$ and leave the state ω_μ invariant.

Even if the maps T belonging to the UMG are continuous, when dealing with quantized UMG it is preferable to make use of the Von Neumann algebra $L^\infty_\mu(\mathcal{X})$, in conjunction with the state ω_μ and another automorphism, different from (2.6–2.7), given by:

$$\Pi : L^\infty_\mu(\mathcal{X}) \ni f(\boldsymbol{x}) \mapsto \Pi(f)(\boldsymbol{x}) := f(T^{-1}\cdot \boldsymbol{x}) \in L^\infty_\mu(\mathcal{X}) \quad \cdot \tag{2.8}$$

The maps Π^j are measure preserving automorphisms on $L^\infty_\mu(\mathcal{X})$. The following definitions are thus justified:

Definitions 2.1.2

– The triplets describing UMG automorphisms will be chosen between either $\left(L^\infty_\mu(\mathcal{X}), \omega_\mu, \Pi\right)$ or $\left(\mathcal{C}^0(\mathcal{X}), \omega_\mu, \Theta_\alpha\right)$.

– Sawtooth Maps will be identified by triplets $\left(L^\infty_\mu(\mathcal{X}), \omega_\mu, \Theta_\alpha\right)$.

2.2. Quantum Automorphisms on the Torus

2.2.1. Dynamical evolution of the Weyl operators

In the quantization of dynamical systems $\left(L^\infty_\mu(\mathcal{X}), \omega_\mu, \Pi\right)$ and the study of their classical limit, the main role is played by the evolution of the Weyl operators. Indeed, from the time evolution in the Weyl pictures, one easily goes to the anti–Wick scheme realized by means of C.S. of the form (1.36).

We will derive this evolution basing our analysis on the Weyl scheme, described in Section 1.2.2, and then we will extend such kind of evolution to the Anti–Wick scheme, that will be used in the rest of our work. Of course in Section 2.4.2 we will provide a proof that our definition of quantum dynamics is "well posed", in the sense that it leads to a well defined classical limit.

We introduce our evolution operator $\Theta^j_N : \mathcal{M}_N \to \mathcal{M}_N$ by giving this requirement:

$$\forall\, f \in \mathcal{A}_{\mathbb{T}^2}, \qquad \Theta^j_N\left(W_{N,\infty}(f)\right) = W_{N,\infty}\left(\Pi^j(f)\right) \ , \tag{2.9}$$

where once more we suppose $\mathcal{A}_{\mathbb{T}^2}$ to consist of functions sufficiently smooth and regular, namely to be Fourier decomposed, Π is the (measure preserving) automorphism on $L^\infty_\mu(\mathcal{X})$ given in (2.8), and $W_{N,\infty}$ is the Weyl quantization operator, whose definition (1.27) leads to:

$$\Theta^j_N\left(W_{N,\infty}(f)\right) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \hat{f}_{\boldsymbol{n}}\, \Theta^j_N\left(W_N(\boldsymbol{n})\right) \quad \cdot \tag{2.10}$$

Conversely, from (2.9) and (2.8), we have

$$\begin{aligned}
\Theta_N^j\left(W_{N,\infty}(f)\right) &= W_{N,\infty}\left(\Pi^j(f)\right) && (2.11)\\
&= W_{N,\infty}\left(f\circ T^{-j}\right) && (2.12)\\
&= \sum_{\boldsymbol{n}\in\mathbb{Z}^2} W_N(\boldsymbol{n})\ (\widehat{f\circ T^{-j}})_{\boldsymbol{n}} && (2.13)\\
&= \sum_{\boldsymbol{n}\in\mathbb{Z}^2} W_N(\boldsymbol{n})\ \iint_{\mathbb{T}^2} \mathrm{d}\mu(\boldsymbol{x})\ f(T^{-j}\cdot\boldsymbol{x})\ e^{-2\pi i\,\sigma(\boldsymbol{n}\,,\,\boldsymbol{x})} && (2.14)
\end{aligned}$$

and changing variable $\boldsymbol{y}:= T^{-j}\cdot\boldsymbol{x}$ (note that $T\in\mathrm{SL}_2\left(\mathbb{T}^2\right)$, so $\det\left(T^{-j}\right)=1$)

$$\begin{aligned}
&= \sum_{\boldsymbol{n}\in\mathbb{Z}^2} W_N(\boldsymbol{n})\ \iint_{\mathbb{T}^2} \mathrm{d}\mu(\boldsymbol{y})\ f(\boldsymbol{y})\ e^{-2\pi i\,\sigma(\boldsymbol{n}\,,\,T^j\cdot\boldsymbol{y})} && (2.15)\\
&= \sum_{\boldsymbol{n}\in\mathbb{Z}^2} W_N(\boldsymbol{n})\ \iint_{\mathbb{T}^2} \mathrm{d}\mu(\boldsymbol{y})\ f(\boldsymbol{y})\ e^{-2\pi i\,\sigma(T^{-j}\cdot\boldsymbol{n}\,,\,\boldsymbol{y})} && (2.16)
\end{aligned}$$

(indeed the simplectic form $\sigma(\,\cdot\,,\,\cdot\,)$ is $\mathrm{SL}_2\left(\mathbb{T}^2\right)$-invariant, i.e. $\sigma(T\boldsymbol{p},T\boldsymbol{q})=\sigma(\boldsymbol{p},\boldsymbol{p})$)

$$\begin{aligned}
&= \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \widehat{f}_{T^{-j}\cdot\boldsymbol{n}}\, W_N(\boldsymbol{n}) && (2.17)\\
&= \sum_{T^j\cdot\boldsymbol{m}\in\mathbb{Z}^2} \widehat{f}_{\boldsymbol{m}}\, W_N(T^j\cdot\boldsymbol{m}) && (2.18)\\
&= \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \widehat{f}_{\boldsymbol{m}}\, W_N(T^j\cdot\boldsymbol{m}) && (2.19)
\end{aligned}$$

where in the latter equality we have used the fact that matrices belonging to $\mathrm{SL}_2\left(\mathbb{T}^2\right)$ map $\mathbb{Z}^2$ onto itself. Comparing (2.10) and (2.19) we obtain the result

$$\Theta_N^j\left(W_N(\boldsymbol{n})\right) = W_N\left(T^j\cdot\boldsymbol{n}\right)\ , \tag{2.20}$$

that will be taken as a definition also in the anti–Wick scheme.

In order to describe the quantum dynamical system during its (discrete) temporal evolution, we need to have the evolution unitarily implemented on the Weyl algebra, that is $\Theta_N^j\left(W_N(\boldsymbol{n})\right) = U_T\, W_N(\boldsymbol{n})\, U_T^*$, with U_T unitary operator on $\mathcal{H}_N$. In other words, the representation generated by the two generators $W_N(\hat{\boldsymbol{e}}_1)$ and $W_N(\hat{\boldsymbol{e}}_2)$, and the one generated by $\Theta_N\left(W_N(\hat{\boldsymbol{e}}_1)\right) = [W_N(T\cdot\hat{\boldsymbol{e}}_1)]$ and $\Theta_N\left(W_N(\hat{\boldsymbol{e}}_2)\right) = [W_N(T\cdot\hat{\boldsymbol{e}}_2)]$, has to be unitarily equivalent. If this is obtained, by point (b) of Theorem 1 in Remark 1.2.2, we see that the two representations have to be labeled by the same u and v. Therefore:

$$[W_N(\hat{\boldsymbol{e}}_1)]^N = [W_N(T\cdot\hat{\boldsymbol{e}}_1)]^N \qquad \text{and} \qquad [W_N(\hat{\boldsymbol{e}}_2)]^N = [W_N(T\cdot\hat{\boldsymbol{e}}_2)]^N \ \cdot \tag{2.21}$$

The latter request restrict the possible couples (u, v) available, as it is showed in the next [29]

Theorem 4 (Degli Esposti) : Let $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2\left(\mathbb{T}^2\right)$ and N be positive integer. Then (2.21) can be fulfilled, more precisely:

for any given automorphism T, all admissible representations are labeled by all $(u, v) \in \mathbb{T}^2$ such that

$$\left(T^{\mathrm{tr}} - \mathbb{1}\right)\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}N\begin{pmatrix} ac \\ bd \end{pmatrix} + \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad m_1, m_2 \in \mathbb{Z} \tag{2.22}$$

Equation (2.22) can be trivially solved[2], and we get for the couples (u, v):

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\mathrm{Tr}\,(T) - 2}\begin{pmatrix} 1-d & c \\ b & 1-a \end{pmatrix}\left[\boldsymbol{p} + \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}\right] \pmod 1, \quad m_1, m_2 \in \mathbb{Z} \tag{2.23}$$

where $\boldsymbol{p} = \boldsymbol{0}$ for even N, whereas for odd N it depends from the parity of the elements of the matrix T. In particular, in agreement with the condition $\det(T) = 1$, we have three allowed sets of matrices T, that are:

$$\mathcal{N}_1 := \left\{\begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix}, \begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix}\right\} \quad , \quad \mathcal{N}_2 := \left\{\begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{odd} \end{pmatrix}, \begin{pmatrix} \text{odd} & \text{even} \\ \text{odd} & \text{odd} \end{pmatrix}\right\} \quad ,$$
$$\mathcal{N}_3 := \left\{\begin{pmatrix} \text{odd} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix}, \begin{pmatrix} \text{odd} & \text{odd} \\ \text{even} & \text{odd} \end{pmatrix}\right\}$$

whose corresponding vector $\boldsymbol{p}$ are $\boldsymbol{p}\left(\mathcal{N}_1\right) = \boldsymbol{0}$, $\boldsymbol{p}\left(\mathcal{N}_2\right) = \frac{1}{2}\,\hat{\boldsymbol{e}}_2$ and $\boldsymbol{p}\left(\mathcal{N}_3\right) = \frac{1}{2}\,\hat{\boldsymbol{e}}_1$.

The set $\mathcal{N}_1$, whose corresponding (unique) couple (u, v) is $(0, 0)$, is also important for historical reasons: indeed this set of matrices was used to develop the first quantization of Cat Maps obtained by Berry and Hannay [41][3].

We end up this Section by noting that unitarity of dynamics guarantees:

$$\tau_N(W_N(T\cdot\boldsymbol{n})) = \tau_N(W_N(\boldsymbol{n})) \ \cdot \tag{2.24}$$

[2] Note that, in equation (2.23), $\mathrm{Tr}\,(T) \neq 2$ because of the choice in Remark 2.1.1.v.

[3] For recent developments of the Berry's approach to the quantization of Cat Maps, see [42–44].

2.3. Discretization of Sawtooth Map families

2.3.1. Dynamical evolution on $\mathcal{D}_N$ arising from the $\{T_\alpha\}$ subfamily of UMG

In this Section we will parallel what we have done in Section 2.2.1 in the framework where the dynamical evolution is dictated by the $\{T_\alpha\}$ subfamily of UMG (see Definitions 2.1.1 and 2.1.2), namely we will use the C*–algebra $\mathcal{C}^0\left(\mathbb{T}^2\right)$ to describe the continuous system and we develop a technique of discretization/dediscretization presented in Section 1.3.1, respectively Section 1.6.

The calculations follow (2.9–2.20): starting from the request

$$\forall\, f \in \mathcal{C}^0\left(\mathbb{T}^2\right), \qquad \widetilde{\Theta}^j_{N,\alpha}\left(\mathcal{J}_{N,\infty}\left(f\right)\right) = \mathcal{J}_{N,\infty}\left(\Theta^j_\alpha\left(f\right)\right)\ , \tag{2.25}$$

we get

$$\widetilde{\Theta}^j_{N,\alpha}\left(W_N(\boldsymbol{n})\right) = W_N\left(\left(T^{\mathrm{tr}}_\alpha\right)^j \cdot \boldsymbol{n}\right)\ \cdot \tag{2.26}$$

Few comment are now in order: in Section 2.2.1 we used Weyl operators defined by means of the symplectic form $\sigma\left(\boldsymbol{n}\,,\,\boldsymbol{x}\right)$ (see (1.24–1.27)), which is invariant under the action of $T \in \mathrm{SL}_2\left(\mathbb{T}^2\right)$, that is $\sigma\left(T\cdot\boldsymbol{n}\,,\,T\cdot\boldsymbol{x}\right) = \sigma\left(\boldsymbol{n}\,,\,\boldsymbol{x}\right)$. Therefore the dynamics $\boldsymbol{x} \mapsto T^{-j}\cdot\boldsymbol{x}$ defined in (2.8) actually affects the indices of Weyl operators in the in the sense that $\boldsymbol{n} \mapsto T^j\cdot\boldsymbol{n}$ (indeed $\sigma\left(\boldsymbol{n}\,,\,T^{-j}\cdot\boldsymbol{x}\right) = \sigma\left(T^j\cdot\boldsymbol{n}\,,\,\cdot\boldsymbol{x}\right)$).

In the present case, Weyl operators are defined using not the symplectic form but scalar product $\langle\boldsymbol{n}|\,\boldsymbol{x}\rangle = \boldsymbol{n}\cdot\boldsymbol{x}$ instead, so that $\left\langle\boldsymbol{n}\middle|T^j_\alpha\cdot\boldsymbol{x}\right\rangle = \left\langle\left(T^{\mathrm{tr}}_\alpha\right)^j\cdot\boldsymbol{n}\middle|\,\boldsymbol{x}\right\rangle$, whence (2.26).

Although we defined the dynamics on Weyl operators, neither the discretizing operator $\mathcal{J}_{N,\infty}$ in Definition 1.3.2 nor the dediscretizing one $\mathcal{J}_{\infty,N}$ in Definition 1.6.1 do depend (explicitly) on Weyl operators. For this reason we introduce now the operator $\widetilde{\Theta}_{N,\alpha}$ on $\mathcal{D}_N$ in a way compatible with (2.26). To this aim we introduce first a new family of maps on the torus $\mathbb{T}^2\left([0,N)^2\right)$, namely $[0,N)^2\pmod{N}$, given by[4]

[4]Although we are now dealing with the $\{T_\alpha\}$ family of maps, definition (2.27) is formulated for Sawtooth maps; when $\alpha \in \mathbb{Z}$, (2.27) obviously simplify to direct matrix action, which is not true when $\alpha \in \mathbb{R}\setminus\mathbb{Z}$, which will be the case later on.

$$\begin{aligned}
\mathbb{T}^2\left([0,N)^2\right) \ni \boldsymbol{x} &\mapsto U_\alpha^0(\boldsymbol{x}) := \boldsymbol{x} \\
&= N\,S_\alpha^0\left(\frac{\boldsymbol{x}}{N}\right) \in \mathbb{T}^2\left([0,N)^2\right)\ , && (2.27a)\\
\mathbb{T}^2\left([0,N)^2\right) \ni \boldsymbol{x} &\mapsto U_\alpha^{\pm 1}(\boldsymbol{x}) := N\,S_\alpha^{\pm 1}\left(\frac{\boldsymbol{x}}{N}\right) \in \mathbb{T}^2\left([0,N)^2\right)\ , && (2.27b)\\
\mathbb{T}^2\left([0,N)^2\right) \ni \boldsymbol{x} &\mapsto U_\alpha^{\pm j}(\boldsymbol{x}) := \underbrace{U_\alpha^{\pm 1}(U_\alpha^{\pm 1}(\cdots U_\alpha^{\pm 1}(U_\alpha^{\pm 1}(}_{j \text{ times}}\boldsymbol{x}))\cdots))\ ,\quad j\in\mathbb{N}\ ,\\
&= N\,S_\alpha^{\pm j}\left(\frac{\boldsymbol{x}}{N}\right) \in \mathbb{T}^2\left([0,N)^2\right)\ \cdot && (2.27c)
\end{aligned}$$

Using the latter set of maps, we can give the following

Definition 2.3.1

$\widetilde{\Theta}_{N,\alpha}$ is the *automorphism of $\mathcal{D}_N$ defined by:

$$\mathcal{D}_N \ni X \mapsto \widetilde{\Theta}_{N,\alpha}(X) := \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} X_{U_\alpha(\boldsymbol{\ell}),U_\alpha(\boldsymbol{\ell})}\,|\boldsymbol{\ell}\rangle\langle\boldsymbol{\ell}| \in \mathcal{D}_N\ \cdot \tag{2.28}$$

Remarks 2.3.1

i. $\widetilde{\Theta}_{N,\alpha}$ is a *automorphism because the map $(\mathbb{Z}/N\mathbb{Z})^2 \ni \boldsymbol{\ell} \longmapsto U_\alpha(\boldsymbol{\ell}) \in (\mathbb{Z}/N\mathbb{Z})^2$ is a bijection. For the same reason the state τ_N is $\widetilde{\Theta}_{N,\alpha}$–invariant.

ii. One can check that, given $f \in \mathcal{C}^0\left(\mathbb{T}^2\right)$,

$$\widetilde{\Theta}_{N,\alpha}(\mathcal{J}_{N,\infty}(f)) := \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} f\left(\frac{U_\alpha(\boldsymbol{\ell})}{N}\right)|\boldsymbol{\ell}\rangle\langle\boldsymbol{\ell}|\ \cdot \tag{2.29}$$

iii. Also, $\widetilde{\Theta}^j_{N,\alpha}\circ\mathcal{J}_{N,\infty} = \mathcal{J}_{N,\infty}\circ\Theta^j_\alpha$ for all $j\in\mathbb{Z}$.

The automorphism $\widetilde{\Theta}_{N,\alpha}$ can be rewritten in the more familiar form

$$\begin{aligned}
\widetilde{\Theta}_{N,\alpha}(X) &= \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} X_{U_\alpha(\boldsymbol{\ell}),U_\alpha(\boldsymbol{\ell})}\,|\boldsymbol{\ell}\rangle\langle\boldsymbol{\ell}| = && (2.30)\\
&= \sum_{U_\alpha^{-1}(\boldsymbol{s})\in(\mathbb{Z}/N\mathbb{Z})^2} X_{\boldsymbol{s},\boldsymbol{s}}\,\left|U_\alpha^{-1}(\boldsymbol{s})\right\rangle\left\langle U_\alpha^{-1}(\boldsymbol{s})\right| =
\end{aligned}$$

$$\text{(see Remark 2.3.2 i and ii)} \qquad = U_{\alpha,N}\left(\sum_{\substack{\text{all equiv.}\\ \text{classes}}} X_{\boldsymbol{s},\boldsymbol{s}}\,|\,\boldsymbol{s}\,\rangle\,\langle\,\boldsymbol{s}\,|\right)U^{*}_{\alpha,N} = \tag{2.31}$$

$$= U_{\alpha,N}\; X \;\; U^{*}_{\alpha,N}\ , \tag{2.32}$$

where the operators $U_{\alpha,N}$ defined by

$$\mathcal{H}_{N^2} \ni \big|\boldsymbol{\ell}\big\rangle \longmapsto U_{\alpha,N}\big|\boldsymbol{\ell}\big\rangle := \big|\,U_{\alpha}^{-1}\left(\boldsymbol{\ell}\right)\big\rangle \ \ \cdot \tag{2.33}$$

are unitary (see below).

Remarks 2.3.2

i) All of T_α, T_α^{-1}, T_α^{tr} and $\left(T_\alpha^{-1}\right)^{\mathrm{tr}}$ belong to $SL_2\left(\mathbb{Z}/N\mathbb{Z}\right)$; in particular these matrices are automorphisms on $\left(\mathbb{Z}/N\mathbb{Z}\right)^2$ (and so U_α and U_α^{-1} as well) so that, in (2.31), one can sum over the equivalence classes.

ii) The same argument as before proves that the operators in (2.33) are unitary, whence $\widetilde{\Theta}_{N,\alpha}$ is a *-automorphism of $\mathcal{D}_N$.

2.3.2. Dynamical evolution on $\mathcal{D}_N$ arising from the action of Sawtooth maps

From a the measure theoretical point of view, the dynamical systems $\left(\mathcal{X},\mu,S_\alpha\right)$ can be thought as extensions of $\left(\mathcal{X},\mu,T_\alpha\right)$. Both kind of systems are defined on the same space $\mathbb{T}^2$, endowed with the same measure μ and the S_α maps reduce to T_α when we restrict the domain of α from $\mathbb{R}$ to $\mathbb{Z}$.

From an algebraic point of view, we note that the algebra describing classical dynamical systems given by $\left(L^\infty_\mu\left(\mathcal{X}\right),\omega_\mu,\Theta_\alpha\right)$ is larger than the one of $\left(\mathcal{C}^0\left(\mathcal{X}\right),\omega_\mu,\Theta_\alpha\right)$ as $\mathcal{C}^0\left(\mathcal{X}\right)\subset L^\infty_\mu\left(\mathcal{X}\right)$, while the state ω_μ is the same and for the dynamics the same consideration of above holds.

For what concern discretization, while the Weyl scheme can be used for $\left(\mathcal{C}^0\left(\mathcal{X}\right),\omega_\mu,\Theta_\alpha\right)$, we note that a straightforward application of the same procedure for $\left(L^\infty_\mu\left(\mathcal{X}\right),\omega_\mu,\Theta_\alpha\right)$ would not work. Indeed, $f\in L^\infty_\mu\left(\mathcal{X}\right)$ could be unbounded on the grid L_N, because the grid has null μ-measure and thus not felt by the essential norm $\left\|\cdot\right\|_\infty$ in (1.2).

Moreover, while in (2.26) one sees that the (matrix) action of T_α on $\boldsymbol{x}$ transfers to the index of the Weyl operators, this property no longer holds in the case of the action of S_α, $\alpha\notin\mathbb{Z}$, which is non–matricial.

In the (algebraic) discrete description, we deal with two families of "quantum" dynamical systems, namely $(\mathcal{D}_N, \tau_N, \widetilde{\Theta}_{N,\alpha})$ for $\{T_\alpha\}$ and $(\mathcal{D}_N, \tau_N,\ \mathbf{?}^5\ )$ for Sawtooth maps, in which the Abelian finite dimensional algebra $\mathcal{D}_N$ and the tracial state τ_N are the same, for both.

Thus it proves convenient to extend the automorphisms $\widetilde{\Theta}_{N,\alpha}$ in a way more suited to Sawtooth maps, which coincide with Definition 2.3.1 when $\alpha \in \mathbb{Z}$. To this aim we introduce first new family of maps V_α from the torus $\mathbb{T}^2\left([0,N)^2\right) := [0,N) \times [0,N) \pmod N$, to $(\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})$ (where $(\mathbb{Z}/N\mathbb{Z})$ denotes the residual class $\pmod N$). These maps are defined as follows:

$$\begin{aligned} \mathbb{T}^2\left([0,N)^2\right) \ni \boldsymbol{x} &\mapsto V_\alpha^0(\boldsymbol{x}) := \lfloor \boldsymbol{x} \rfloor \\ &\qquad = \left\lfloor U_\alpha^0(\boldsymbol{x}) \right\rfloor \in (\mathbb{Z}/N\mathbb{Z})^2 , \end{aligned} \tag{2.34a}$$

$$\mathbb{T}^2\left([0,N)^2\right) \ni \boldsymbol{x} \mapsto V_\alpha(\boldsymbol{x}) := \lfloor U_\alpha(\boldsymbol{x}) \rfloor \in (\mathbb{Z}/N\mathbb{Z})^2 , \tag{2.34b}$$

$$\begin{aligned} \mathbb{T}^2\left([0,N)^2\right) \ni \boldsymbol{x} &\mapsto V_\alpha^j(\boldsymbol{x}) := \underbrace{V_\alpha(\,V_\alpha(\cdots V_\alpha(\,V_\alpha(}_{j \text{ times}}\boldsymbol{x}\,))\cdots)) \in (\mathbb{Z}/N\mathbb{Z})^2 , \quad j \in \mathbb{N}^+ , \\ &\qquad = \underbrace{\lfloor U_\alpha(\,\lfloor U_\alpha(\cdots \lfloor U_\alpha(\,\lfloor U_\alpha(}_{j \text{ times}}\boldsymbol{x}\,)\rfloor\,)\rfloor \cdots)\rfloor\,)\rfloor \cdot \end{aligned} \tag{2.34c}$$

By means of the latter set $\left\{V_\alpha^j\right\}_{j\in\mathbb{Z}}$, we can proceed to

Definition 2.3.2

$\widetilde{\Theta}_{N,\alpha}$ is the *-automorphism of $\mathcal{D}_N$ defined by:

$$\mathcal{D}_N \ni X \mapsto \widetilde{\Theta}_{N,\alpha}(X) := \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} X_{V_\alpha(\boldsymbol{\ell}),V_\alpha(\boldsymbol{\ell})} \left|\boldsymbol{\ell}\right\rangle \left\langle\boldsymbol{\ell}\right| \in \mathcal{D}_N \cdot \tag{2.35}$$

Remarks 2.3.3

i. Note that $\widetilde{\Theta}_\alpha^j := \underbrace{\widetilde{\Theta}_\alpha \circ \widetilde{\Theta}_\alpha \circ \cdots \circ \widetilde{\Theta}_\alpha}_{j \text{ times}}$ is implemented by $V_\alpha^j(\boldsymbol{\ell})$ given in (2.34c).

ii. $\widetilde{\Theta}_{N,\alpha}$ is a *-automorphism because the map $(\mathbb{Z}/N\mathbb{Z})^2 \ni \boldsymbol{\ell} \longmapsto V_\alpha(\boldsymbol{\ell}) \in (\mathbb{Z}/N\mathbb{Z})^2$ is a bijection. For the same reason the state τ_N is $\widetilde{\Theta}_{N,\alpha}$–invariant and V_α is invertible too.

iii. When $\alpha \in \mathbb{Z}$, $(\mathbb{Z}/N\mathbb{Z})^2 \ni \boldsymbol{\ell} \longmapsto V_\alpha(\boldsymbol{\ell}) = T_\alpha \boldsymbol{\ell} \in (\mathbb{Z}/N\mathbb{Z})^2$, namely the action of the map V_α becomes that of a matrix $\pmod N$. Moreover, in that case, U_α and V_α coincide and Definitions 2.3.1 and 2.3.2 so do.

[5] Here the dynamics is not yet set.

A the same argument as in (2.30–2.32), proves that the automorphism $\widetilde{\Theta}_{N,\alpha}$ is unitarily implemented,

$$\widetilde{\Theta}_{N,\alpha} = U'_{\alpha,N}\ X\ \ U'^{*}_{\alpha,N}\ , \tag{2.36}$$

with $U'_{\alpha,N}$ unitary and given by

$$\mathcal{H}_{N^2} \ni |\boldsymbol{\ell}\rangle \longmapsto U'_{\alpha,N}|\boldsymbol{\ell}\rangle := \left|V_\alpha^{-1}(\boldsymbol{\ell})\right\rangle \tag{2.37}$$

(note that Remark 2.3.3.ii. allows us to use V_α^{-1}).

2.4. Classical/Continuous limit of the dynamics

One of the main issues in the semi-classical analysis is to compare if and how the quantum and classical time evolutions mimic each other when a quantization parameter goes to zero.

In the case of classically chaotic quantum systems, the situation is strikingly different from the case of classically integrable quantum systems. In the former case, classical and quantum mechanics agree on the level of coherent states only over times which scale as $-\log \hbar$.

As we shall see, such kind of scaling is not related with non commutativity. The quantization–like procedure we developed until now, depending on the lattice spacing $\frac{1}{N}$, has been set with the purpose to exhibit such a behavior also in the classical limit (actually continuous limit) connecting functional Abelian algebras $\mathcal{A}_{\mathcal{X}}$ with Abelian (finite dimensional) ones, consisting of diagonal matrices.

2.4.1. A useful localization property leading to a well defined classical limit

As before, let T denote the evolution on the classical phase space $\mathcal{X}$ and U_T the unitary single step evolution on $\mathbb{C}^N$ which represent its "quantization". We formally state the semi–classical correspondence of classical and quantum evolution using coherent states:

Condition 2.4.1 (Dynamical localization)

There exists an $\alpha > 0$ such that for all choices of $\varepsilon > 0$ and $d_0 > 0$ there exists an $N_0 \in \mathbb{N}$ with the following property: if $N > N_0$ and $k \leq \alpha \log N$, then $N|\langle C_N(\boldsymbol{x}), U_T^k C_N(\boldsymbol{y})\rangle|^2 \leq \varepsilon$ whenever $d(T^k\boldsymbol{x}, \boldsymbol{y}) \geq d_0$.

Remark 2.4.1

The condition of dynamical localization is what is expected of a good choice of coherent states, namely, on a time scale logarithmic in the inverse of the semi-classical parameter, evolving coherent states should stay localized around the classical trajectories. Informally, when $N \to \infty$, the quantities

$$K_k(\boldsymbol{x}, \boldsymbol{y}) := \langle C_N(\boldsymbol{x}), U_T^k C_N(\boldsymbol{y})\rangle \tag{2.38}$$

should behave as if $N|K_k(\boldsymbol{x}, \boldsymbol{y})|^2 \simeq \delta(T^k\boldsymbol{x} - \boldsymbol{y})$ (note that this hypothesis makes our quantization consistent with the notion of *regular quantization* described in Section V of [23]). The constraint $k \leq \alpha \log N$ is typical of hyperbolic classical behaviour and comes heuristically as follows. The maximal localization of coherent states cannot exceed the minimal coarse-graining dictated by $1/N$; if, while evolving, coherent states stayed localized forever around the classical trajectories, they would get more and more localized along the contracting direction. Since for hyperbolic systems the increase of localization is exponential with Lyapounov exponent $\log \lambda > 0$, this sets the upper bound and indicates that $\alpha \simeq 1/\log \lambda$.

Proposition 2.4.1

Let $(\mathcal{M}_N, \Theta_N, \tau_N)$ be a general quantum dynamical system as defined in Section 1.2 and suppose that it satisfies Condition 2.4.1. Let $\|X\|_2 := \sqrt{\tau_N(X^*X)}$, $X \in \mathcal{M}_N$ denote the normalized Hilbert-Schmidt norm. In the ensuing topology

$$\lim_{\substack{k,\ N\to\infty \\ k<\alpha \log N}} \|\Theta_N^k \circ \mathcal{J}_{N\infty}(f) - \mathcal{J}_{N\infty} \circ \Theta^k(f)\|_2 = 0. \tag{2.39}$$

Proof of Proposition 2.4.1:

One computes

$$\begin{aligned}
&\|\Theta_N^k \circ \mathcal{J}_{N\infty}(f) - \mathcal{J}_{N\infty} \circ \Theta^k(f)\|_2^2 \\
&= 2N \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\, \overline{f(\boldsymbol{x})}\, f(\boldsymbol{y})\, |\langle C_N(\boldsymbol{x}), C_N(\boldsymbol{y})\rangle|^2 \\
&\quad - 2N \operatorname{Re}\left[\int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\, \overline{f(\boldsymbol{y})}\, f(T^k\boldsymbol{x}) |\langle C_N(\boldsymbol{x}), U_T^k C_N(\boldsymbol{y})\rangle|^2\right].
\end{aligned} \tag{2.40}$$

The double integral in the first term goes to $\int \mu(\mathrm{d}\boldsymbol{x})|f(\boldsymbol{x})|^2$. So, we need to show that the second integral, which we shall denote by $I_N(k)$, does the same. We will concentrate on the case of continuous f, the extension to essentially bounded f is straightforward. Explicitly, selecting a ball $B(T^k\boldsymbol{x}, d_0)$, one derives

$$\begin{aligned}
&\left| I_N(k) - \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\, |f(\boldsymbol{y})|^2 \right| \\
&= \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int \mu(\mathrm{d}\boldsymbol{y})\, \overline{f(\boldsymbol{y})}\, \big(f(T^k\boldsymbol{x}) - f(\boldsymbol{y})\big)\, N|\langle C_N(\boldsymbol{x}), U_T^k C_N(\boldsymbol{y})\rangle|^2 \right| \\
&\leq \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{B(T^k\boldsymbol{x}, d_0)} \mu(\mathrm{d}\boldsymbol{y})\, \overline{f(\boldsymbol{y})} \big(f(T^k\boldsymbol{x}) - f(\boldsymbol{y})\big) N|\langle C_N(\boldsymbol{x}), U_T^k C_N(\boldsymbol{y})\rangle|^2 \right| \\
&+ \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{\mathcal{X}\setminus B(T^k\boldsymbol{x}, d_0)} \mu(\mathrm{d}\boldsymbol{y}) \overline{f(\boldsymbol{y})} \big(f(T^k\boldsymbol{x}) - f(\boldsymbol{y})\big) N|\langle C_N(\boldsymbol{x}), U_T^k C_N(\boldsymbol{y})\rangle|^2 \right| .
\end{aligned}$$

Applying the mean value theorem and approximating the integral of the kernel as in the proof of Proposition 1.5.1, we get that $\exists\, \boldsymbol{c} \in B(T^k\boldsymbol{x}, d_0)$ such that

$$\begin{aligned}
&\left| I_N(k) - \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\, |f(\boldsymbol{y})|^2 \right| \\
&\leq \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, \overline{f(\boldsymbol{c})}\, \big(f(T^k\boldsymbol{x}) - f(\boldsymbol{c})\big) \int_{B(T^k\boldsymbol{x}, d_0)} \mu(\mathrm{d}\boldsymbol{y})\, N|\langle C_N(\boldsymbol{x}), U_T^k C_N(\boldsymbol{y})\rangle|^2 \right| \\
&+ \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{\mathcal{X}\setminus B(T^k\boldsymbol{x}, d_0)} \mu(\mathrm{d}\boldsymbol{y}) \overline{f(\boldsymbol{y})} \big(f(T^k\boldsymbol{x}) - f(\boldsymbol{y})\big) N|\langle C_N(\boldsymbol{x}), U_T^k C_N(\boldsymbol{y})\rangle|^2 \right|
\end{aligned}$$

and using property (1.4.1.3)

$$\begin{aligned}
&\leq \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, \overline{f(\boldsymbol{c})}\, \big(f(T^k\boldsymbol{x}) - f(\boldsymbol{c})\big) \right| \\
&+ \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{\mathcal{X}\setminus B(T^k\boldsymbol{x}, d_0)} \mu(\mathrm{d}\boldsymbol{y}) \overline{f(\boldsymbol{y})} \big(f(T^k\boldsymbol{x}) - f(\boldsymbol{y})\big) N|\langle C_N(\boldsymbol{x}), U_T^k C_N(\boldsymbol{y})\rangle|^2 \right| .
\end{aligned}$$

By uniform continuity we can bound the first term by some arbitrary small ε, provided we choose d_0 small enough. Now, for the second integral we use our localization condition 2.4.1. As the constraint $k \leq \alpha \log N$ has to be enforced, we have to take a joint limit of time and size of the system with this constraint. In that case the second integral can also be bounded by an arbitrarily small ε', provided N is large enough. ■

2.4.2. Classical Limit for Quantum Cat Maps

We shall not prove the dynamical localization condition 2.4.1 for the quantum cat maps but instead provide a direct derivation of formula (2.39) based on the simple expression (2.20) of the dynamics when acting on Weyl operators. For this reason, we remind the reader the Weyl quantization operator $W_{N,\infty}$, already introduced in (1.27), together with some other useful tools.

Definitions 2.4.1

Let f be a function in $\mathcal{A}_{\mathbb{T}^2} \subset L^\infty_\mu\left(\mathbb{T}^2\right)$, $\mathcal{A}_{\mathbb{T}^2}$ denoting the subset of $L^\infty_\mu\left(\mathbb{T}^2\right)$ characterized by functions whose Fourier series $\hat{f}$ have only finitely many non-zero terms. We shall denote by $\mathrm{Supp}(\hat{f})$ the support of $\hat{f}$ in $\mathbb{Z}^2$. Then, in the Weyl quantization scheme, one associates to f the $N \times N$ matrix

$$W_{N,\infty} : \mathcal{A}_{\mathbb{T}^2} \ni f \longmapsto W_{N,\infty}(f) = \sum_{\boldsymbol{k} \in \mathrm{Supp}(\hat{f})} \hat{f}_{\boldsymbol{k}} \, W_N(\boldsymbol{k}) \in \mathcal{M}_N \ .$$

Our aim is to prove:

Proposition 2.4.2

Let $\left(\mathcal{M}_N, \tau_N, \Theta_N\right)$ be a sequence of quantum cat maps tending with $N \to \infty$ to a classical cat map with Lyapounov exponent $\log \lambda$; then

$$\lim_{\substack{k,\ N \to \infty \\ k < \log N/(2\log\lambda)}} \|\Theta_N^k \circ \mathcal{J}_{N\infty}(f) - \mathcal{J}_{N\infty} \circ \Theta^k(f)\|_2 = 0 \quad ,$$

where $\|\cdot\|_2$ is the Hilbert-Schmidt norm of Proposition (2.4.1)

First we prove an auxiliary result.

Lemma 2.4.1

If $\boldsymbol{n} = (n_1, n_2) \in \mathbb{Z}^2$ is such that $0 \leq n_i \leq N-1$ and $\lim_N \frac{n_i}{\sqrt{N-1}} = 0$, then the expectation of Weyl operators $W_N(\boldsymbol{n})$ with respect to the state $|C_N\rangle$ given in (1.37) is such that

$$\lim_{N\to\infty} \langle C_N, W_N(\boldsymbol{n})\, C_N\rangle = 1 \cdot$$

Proof of Lemma 2.4.1:

The idea of the proof is to use the fact that, for large N, the binomial coefficients $\binom{N-1}{j}$ contribute to the binomial sum only when j stays within a neighborhood of $(N-1)/2$ of width $\simeq \sqrt{N}$, in which case they can be approximated by a normalized Gaussian function. We also notice that, by expanding the exponents in the bounds (1.48) and (1.49), the exponential decay fails only if $n_{1,2}$ grow with N slower than $\sqrt{N}$, which is surely the case for fixed finite n, whereby it also follows that we can disregard the second term in the sum comprising the contributions (1.39). We then write the j's in the binomial coefficients as

$$j = \left\lfloor \frac{N-1}{2} \right\rfloor + k = \frac{N-1}{2} + k - \alpha, \quad \alpha \in \left\{0, \frac{1}{2}\right\},$$

and consider only $k = \mathrm{O}(\sqrt{N})$. Stirling's formula [45]

$$L! = L^{L+1/2}\,\mathrm{e}^{-L}\,\sqrt{2\pi}\,\Big(1 + \mathrm{O}(L^{-1})\Big),$$

allows us to rewrite the first term in the r.h.s. of (1.39) as

$$\exp\left(-\frac{1}{2}\frac{n_1^2}{N-1}\right) \sum_{k=-\left[\frac{N-1}{2}\right]}^{N-1-\left[\frac{N-1}{2}\right]+n_1} \frac{2\,\mathrm{e}^{\frac{2\pi i}{N} n_2\left(k+\left[\frac{N-1}{2}\right]\right)}}{\sqrt{2\pi(N-1)}}$$
$$\times \exp\left(-\frac{2(k-\alpha+\frac{n_1}{2})^2}{N-1}\right)\Big(1 + \mathrm{O}(N^{-1}) + \mathrm{O}((k+n_1)^3\,N^{-2})\Big). \tag{2.41}$$

For any fixed, finite $\boldsymbol{n}$, both the sum and the factor in front tend to 1, the sum becoming the integral of a normalized Gaussian. ■

Proof of Proposition 2.4.2:

Given $f \in L^\infty_\mu\left(\mathbb{T}^2\right)$ and $\varepsilon > 0$, we choose N_0 such that the Fourier approximation f_ε of f with $\#(\mathrm{Supp}(\hat{f})) = N_0$ is such that $\|f - f_\varepsilon\| \leq \varepsilon$, where $\|\cdot\|$ denotes the usual Hilbert space norm. Next, we estimate

$$\begin{aligned} I_N(f) &:= \left\|\Theta_N^k \circ \mathcal{J}_{N\infty}(f) - \mathcal{J}_{\infty N} \circ \Theta^k(f)\right\|_2 \\ &\le \left\|\Theta_N^k \circ \mathcal{J}_{N\infty}(f - f_\varepsilon)\right\|_2 + \left\|\mathcal{J}_{N\infty} \circ \Theta^k(f - f_\varepsilon)\right\|_2 \\ &\quad + \left\|\Theta_N^k \circ \mathcal{J}_{N\infty}(f_\varepsilon) - \mathcal{J}_{N\infty} \circ \Theta^k(f_\varepsilon)\right\|_2 \\ &\le 2\|f - f_\varepsilon\| + I_N(f_\varepsilon). \end{aligned}$$

This follows from Θ_N-invariance of the norm $\|\cdot\|_2$, from T-invariance of the measure μ and from the fact that the positivity inequality for unital completely positive maps such as $\mathcal{J}_{N\infty}$ gives:

$$\begin{aligned} \left\|\mathcal{J}_{N\infty}(g)\right\|_2^2 &= \tau_N\left(\mathcal{J}_{N\infty}(g)^* \mathcal{J}_{N\infty}(g)\right) \le \tau_N\Big(\mathcal{J}_{N\infty}(|g|^2)\Big) \\ &= \int_{\mathbb{T}^2} d\boldsymbol{x}\, |g|^2(\boldsymbol{x}) = \|g\|^2 \quad . \end{aligned}$$

We now use that f_ε is a function with finitely supported Fourier transform and, inserting the Weyl quantization of f_ε, we estimate

$$I_N(f_\varepsilon) \le \left\|\mathcal{J}_{N\infty}(f_\varepsilon) - W_{N,\infty}(f_\varepsilon)\right\|_2 + \left\|\mathcal{J}_{N\infty} \circ \Theta^k(f_\varepsilon) - \Theta_N^k(W_{N,\infty}(f_\varepsilon))\right\|_2. \tag{2.42}$$

Then, we concentrate on the square of the second term, which we denote by $G_{N,k}(f_\varepsilon)$ and explicitly reads

$$\begin{aligned} G_{N,k}(f_\varepsilon) &= \tau_N\left(\mathcal{J}_{N\infty} \circ \Theta^k(f_\varepsilon)^* \mathcal{J}_{N\infty} \circ \Theta^k(f_\varepsilon)\right) + \tau_N\left(W_{N,\infty}(f_\varepsilon)^* W_{N,\infty}(f_\varepsilon)\right) \\ &\quad - 2\,\mathrm{Re}\,\Big(\tau_N\left(\mathcal{J}_{N\infty} \circ \Theta^k(f_\varepsilon)^* \Theta_N^k(W_{N,\infty}(f_\varepsilon))\right)\Big). \end{aligned} \tag{2.43}$$

The first term tends to $\|f_\varepsilon\|^2$ as $N \to \infty$, because of Proposition 1.5.2 and the same is true of the second term; indeed,

$$\tau_N\big(W_{N,\infty}(f_\varepsilon)^* W_{N,\infty}(f_\varepsilon)\big) = \sum_{\boldsymbol{k},\boldsymbol{q} \in \mathrm{Supp}(\hat{f}_\varepsilon)} \overline{\hat{f}_{\varepsilon\,\boldsymbol{k}}}\, \hat{f}_{\varepsilon\,\boldsymbol{q}}\, \mathrm{e}^{\frac{i\pi}{N}\sigma(\boldsymbol{q},\boldsymbol{k})}\, \tau_N\big(W_N(\boldsymbol{q} - \boldsymbol{k})\big).$$

Now, since $\mathrm{Supp}(\hat{f}_\varepsilon)$ is finite, the vector $\boldsymbol{k} - \boldsymbol{q}$ is uniformly bounded with respect to N. Therefore, with N large enough, (1.20) forces $\boldsymbol{k} = \boldsymbol{q}$, whence the claim. It remains to show that the same for the third term in (2.43) which amounts to twice the real part of

$$\int_{\mathbb{T}^2} \mathrm{d}\boldsymbol{x}\, \overline{f_\varepsilon(T^{-k}\boldsymbol{x})} \langle C_N(\boldsymbol{x}), \Theta_N^k(W_{N,\infty}(f_\varepsilon))C_N(\boldsymbol{x})\rangle$$
$$= \sum_{\boldsymbol{p}\in \mathrm{Supp}(\hat{f}_\varepsilon)} \overline{\hat{f}_{\varepsilon\,\boldsymbol{q}}} \langle C_N, W_N(T^k\boldsymbol{p})C_N\rangle \int_{\mathbb{T}^2} \mathrm{d}\boldsymbol{x} \overline{f_\varepsilon(T^{-k}\boldsymbol{x})} \exp\Big(\frac{2\pi i}{N}\sigma(T^k\boldsymbol{p}, [N\boldsymbol{x}])\Big).$$

According to Lemma 2.4.1, the matrix element $\langle C_N, W_N(T^k\boldsymbol{p})C_N\rangle$ tends to 1 as $N\to\infty$ whenever the vectorial components $(T^k\boldsymbol{p})_j$, $j=1,2$, satisfy

$$\lim_N \frac{(T^k\boldsymbol{p})_j^2}{N} = C_{\boldsymbol{u}}(\boldsymbol{p})(\boldsymbol{u})_j \lim_N \frac{\lambda^{2k}}{N} = 0,$$

where we expanded $\boldsymbol{p} = C_{\boldsymbol{u}}(\boldsymbol{p})\boldsymbol{u} + C_{\boldsymbol{v}}(\boldsymbol{p})\boldsymbol{v}$ along the stretching and squeezing eigendirections of T (see Remark 2.1.1.v). This fact sets the logarithmic time scale $k < \frac{1}{2}\frac{\log N}{\log\lambda}$. Notice that, when $k=0$, $G_{N,k}(f_\varepsilon)$ equals the first term in (2.42) and this concludes the proof. ∎

Remark 2.4.2

The previous result essentially points to the fact that the time evolution and the classical limit do commute over time scales that are logarithmic in the semi–classical parameter N. The upper bound, which goes like const. $\times\ \frac{\log N}{\log\lambda}$, is typical of quantum chaos and is known as logarithmic *breaking–time*. Such a scaling appear numerically in Section 4.2.1.2 also for discrete classical cat maps, converging in a suitable classical limit to continuous cat maps.

2.4.3. Continuous limit for Sawtooth Maps

In Section 2.3.1 we provided a discretization procedure for algebraic classical dynamical systems by constructing two discretization/dediscretization operators $\mathcal{J}_{N,\infty} : \mathcal{C}^0(\mathcal{X}) \mapsto \mathcal{D}_N$ and $\mathcal{J}_{\infty,N} : \mathcal{D}_N \mapsto \mathcal{C}^0(\mathcal{X})$ such that $\widetilde{\Theta}^j_{N,\alpha} \circ \mathcal{J}_{N,\infty} = \mathcal{J}_{N,\infty}\circ\Theta^j_\alpha$, where $\widetilde{\Theta}^j_{N,\alpha}$, Θ^j_α are the quantized, respectively classical dynamical maps at timestep j.

However, the pictures drastically changes when we pass from T_α to S_α: then we are forced to enlarge the algebra from $\mathcal{C}^0(\mathcal{X})$ to $L^\infty_\mu(\mathbb{T}^2)$, to define a different discretization scheme and a new *automorphism $\widetilde{\Theta}_{N,\alpha}$ on $\mathcal{D}_N$.

The origin of the inequality $\widetilde{\Theta}^j_\alpha \circ \mathcal{J}_{N,\infty} \neq \mathcal{J}_{N,\infty}\circ\Theta^j_\alpha$ (when $\alpha\in\mathbb{R}\setminus\mathbb{Z}$) lies in the discontinuous character of the fractional part that appears in (2.1b). Nevertheless, the equality is obtained when $N\longmapsto\infty$; more precisely, in Proposition 2.4.4, we shall rigorously prove that $\widetilde{\Theta}^j_\alpha \circ \mathcal{J}_{N,\infty}(f) \longmapsto \mathcal{J}_{N,\infty} \circ \Theta^j_\alpha(f)$, for all $L^\infty_\mu(\mathbb{T}^2)$ with respect to the topology on $\mathcal{D}_N$ determined by the state τ_N through the Hilbert-Schmidt norm $\|\cdot\|_2$ defined in Proposition 2.4.1.

The above choice of topology is dictated by the fact that in the continuous limit the set of discontinuities of the j–power of Sawtooth Maps are a subset of zero measure. For later purposes, we now give a brief review of the discontinuities of the maps S_α [39, 40, 46].

As already noted in Remark 2.1.1, S_α is discontinuous on the circle γ_0; therefore S^n_α will be discontinuous on the preimages

$$\gamma_m := S^{-m}_\alpha(\gamma_0) \quad \text{for} \quad 0 \leqslant m < n \ , \tag{2.44}$$

while the discontinuities of S^{-n}_α lie on the sets

$$\gamma_{-m} := S^m_\alpha(\gamma_0) \quad \text{for} \quad 0 < m \leqslant n \ \cdot$$

Apart from γ_{-1}, which is a closed curve on the torus intersecting γ_0 transversally, each set of the type γ_m (for γ_{-m} the argument is similar) is the (disjoint) union of segments parallel to each other whose endpoints lie either on the same segment belonging to γ_p, $p < m$, or on two different segments belonging to γ_p and $\gamma_{p'}$, with $p' \leqslant p < m$ [40].It proves convenient to introduce the *discontinuity set* of S^n_α,

$$\Gamma_n := \bigcup_{p=0}^{n-1} \gamma_p \ , \tag{2.45}$$

which is a one dimensional sub–manifold of $\mathcal{X} = \mathbb{T}^2$ and its complementary set, $G_n := \mathbb{T}^2 \setminus \Gamma_n$.

We now enlarge the above definition from continuous Sawtooth Maps, to discretized ones.

Definitions 2.4.2

Given $\boldsymbol{x} \in \mathbb{T}^2$, we shall denote by $\hat{\boldsymbol{x}}_N$ the element of $(\mathbb{Z}/N\mathbb{Z})^2$ given by:

$$\hat{\boldsymbol{x}}_N := \left(\lfloor N x_1 + \tfrac{1}{2} \rfloor \ , \ \lfloor N x_2 + \tfrac{1}{2} \rfloor \right) \ , \tag{2.46}$$

by segment (A, B), the shortest curve joining $A, B \in \mathbb{T}^2$, by $l(\gamma_p)$ the length of the curve γ_p and by[6]

$$\overline{\gamma}_p(\varepsilon) := \left\{ \boldsymbol{x} \in \mathbb{T}^2 \;\middle|\; d_{\mathbb{T}^2}(\boldsymbol{x}, \gamma_p) \leqslant \varepsilon \right\} \tag{2.47}$$

the strip around γ_p of width ε.

Further, we shall denote by

$$\overline{\Gamma}_n(\varepsilon) := \bigcup_{p=0}^{n-1} \overline{\gamma}_p(\varepsilon) \tag{2.48}$$

the union of the strips up to $p = n - 1$ and by $G_n^N(\varepsilon)$ the subset of points

$$G_n^N(\varepsilon) := \left\{ \boldsymbol{x} \in \mathbb{T}^2 \;\middle|\; \frac{\hat{\boldsymbol{x}}_N}{N} \notin \overline{\Gamma}_n(\varepsilon) \right\} . \tag{2.49}$$

To prove that the discretized Sawtooth maps tend to Sawtooth maps in the continuum when $N \longrightarrow \infty$, the main problem is to control the discontinuities. In order to do that, we shall subdivide the lattice points in a good and a bad set and prove that the images of points in the good set, under the evolution U_α^q, V_α^q remain close to each other. This is not true for the bad set, however we shall show that it tends with N to a set of zero Lebesgue measure and thus becomes ineffective.

To concretely implement the above strategy we need the next two Lemmas whose proofs are given in Appendix B.

Lemma 2.4.2

Using the notation of Definition 2.4.2, we have:

1) Given the matrix $S_\alpha = \begin{pmatrix} 1+\alpha & 1 \\ \alpha & 1 \end{pmatrix}$, $\alpha \in \mathbb{R}$, and its inverse $S_\alpha^{-1} = \begin{pmatrix} 1 & -1 \\ -\alpha & 1+\alpha \end{pmatrix}$, there exists a real number $\eta > 1$ depending only on α such that, for any $A,\ B \in \mathbb{R}^2$, we have, with $\boldsymbol{v} = A - B$,

$$\left\| S_\alpha^{\pm 1} \cdot \boldsymbol{v} \right\|_{\mathbb{R}^2} \leqslant \eta \quad \|\boldsymbol{v}\|_{\mathbb{R}^2} , \tag{2.50a}$$
$$\left\| S_\alpha^{\pm 1} \cdot \boldsymbol{v} \right\|_{\mathbb{R}^2} \geqslant \eta^{-1} \|\boldsymbol{v}\|_{\mathbb{R}^2} , \tag{2.50b}$$

where $S_\alpha^{\pm 1} \cdot \boldsymbol{v}$ denotes the matricial action of $S_\alpha^{\pm 1}$ on $\boldsymbol{v}$.

2) Let $A, B \in \mathbb{T}^2$ and $d_{\mathbb{T}^2}(A, B) < \frac{1}{2}\, \eta^{-1}$:

[6] The distance $d_{\mathbb{T}^2}(\cdot, \cdot)$ on the torus has been introduced in Definition 1.4.2.

2a) If the segment (A, B) does not cross γ_{-1}, then

$$d_{\mathbb{T}^2}\left(S_\alpha^{-1}(A), S_\alpha^{-1}(B)\right) \leqslant \eta\, d_{\mathbb{T}^2}(A, B) \ \cdot \tag{2.51a}$$

2b) If the (A, B) does not cross γ_0 , then

$$d_{\mathbb{T}^2}\left(S_\alpha \ (A), S_\alpha \ (B)\right) \leqslant \eta\, d_{\mathbb{T}^2}(A, B) \ \cdot \tag{2.51b}$$

3) For any given $\alpha \in \mathbb{R}$, $p \in \mathbb{N}^+$ and $0 \leqslant \varepsilon \leqslant \frac{1}{2}\eta^{-1}$,

$$\boldsymbol{x} \in \overline{\gamma}_{p-1}(\varepsilon) \Longrightarrow S_\alpha^{-1}(\boldsymbol{x}) \in \left(\overline{\gamma}_p(\eta\,\varepsilon) \cup \overline{\gamma}_0(\eta\,\varepsilon)\right) \ \cdot \tag{2.52}$$

4) For any given $\alpha \in \mathbb{R}$, $n \in \mathbb{N}^+$ and $0 \leqslant \varepsilon \leqslant \frac{1}{2}$, with U_α^q as in (2.27),

$$\boldsymbol{x} \notin \overline{\Gamma}_n(\varepsilon) \Longrightarrow d_{\mathbb{T}^2}\left(\frac{U_\alpha^q(N\boldsymbol{x})}{N}, \gamma_0\right) > \varepsilon\,\eta^{-q} \ , \ \forall\ 0 \leqslant q < n \ \cdot \tag{2.53}$$

5) For any given $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}^+$, if

$$N > \widetilde{N} + 3 = 2\sqrt{2}\,n\,\eta^{2n} + 3 \quad \text{and} \quad \boldsymbol{x} \in G_n^N\left(\frac{\widetilde{N}}{2N}\right) \text{ then}$$

$$d_{\mathbb{T}^2}\left(\frac{U_\alpha^p(N\boldsymbol{x})}{N}, \frac{V_\alpha^p(\hat{\boldsymbol{x}}_N)}{N}\right) \leqslant \frac{\sqrt{2}}{N}\left(\frac{\eta^{p+1}-1}{\eta-1}\right) \quad , \quad \forall p \leqslant n \ \cdot \tag{2.54}$$

Lemma 2.4.3

With the notation of Definition 2.4.2, the following relations hold for all $p \in \mathbb{N}$, $n \in \mathbb{N}^+$ and $\varepsilon \in \mathbb{R}^+$:

$$l(\gamma_p) \leqslant \eta^p \ , \tag{2.55a}$$

$$\mu\left(\overline{\gamma}_p(\varepsilon)\right) \leqslant 2\,\varepsilon\,\eta^p + \pi\varepsilon^2 \ , \tag{2.55b}$$

$$\mu\left(\overline{\Gamma}_n(\varepsilon)\right) \leqslant \varepsilon\,n\,(2\,\eta^n + \pi\,\varepsilon) \ \cdot \tag{2.55c}$$

Denoting with $[E]^\circ$ the complementary set of E on the torus, namely $[E]^\circ := \mathbb{T}^2 \setminus E$, it holds

$$\left[\overline{\Gamma}_n\left(\varepsilon + \frac{1}{\sqrt{2}N}\right)\right]^\circ \subseteq G_n^N(\varepsilon) \ \cdot \tag{2.55d}$$

Moreover, if $N \in \mathbb{N}^+$ and $\widetilde{N} = 2\sqrt{2}\, n\, \eta^{2n}$ (cfr. Lemma 2.4.2.5), we have:

$$N > \widetilde{N} + 3 \Longrightarrow \mu\left(\left[G_n^N\left(\frac{\widetilde{N}}{2N}\right)\right]^\circ\right) \leqslant \frac{12\sqrt{2}\, n^2\, \eta^{3n}}{N} \,. \tag{2.55e}$$

By using Lemma 2.4.2, we prove now a dynamical localization Property, not too far from the one given in page 41, which involves the unitary single step evolution operator $U'_{\alpha,N}$ defined in (2.37).

Proposition 2.4.3 (Dynamical localization on $\{|C_N^3(\boldsymbol{x})\rangle\}$ states)

For any given $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, for all $d_0 > 0$ there exists an $N_0 \in \mathbb{N}$ with the following property: if $N > N_0$ and $\boldsymbol{x} \in G_n^N\left(\frac{\widetilde{N}}{2N}\right)$, then $\left\langle C_N^3(\boldsymbol{x}) \,\middle|\, U'^{\,n}_{\alpha,N}\, C_N^3(\boldsymbol{y})\right\rangle = 0$ whenever $d_{\mathbb{T}^2}\left(S_\alpha^n\left(\boldsymbol{x}\right), \boldsymbol{y}\right) \geqslant d_0$.

Proof of Proposition 2.4.3 :

Using the definitions of states $\{|C_N^3(\boldsymbol{x})\rangle\}$ given in (1.52), together with the unitary evolution operator $U'_{\alpha,N}$ defined in (2.37), $\left\langle C_N^3(\boldsymbol{x}) \,\middle|\, U'^{\,n}_{\alpha,N}\, C_N^3(\boldsymbol{y})\right\rangle$ can be easily computed, as follows:

$$\left\langle C_N^3(\boldsymbol{x}) \,\middle|\, U'^{\,n}_{\alpha,N}\, C_N^3(\boldsymbol{y})\right\rangle = \left\langle \hat{\boldsymbol{x}}_N \,\middle|\, V_\alpha^{-n}\left(\hat{\boldsymbol{y}}_N\right) \right\rangle = \delta^{(N)}_{V_\alpha^n(\hat{\boldsymbol{x}}_N)\,,\,\hat{\boldsymbol{y}}_N} \,. \tag{2.56}$$

Using the triangular inequality, we get:

$$d_{\mathbb{T}^2}\left(\frac{U_\alpha^n\left(N\boldsymbol{x}\right)}{N}\,,\,\boldsymbol{y}\right) \leqslant d_{\mathbb{T}^2}\left(\frac{U_\alpha^n\left(N\boldsymbol{x}\right)}{N}\,,\,\frac{V_\alpha^n\left(\hat{\boldsymbol{x}}_N\right)}{N}\right) + \\ + d_{\mathbb{T}^2}\left(\frac{V_\alpha^n\left(\hat{\boldsymbol{x}}_N\right)}{N}\,,\,\frac{\hat{\boldsymbol{y}}_N}{N}\right) + d_{\mathbb{T}^2}\left(\frac{\hat{\boldsymbol{y}}_N}{N}\,,\,\boldsymbol{y}\right) \tag{2.57}$$

or equivalently,

$$d_{\mathbb{T}^2}\left(\frac{V_\alpha^n\left(\hat{\boldsymbol{x}}_N\right)}{N}\,,\,\frac{\hat{\boldsymbol{y}}_N}{N}\right) \geqslant d_{\mathbb{T}^2}\left(S_\alpha^n\left(\boldsymbol{x}\right)\,,\,\boldsymbol{y}\right) - \\ d_{\mathbb{T}^2}\left(\frac{U_\alpha^n\left(N\boldsymbol{x}\right)}{N}\,,\,\frac{V_\alpha^n\left(\hat{\boldsymbol{x}}_N\right)}{N}\right) - d_{\mathbb{T}^2}\left(\frac{\hat{\boldsymbol{y}}_N}{N}\,,\,\boldsymbol{y}\right) \tag{2.58}$$

Now we will use these observations:

- From (B.37) in Appendix B we have

$$d_{\mathbb{T}^2}\left(\boldsymbol{y}\,,\,\frac{\hat{\boldsymbol{y}}_N}{N}\right) \leqslant \frac{1}{\sqrt{2}N} \quad , \quad \forall\, \boldsymbol{y} \in \mathbb{T}^2 \,; \tag{2.59}$$

- The fact that $\boldsymbol{x} \in G_n^N\left(\frac{\widetilde{N}}{2N}\right)$ permits us to use point 5 of Lemma 2.4.2, that is

$$N > \widetilde{N} = 2\sqrt{2}\, n\, \eta^{2n} \implies d_{\mathbb{T}^2}\left(\frac{U_\alpha^n\,(N\boldsymbol{x})}{N}, \frac{V_\alpha^n\,(\hat{\boldsymbol{x}}_N)}{N}\right) \leqslant \frac{\sqrt{2}}{N}\left(\frac{\eta^{n+1}-1}{\eta-1}\right) \; ; \quad (2.60)$$

- $d_{\mathbb{T}^2}\left(S_\alpha^n\,(\boldsymbol{x})\;,\, \boldsymbol{y}\right) \geqslant d_0$ by hypothesis.

Then we can find a $N_0 > \tilde{N} + 3$ such that

$$N > N_0 \quad \implies \quad d_{\mathbb{T}^2}\left(\frac{V_\alpha^n\,(\hat{\boldsymbol{x}}_N)}{N}\,,\, \frac{\hat{\boldsymbol{y}}_N}{N}\right) > \frac{1}{N} \quad (2.61)$$

Indeed, inserting (2.59–2.60) in equation (2.58), we get

$$d_{\mathbb{T}^2}\left(\frac{V_\alpha^n\,(\hat{\boldsymbol{x}}_N)}{N}\,,\, \frac{\hat{\boldsymbol{y}}_N}{N}\right) \geqslant d_0 - \frac{\sqrt{2}}{N}\left(\frac{\eta^{n+1}-1}{\eta-1}\right) - \frac{1}{\sqrt{2}N}\;; \quad (2.62)$$

therefore, we can choose N_0 such that for $N > N_0 > \widetilde{N} + 3$ the right hand side of the above inequality is larger than $\frac{1}{N}$. Explicitly, it is sufficient to choose:

$$N_0 = \max\left\{\frac{1}{d_0}\left[1 + \sqrt{2}\left(\frac{\eta^{n+1}-1}{\eta-1}\right) + \frac{1}{\sqrt{2}}\right], \widetilde{N} + 3\right\}\;. \quad (2.63)$$

Combining (2.56) and (2.61) the proof is completed, by noting that if the toral distance of two grid points exceeds $\frac{1}{N}$, then the distance between the components of the integer vector labeling the two points is different from zero and then the periodic Kronecker delta in (2.56) vanishes. ■

We now use the previous two Lemmas to prove the main result of this section.

Proposition 2.4.4

Let $\left(\mathcal{D}_N, \tau_N, \widetilde{\Theta}_{N,\alpha}\right)$ be a quantum dynamical system as defined in Section 2.3.2 and suppose that it satisfies Condition 2.4.3. For any fixed integer k, in the topology given by the Hilbert-Schmidt norm of Proposition (2.4.1), we have

$$\lim_{N\to\infty} \|\widetilde{\Theta}_{N,\alpha}^k \circ \mathcal{J}_{N\infty}(f) - \mathcal{J}_{N\infty} \circ \Theta_\alpha^k(f)\|_2 = 0. \quad (2.64)$$

Proof of Proposition 2.4.4:

In this proof we will parallel the same strategy used in the proof of Proposition 2.4.1. As done there, we will consider the case of f continuous, being the extension to f–essentially bounded just an application of Lusin's (Theorem 2, Corollary 1, page 21). Then we have to show that

$$I_N(k) := N \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\, \overline{f(\boldsymbol{y})}\, f(S_\alpha^k \boldsymbol{x}) |\langle C_N^3(\boldsymbol{x}), U'^{\,k}_{\alpha,N} C_N^3(\boldsymbol{y})\rangle|^2 \tag{2.65}$$

goes to $\int \mu(\mathrm{d}\boldsymbol{z})|f(\boldsymbol{z})|^2$. Resorting to $G_n^N\left(\frac{\widetilde{N}}{2N}\right)$ in Definition 2.4.2, and to its complementary set $\left[G_n^N\left(\frac{\widetilde{N}}{2N}\right)\right]^\circ = \mathcal{X} \setminus G_n^N\left(\frac{\widetilde{N}}{2N}\right)$, we can write

$$\begin{aligned}
&\left| I_N(k) - \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\, |f(\boldsymbol{y})|^2 \right| \\
&= \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{y})\, \overline{f(\boldsymbol{y})} \left(f(S_\alpha^k \boldsymbol{x}) - f(\boldsymbol{y})\right) N |\langle C_N^3(\boldsymbol{x}), U'^{\,k}_{\alpha,N} C_N^3(\boldsymbol{y})\rangle|^2 \right| \\
&\leq \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{\left[G_n^N\left(\frac{\widetilde{N}}{2N}\right)\right]^\circ} \mu(\mathrm{d}\boldsymbol{y})\, \overline{f(\boldsymbol{y})} \left(f(S_\alpha^k \boldsymbol{x}) - f(\boldsymbol{y})\right) N |\langle C_N^3(\boldsymbol{x}), U'^{\,k}_{\alpha,N} C_N^3(\boldsymbol{y})\rangle|^2 \right| \\
&+ \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{G_n^N\left(\frac{\widetilde{N}}{2N}\right)} \mu(\mathrm{d}\boldsymbol{y}) \overline{f(\boldsymbol{y})} \left(f(S_\alpha^k \boldsymbol{x}) - f(\boldsymbol{y})\right) N |\langle C_N^3(\boldsymbol{x}), U'^{\,k}_{\alpha,N} C_N^3(\boldsymbol{y})\rangle|^2 \right|.
\end{aligned} \tag{2.66}$$

For the first integral in the r.h.s. of the previous expression we have:

$$\begin{aligned}
&\left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{\left[G_n^N\left(\frac{\widetilde{N}}{2N}\right)\right]^\circ} \mu(\mathrm{d}\boldsymbol{y})\, \overline{f(\boldsymbol{y})} \left(f(S_\alpha^k \boldsymbol{x}) - f(\boldsymbol{y})\right) N |\langle C_N^3(\boldsymbol{x}), U'^{\,k}_{\alpha,N} C_N^3(\boldsymbol{y})\rangle|^2 \right| \\
&\leq 2(\|f\|_\infty)^2 \left| \int_{\left[G_n^N\left(\frac{\widetilde{N}}{2N}\right)\right]^\circ} \mu(\mathrm{d}\boldsymbol{y}) \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, N |\langle C_N^3(\boldsymbol{x}), U'^{\,k}_{\alpha,N} C_N^3(\boldsymbol{y})\rangle|^2 \right|
\end{aligned}$$

(note that exchange of integration order is harmless because of the existence of the integral)

$$\leq 2(\|f\|_\infty)^2 \mu\left(\left[G_n^N\left(\frac{\widetilde{N}}{2N}\right)\right]^\circ\right) \leqslant \frac{24\sqrt{2}\, n^2\, \eta^{3n}}{N} (\|f\|_\infty)^2$$

where we have used Properties (2) and (3) of Definition 1.4.1 and equation (2.55e) from Lemma 2.4.3; this term becomes negligible for large $N > \widetilde{N}$. Now it remains to prove that the second term in (2.66) is also negligible for large N: selecting a ball $B(S_\alpha^k \boldsymbol{x}, d_0)$, one derives

$$\left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{G_n^N\left(\frac{\tilde{N}}{2N}\right)} \mu(\mathrm{d}\boldsymbol{y})\, \overline{f(\boldsymbol{y})}\, \big(f(S_\alpha^k \boldsymbol{x}) - f(\boldsymbol{y})\big)\, N |\langle C_N^3(\boldsymbol{x}), U'^k_{\alpha,N} C_N^3(\boldsymbol{y})\rangle|^2 \right|$$
$$\leq \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{G_n^N\left(\frac{\tilde{N}}{2N}\right) \cap B(S_\alpha^k \boldsymbol{x}, d_0)} \mu(\mathrm{d}\boldsymbol{y})\, \overline{f(\boldsymbol{y})} \big(f(S_\alpha^k \boldsymbol{x}) - f(\boldsymbol{y})\big) N |\langle C_N^3(\boldsymbol{x}), U'^k_{\alpha,N} C_N^3(\boldsymbol{y})\rangle|^2 \right|$$
$$+ \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{G_n^N\left(\frac{\tilde{N}}{2N}\right) \cap \left(\mathcal{X} \setminus B(S_\alpha^k \boldsymbol{x}, d_0)\right)} \mu(\mathrm{d}\boldsymbol{y}) \overline{f(\boldsymbol{y})} \big(f(S_\alpha^k \boldsymbol{x}) - f(\boldsymbol{y})\big) N |\langle C_N^3(\boldsymbol{x}), U'^k_{\alpha,N} C_N^3(\boldsymbol{y})\rangle|^2 \right| .$$

Applying the mean value theorem in the first double integral and approximating the integral of the kernel as in the proof of Proposition 1.5.1, we get that $\exists\, \boldsymbol{c} \in B(S_\alpha^k \boldsymbol{x}, d_0)$ such that

$$\left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{G_n^N\left(\frac{\tilde{N}}{2N}\right)} \mu(\mathrm{d}\boldsymbol{y})\, \overline{f(\boldsymbol{y})}\, \big(f(S_\alpha^k \boldsymbol{x}) - f(\boldsymbol{y})\big)\, N |\langle C_N^3(\boldsymbol{x}), U'^k_{\alpha,N} C_N^3(\boldsymbol{y})\rangle|^2 \right|$$
$$\leq \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, \overline{f(\boldsymbol{c})}\, \big(f(S_\alpha^k \boldsymbol{x}) - f(\boldsymbol{c})\big) \int_{G_n^N\left(\frac{\tilde{N}}{2N}\right) \cap B(S_\alpha^k \boldsymbol{x}, d_0)} \mu(\mathrm{d}\boldsymbol{y})\, N |\langle C_N^3(\boldsymbol{x}), U'^k_{\alpha,N} C_N^3(\boldsymbol{y})\rangle|^2 \right|$$
$$+ \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{G_n^N\left(\frac{\tilde{N}}{2N}\right) \cap \left(\mathcal{X} \setminus B(S_\alpha^k \boldsymbol{x}, d_0)\right)} \mu(\mathrm{d}\boldsymbol{y}) \overline{f(\boldsymbol{y})} \big(f(S_\alpha^k \boldsymbol{x}) - f(\boldsymbol{y})\big) N |\langle C_N^3(\boldsymbol{x}), U'^k_{\alpha,N} C_N^3(\boldsymbol{y})\rangle|^2 \right|$$

and using property (1.4.1.3)

$$\leq \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, \overline{f(\boldsymbol{c})}\, \big(f(S_\alpha^k \boldsymbol{x}) - f(\boldsymbol{c})\big) \right|$$
$$+ \left| \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \int_{G_n^N\left(\frac{\tilde{N}}{2N}\right) \cap \left(\mathcal{X} \setminus B(S_\alpha^k \boldsymbol{x}, d_0)\right)} \mu(\mathrm{d}\boldsymbol{y}) \overline{f(\boldsymbol{y})} \big(f(S_\alpha^k \boldsymbol{x}) - f(\boldsymbol{y})\big) N |\langle C_N^3(\boldsymbol{x}), U'^k_{\alpha,N} C_N^3(\boldsymbol{y})\rangle|^2 \right| .$$

By uniform continuity we can bound the first term by some arbitrary small ε, provided we choose d_0 small enough. For the second integral, we use the localization condition 2.4.3 that allow us to find $N_0 = N_0(d_0, k)$ depending on the d_0 just chosen to bound the first term and on the given fixed timestep k, such that the second integral vanishes. ∎

Chapter 3

Entropies

3.1. Classical Dynamical Entropy

Intuitively, one expects the instability proper to the presence of a positive Lyapounov exponent to correspond to some degree of unpredictability of the dynamics: classically, the metric entropy of Kolmogorov provides the link [8].

3.1.1. Kolmogorov Metric Entropy

For continuous classical systems $\big(\mathcal{X}, \mu, T\big)$ such as those introduced in Section 1.1.1, the construction of the dynamical entropy of Kolmogorov is based on subdividing $\mathcal{X}$ into measurable disjoint subsets $\{E_\ell\}_{\ell=1,2,\cdots,D}$ such that $\bigcup_\ell E_\ell = \mathcal{X}$ which form finite partitions (coarse graining) $\mathcal{E}$.

Under the a dynamical maps $T : \mathcal{X} \rightarrow \mathcal{X}$, any given $\mathcal{E}$ evolves into $T^j(\mathcal{E})$ with atoms $T^{-j}(E_\ell) = \{x \in \mathcal{X} : T^j x \in E_\ell\}$; one can then form finer partitions

$$\mathcal{E}_{[0,n-1]} := \bigvee_{j=0}^{n-1} T^j(\mathcal{E}) \qquad = \mathcal{E} \ \bigvee \ T(\mathcal{E}) \ \bigvee \cdots \bigvee \ T^{n-1}(\mathcal{E})$$

whose atoms

$$E_{i_0\,i_1\cdots i_{n-1}} := \bigcap_{j=0}^{n-1} T^{-j} E_{i_j} \qquad = E_{i_0} \bigcap T^{-1}(E_{i_1}) \bigcap \cdots \bigcap T^{-n+1}(E_{i_{n-1}})$$

have volumes

$$\mu_{i_0\,i_1\cdots i_{n-1}} := \mu\left(E_{i_0\,i_1\cdots i_{n-1}}\right) \; . \tag{3.1}$$

Definition 3.1.1

We shall set $\boldsymbol{i} = \{i_0\, i_1 \cdots i_{n-1}\}$ and denote by Ω_D^n the set of D^n n_tuples with i_j taking values in $\{1, 2, \cdots, D\}$.

The atoms of the partitions $\mathcal{E}_{[0,n-1]}$ describe segments of trajectories up to time n encoded by the atoms of $\mathcal{E}$ that are traversed at successive times; the volumes $\mu_{\boldsymbol{i}} = \mu\left(E_{\boldsymbol{i}}\right)$ corresponds to probabilities for the system to belong to the atoms $E_{i_0}, E_{i_1}, \cdots, E_{i_{n-1}}$ at successive times $0 \leqslant j \leqslant n-1$. The n_tuples $\boldsymbol{i}$ by themselves provide a description of the system in a symbolic dynamic. Of course, once the evolved partition $\mathcal{E}_{[0,n-1]}$ is specified, not all strings (or words) $\boldsymbol{i} \in \Omega_D^n$ would represent the possible trajectories of the dynamical system. Therefore we can split the set Ω_D^n in two: a set containing all admissible words, denoted by $k\left(\mathcal{E}_{[0,n-1]}\right)$, and its complementary set. Of course a word belongs to $k\left(\mathcal{E}_{[0,n-1]}\right)$, if the corresponding $E_{\boldsymbol{i}} \in \mathcal{E}_{[0,n-1]}$ contains (at least) one point. The study of the cardinality of the set of admissible words $k\left(\mathcal{E}_{[0,n-1]}\right)$, in the limit $n \longmapsto \infty$, provide the simplest possibility of estimating the complexity of a dynamical system [18]. Moreover, we can expect that not all trajectories would have the same weight for the system: in particular there could be “rare” trajectories, encoded by $\boldsymbol{i} \in k\left(\mathcal{E}_{[0,n-1]}\right)$, whose weight (given by the measure of the corresponding set $\mu_{\boldsymbol{i}} = \mu\left(E_{\boldsymbol{i}}\right)$) is negligible.

The richness in diverse trajectories, that is the degree of irregularity of the motion (as seen with the accuracy of the given coarse-graining) correspond intuitively to our idea of “complexity” and can be measured better by the Shannon entropy [18]

$$S_\mu(\mathcal{E}_{[0,n-1]}) := -\sum_{\boldsymbol{i}\in\Omega_D^n} \mu_{\boldsymbol{i}} \log \mu_{\boldsymbol{i}} \; . \tag{3.2}$$

In the long run, $\mathcal{E}$ attributes to the dynamics an entropy per unit time–step

$$h_\mu(T, \mathcal{E}) := \lim_{n\to\infty} \frac{1}{n} S_\mu(\mathcal{E}_{[0,n-1]}) \; . \tag{3.3}$$

This limit is well defined [3] and the "average entropy production" $h_\mu(T,\mathcal{E})$ measure how predictable the dynamics is on the coarse grained scale provided by the finite partition $\mathcal{E}$. To remove the dependence on $\mathcal{E}$, the Kolmogorov entropy $h_\mu(T)$ of $\big(\mathcal{X},\mu,T\big)$ (or KS entropy) is defined as the supremum over all finite measurable partitions [3, 18]:

$$h_\mu(T) := \sup_{\mathcal{E}} h_\mu(T,\mathcal{E}) \qquad \cdot \tag{3.4}$$

If we go now to the problem of estiming a probability for the "possible" words in $k\left(\mathcal{E}_{[0,n-1]}\right)$, an important Theorem comes to our aid. Let $\mathcal{E}_{[0,n-1]}(\boldsymbol{x}) \in \mathcal{E}_{[0,n-1]}$ denote the atom of containing $\boldsymbol{x} \in \mathcal{X}$. Then it holds [47]:

Theorem 5 (Shannon–Mc Millan–Breiman) : If $\big(\mathcal{X},\mu,T\big)$ is ergodic, then

$$\lim_{n\longrightarrow\infty} -\frac{1}{n}\log\mu\left(\mathcal{E}_{[0,n-1]}(\boldsymbol{x})\right) = h_\mu(T,\mathcal{E}) \qquad \mu - \text{a.e.} \tag{3.5}$$

This Theorem implies that most of the words in $k\left(\mathcal{E}_{[0,n-1]}\right)$ have (asymptotically with n) the probability $e^{-nh_\mu(T)}$; this is more precisely stated by the next [47]

Theorem 6 (Asymptotic equipartition property) : If $\big(\mathcal{X},\mu,T\big)$ is ergodic, then for any $\varepsilon > 0$ there exists a positive integer n_ε such that if $n \geqslant n_\varepsilon$ then $k\left(\mathcal{E}_{[0,n-1]}\right)$ decomposes into two sets H and L such that

$$\sum_{\boldsymbol{i}\in L}\mu_{\boldsymbol{i}} < \varepsilon$$

and such that

$$e^{-n(h_\mu(T)+\varepsilon)} < \mu_{\boldsymbol{i}} < e^{-n(h_\mu(T)-\varepsilon)}$$

for any $\boldsymbol{i} \in H$.

Using (1) one can expect that the volumes (3.1) containing points with close–by trajectories decrease as $\log\mu\left(\mathcal{E}_{[0,n-1]}(\boldsymbol{x})\right) \simeq -n\sum_j \log\lambda_j^+$, where $\log\lambda_j^+$ are the positive Lyapounov exponents, and this would fix a relation between the Lyapounov exponents and (using Theorem 5) the KS entropy; this is indeed the statement of the next important [17]

Theorem 7 (Pesin) : For smooth, ergodic $\big(\mathcal{X},\mu,T\big)$:

$$h_\mu(T) = \sum_j \log\lambda_j^+ \;\cdot \tag{3.6}$$

3.1.2. Symbolic Models as Classical Spin Chains

Finite partitions $\mathcal{E}$ of $\mathcal{X}$ provide symbolic models for the dynamical systems $\big(\mathcal{A}_\mathcal{X}, \omega_\mu, \Theta\big)$ of Section 1.1.2, whereby the trajectories $\big\{T^j x\big\}_{j\in\mathbb{Z}}$ are encoded into sequences $\{i_j\}_{j\in\mathbb{Z}}$ of indices relative to the atoms E_{i_j} visited at successive times j; the dynamics corresponds to the right–shift along the symbolic sequences. The encoding can be modelled as the shift along a classical spin chain endowed with a shift–invariant state [20]. This will help to understand the quantum dynamical entropy which will be introduced in the next Section.

Let D be the number of atoms of a partition $\mathcal{E}$ of $\mathcal{X}$, we shall denote by $\boldsymbol{A}_D$ the diagonal $D \times D$ matrix algebra generated by the characteristic functions e_{E_ℓ} of the atoms E_ℓ and by $\boldsymbol{A}_D^{[0,n-1]}$ the n-fold tensor product of n copies of $(\boldsymbol{A}_D)$, that is the $D^n \times D^n$ diagonal matrix algebra $\boldsymbol{A}_D^{[0,n-1]} := (\boldsymbol{A}_D)_0 \otimes (\boldsymbol{A}_D)_1 \cdots \otimes (\boldsymbol{A}_D)_{n-1}$. Its typical elements are of the form $a_0 \otimes a_1 \cdots \otimes a_{n-1}$ each a_j being a diagonal $D \times D$ matrix. Every $\boldsymbol{A}_D^{[p,q]} := \otimes_{j=p}^{q} (\boldsymbol{A}_D)_j$ can be embedded into the infinite tensor product $\boldsymbol{A}_D^\infty := \otimes_{k=0}^\infty (\boldsymbol{A}_D)_k$ as

$$(\mathbb{1})_0 \otimes \cdots \otimes (\mathbb{1})_{p-1} \otimes (\boldsymbol{A}_D)_p \otimes \cdots \otimes (\boldsymbol{A}_D)_q \otimes (\mathbb{1})_{q+1} \otimes (\mathbb{1})_{q+2} \otimes \cdots \tag{3.7}$$

The algebra $\boldsymbol{A}_D^\infty$ is a classical spin chain with a classical D–spin at each site.

By means of the discrete probability measure $\{\mu_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\Omega_D^n}$, one can define a compatible family of states on the "local" algebras $\boldsymbol{A}_D^{[0,n-1]}$:

$$\rho_\mathcal{E}^{[0,n-1]}\left(a_0 \otimes \cdots \otimes a_{n-1}\right) = \sum_{\boldsymbol{i}\in\Omega_D^n} \mu_{\boldsymbol{i}}\,(a_0)_{i_0 i_0} \cdots (a_{n-1})_{i_{n-1} i_{n-1}} \ . \tag{3.8}$$

Indeed, let $\rho\lceil\boldsymbol{N}$ denote the restriction to a subalgebra $\boldsymbol{N} \subseteq \boldsymbol{M}$ of a state ρ on a larger algebra $\boldsymbol{M}$. Since $\sum_{i_{n-1}} \mu_{i_0 i_1 \cdots i_{n-1}} = \mu_{i_0 i_1 \cdots i_{n-2}}$, when n varies the states $\rho_\mathcal{E}^{[0,n-1]}$ form a compatible family, in the sense that[1] $\rho_\mathcal{E}^{[0,n-1]}\lceil\boldsymbol{A}_D^{[0,n-2]} = \rho_\mathcal{E}^{[0,n-2]}$. Then, the "local states" $\rho_\mathcal{E}^{[0,n-1]}$ define a "global" state $\rho_\mathcal{E}$ on the infinite tensor product $\boldsymbol{A}_D^\infty$ so that $\rho_\mathcal{E}\lceil\boldsymbol{A}_D^{[0,n-1]} = \rho_\mathcal{E}^{[0,n-1]}$.

Remark 3.1.1

$\boldsymbol{A}_D^\infty$ is a D-spin chain which is classical since the algebras at the integer sites consist of diagonal matrices. The state $\rho_\mathcal{E}$ defines the statistical properties of such a chain, for instance the correlations among spins at different sites.

[1]

$$\boldsymbol{A}_D^{[0,n-2]} \ni a \xrightarrow{\rho_\mathcal{E}^{[0,n-1]}\lceil\boldsymbol{A}_D^{[0,n-2]}} \rho_\mathcal{E}^{[0,n-1]}\left(a \otimes (\mathbb{1})_{n-1}\right) = \rho_\mathcal{E}^{[0,n-2]}(a)$$

Furthermore, from (3.1) T-invariance of μ yields

$$\begin{aligned}\sum_{i_0}\mu_{i_0\,i_1\cdots i_{n-1}} &= \sum_{i_0}\mu\left(E_{i_0}\cap T^{-1}(E_{i_1})\cap\cdots\cap T^{-n+1}(E_{i_{n-1}})\right)\\ &= \mu\left(\mathcal{X}\cap T^{-1}(E_{i_1})\cap\cdots\cap T^{-n+1}(E_{i_{n-1}})\right)\\ &= \mu\left(E_{i_1}\cap T^{-1}(E_{i_2})\cap\cdots\cap T^{-n+2}(E_{i_{n-1}})\right)\\ &= \mu_{i_1\,i_2\cdots i_{n-1}}\end{aligned}$$

It thus follows that the under the right-shift $\sigma:\boldsymbol{A}_D^\infty\mapsto\boldsymbol{A}_D^\infty$,

$$\sigma\left(\boldsymbol{A}_D^{[p,q]}\right)=\boldsymbol{A}_D^{[p+1,q+1]}\ , \tag{3.9}$$

the state $\rho_{\mathcal{E}}$ of the classical spin chain is translation invariant:

$$\begin{aligned}\rho_{\mathcal{E}}\circ\sigma\left(a_0\otimes\cdots\otimes a_{n-1}\right) &= \rho_{\mathcal{E}}\left((\mathbb{1})_0\otimes(a_0)_1\otimes\cdots\otimes(a_{n-1})_n\right)\\ &= \rho_{\mathcal{E}}\left(a_0\otimes\cdots\otimes a_{n-1}\right)\end{aligned} \tag{3.10}$$

Finally, denoting by $|j\rangle$ the basis vectors of the representation where the matrices $a\in\boldsymbol{A}_D$ are diagonal, that is $a_m=\sum_{j_m=1}^{D}(a_m)_{j_mj_m}|j_m\rangle\langle j_m|$, local states amount to diagonal density matrices

$$\rho_{\mathcal{E}}^{[0,n-1]}=\sum_{\boldsymbol{i}\in\Omega_D^n}\mu_{\boldsymbol{i}}\,|i_0\rangle\langle i_0|\otimes|i_1\rangle\langle i_1|\otimes\cdots\otimes|i_{n-1}\rangle\langle i_{n-1}|\ , \tag{3.11}$$

and the Shannon entropy (3.2) to the Von Neumann entropy[2]

$$S_\mu(\mathcal{E}_{[0,n-1]})=-\,\mathrm{Tr}\left[\rho_{\mathcal{E}}^{[0,n-1]}\log\rho_{\mathcal{E}}^{[0,n-1]}\right]=:H_\mu\left[\mathcal{E}_{[0,n-1]}\right]\ . \tag{3.12}$$

3.2. Quantum Dynamical Entropies

From an algebraic point of view, the difference between a triplet $\left(\mathcal{M},\omega,\Theta\right)$ (see Section 1.1.1) describing a quantum dynamical system and a triplet $\left(\mathcal{A}_{\mathcal{X}},\omega_\mu,\Theta\right)$ as in Definitions 2.1.2 is that ω and Θ are now a Θ–invariant state, respectively an automorphism over a non–commutative (C* or Von Neumann) algebra of operators.

[2]See (3.13) for the definition.

Remarks 3.2.1

a. In standard quantum mechanics the algebra $\mathcal{M}$ is the von Neumann algebra $B(\mathcal{H})$ of all bounded linear operators on a suitable Hilbert space $\mathcal{H}$. If $\mathcal{H}$ has finite dimension D, $\mathcal{M}$ is the algebra of $D \times D$ matrices.

b. The typical states ω are density matrices ρ, namely operators with positive eigenvalues ρ_ℓ such that $\mathrm{Tr}(\rho) = \sum_\ell \rho_\ell = 1$. Given the state ρ, the mean value of any observable $X \in B(\mathcal{H})$ is given by $\rho(X) := \mathrm{Tr}(\rho X)$.

c. The ρ_ℓ in points (b.) are interpreted as the probabilities of finding the system in the corresponding eigenstates. The uncertainty prior to the measurement is measured by the Von Neumann entropy of ρ:

$$H(\rho) := -\mathrm{Tr}(\rho \log \rho) = \sum_\ell \rho_\ell \log \rho_\ell \cdot \tag{3.13}$$

d. The usual dynamics on $\mathcal{M}$ is of the form $\Theta(X) = UXU^*$, where U is a unitary operator. If one has a Hamiltonian operator that generates the continuous group $U_t = \exp i\,t\,H/\hbar$ then $U := U_{t=1}$ and the time-evolution is discretized by considering powers U^j.

The idea behind the notion of dynamical entropy is that information can be obtained by repeatedly observing a system in the course of its time evolution. Due to the uncertainty principle, or, in other words, to non-commutativity, if observations are intended to gather information about the intrinsic dynamical properties of quantum systems, then non-commutative extensions of the KS-entropy ought first to decide whether quantum disturbances produced by observations have to be taken into account or not.

Concretely, let us consider a quantum system described by a density matrix ρ acting on a Hilbert space $\mathcal{H}$. Via the wave packet reduction postulate, generic measurement processes may reasonably well be described by finite sets $\mathcal{Y} = \{y_0, y_1, \ldots, y_{D-1}\}$ of bounded operators $y_j \in \mathcal{B}(\mathcal{H})$ such that $\sum_j y_j^* y_j = \mathbb{1}$. These sets are called partitions of unity (*p.u.*, for sake of shortness) and describe the change in the state of the system caused by the corresponding measurement process:

$$\rho \mapsto \Gamma^*_{\mathcal{Y}}(\rho) := \sum_j y_j\,\rho\,y_j^*. \tag{3.14}$$

It looks rather natural to rely on partitions of unity to describe the process of collecting information through repeated observations of an evolving quantum system [20]. Yet, most of these measurements interfere with the quantum evolution, possibly acting as a source of unwanted extrinsic randomness. Nevertheless, the effect is typically quantal and rarely avoidable. Quite interestingly, as we shall see later, pursuing these ideas leads to quantum stochastic processes with a quantum dynamical entropy of their own, the ALF-entropy, that is also useful in a classical context.

An alternative approach [19] leads to the CNT-entropy. This approach lacks the operational appeal of the ALF-construction, but is intimately connected with the intrinsic relaxation properties of quantum systems [19, 48] and possibly useful in the rapidly growing field of quantum communication. The CNT-entropy is based on decomposing quantum states rather than on reducing them as in (3.14). Explicitly, if the state ρ is not a one dimensional projection, any partition of unity $\mathcal{Y}$ yields a decomposition

$$\rho = \sum_j \mathrm{Tr}\big(\rho\, y_j^* y_j\big)\, \frac{\sqrt{\rho}\, y_j^* y_j \sqrt{\rho}}{\mathrm{Tr}\big(\rho\, y_j^* y_j\big)}. \tag{3.15}$$

When $\Gamma^*_{\mathcal{Y}}(\rho) = \rho$, reductions also provide decompositions, but not in general.

3.2.1. Decompositions of states and CNT–Entropy

The CNT-entropy is based on decomposing quantum states into convex linear combinations of other states. The information content attached to the quantum dynamics is not based on modifications of the quantum state or on perturbations of the time evolution. Let $(\mathcal{M}, \Theta, \omega)$ represent a quantum dynamical system in the algebraic setting and assume ω to be decomposable. The construction runs as follows.

- Classical partitions are replaced by finite dimensional C*-algebras $\mathcal{N}$ with identity embedded into $\mathcal{M}$ by completely positive[3], unity preserving (cpu) maps $\gamma : \mathcal{N} \mapsto \mathcal{M}$. Given γ, consider the cpu maps $\gamma_\ell := \Theta^\ell \circ \gamma$ that result from successive iterations of the dynamical automorphism Θ, and associate to each of them an index set I_ℓ. These index sets I_ℓ will be coupled to the cpu maps γ_ℓ through the variational problem (3.18).

- If $0 \leqslant \ell < n$ then consider multi-indices $\boldsymbol{i} = (i_0, i_1, \ldots, i_{n-1}) \in I^{[0,n-1]} := I_0 \times \cdots \times I_{n-1}$ as labels of states $\omega_{\boldsymbol{i}}$ on $\mathcal{M}$ and of weights $0 < \mu_{\boldsymbol{i}} < 1$ such that $\sum_{\boldsymbol{i}} \mu_{\boldsymbol{i}} = 1$ and $\omega = \sum_{\boldsymbol{i}} \mu_{\boldsymbol{i}}\, \omega_{\boldsymbol{i}}$. These states are given by elements $0 \leqslant x'_{\boldsymbol{i}} \in \mathcal{M}'$, the commutant of

[3] A completely positive map γ is a map such that for every identity map $\mathbb{1}_N : \mathbb{C}^N \mapsto \mathbb{C}^N$, the tensor product $\gamma \otimes \mathbb{1}_N$ is positive.

$\mathcal{M}$, such that $\sum_{\boldsymbol{i}} x'_{\boldsymbol{i}} = \mathbb{1}_N$. Explicitly

$$y \in \mathcal{M} \longmapsto \omega_{\boldsymbol{i}}(y) := \frac{\omega(x'_{\boldsymbol{i}}\, y)}{\omega(x'_{\boldsymbol{i}})} \ , \quad \mu_{\boldsymbol{i}} := \omega(x'_{\boldsymbol{i}}) \cdot \tag{3.16}$$

The decomposition has be done with elements x' in the commutant in order to ensure the positivity of the expectations $\omega_{\boldsymbol{i}}$[4].

- From $\omega = \sum_{\boldsymbol{i}} \mu_{\boldsymbol{i}}\, \omega_{\boldsymbol{i}}$, one obtains subdecompositions $\omega = \sum_{i_\ell \in I_\ell} \mu^\ell_{i_\ell}\, \omega^\ell_{i_\ell}$, where

$$\omega^\ell_{i_\ell} := \sum_{\substack{\boldsymbol{i} \\ i_\ell \text{ fixed}}} \frac{\mu_{\boldsymbol{i}}}{\mu^\ell_{i_\ell}}\, \omega_{\boldsymbol{i}} \qquad \text{and} \qquad \mu^\ell_{i_\ell} := \sum_{\substack{\boldsymbol{i} \\ i_\ell \text{ fixed}}} \mu_{\boldsymbol{i}}. \tag{3.17}$$

- Since $\mathcal{N}$ is finite dimensional, the states $\omega \circ \Theta^\ell \circ \gamma = \omega \circ \gamma$ and $\omega^\ell_{i_\ell} \circ \Theta^\ell \circ \gamma$, have finite von Neumann entropies $S(\omega \circ \gamma)$ and $S(\omega^\ell_{i_\ell} \circ \Theta^\ell \circ \gamma)$. With $\eta(x) := -x \ \log x$ if $0 < x \leq 1$ and $\eta(0) = 0$, one defines the n subalgebra functional

$$H_\omega(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}) := \sup_{\omega = \sum_{\boldsymbol{i}} \mu_{\boldsymbol{i}}\, \omega_{\boldsymbol{i}}} \Bigg\{ \sum_{\boldsymbol{i}} \eta(\mu_{\boldsymbol{i}}) - \sum_{\ell=0}^{n-1} \sum_{i_\ell \in I_\ell} \eta(\mu^\ell_{i_\ell}) + \sum_{\ell=0}^{n-1} \Big(S(\omega \circ \gamma_\ell) - \sum_{i_\ell \in I_\ell} \mu^\ell_{i_\ell}\, S(\omega^\ell_{i_\ell} \circ \gamma_\ell) \Big) \Bigg\} . \tag{3.18}$$

We list a number of properties of n-subalgebra functionals, see [19], that will be used in the sequel:

- positivity: $0 \leq H_\omega(\gamma_0, \gamma_1, \ldots, \gamma_{n-1})$
- subadditivity: $$H_\omega(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}) \leq H_\omega(\gamma_0, \gamma_1, \ldots, \gamma_{\ell-1}) + H_\omega(\gamma_\ell, \gamma_{\ell+1}, \ldots, \gamma_{n-1})$$
- time invariance: $H_\omega(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}) = H_\omega(\gamma_\ell, \gamma_{\ell+1}, \ldots, \gamma_{\ell+n-1})$
- boundedness: $H_\omega(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}) \leq n H_\omega(\gamma) \leq n S(\omega \circ \gamma)$
- The n-subalgebra functionals are invariant under interchange and repetitions of arguments:

$$H_\omega(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}) = H_\omega(\gamma_{n-1}, \ldots, \gamma_0, \gamma_0). \tag{3.19}$$

[4] Indeed $\mathcal{M} \ni y \geqslant 0 \Longrightarrow y = z^* z$ for some $z \in \mathcal{M}$; then it follows $\omega_{\boldsymbol{i}}(y) = \omega(x'_{\boldsymbol{i}} z^* z) = \omega(z^* x'_{\boldsymbol{i}} z) \geqslant 0$, for $0 \leqslant x'_{\boldsymbol{i}} \in \mathcal{M}'$.

- monotonicity: If $i_\ell : \mathcal{N}_\ell \mapsto \mathcal{N}$, $0 \leq \ell \leq n-1$, are cpu maps from finite dimensional algebras $\mathcal{N}_l$ into $\mathcal{N}$, then the maps $\tilde{\gamma}_\ell := \gamma \circ i_\ell$ are cpu and

$$H_\omega(\tilde{\gamma}_0, \Theta \circ \tilde{\gamma}_1, \ldots, \Theta^{n-1} \circ \tilde{\gamma}_{n-1}) \leq H_\omega(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}). \tag{3.20}$$

- continuity: Let us consider for $\ell = 0, 1, \ldots, n-1$ a set of cpu maps $\tilde{\gamma}_\ell : \mathcal{N} \mapsto \mathcal{M}$ such that $\|\gamma_\ell - \tilde{\gamma}_\ell\|_\omega \leq \epsilon$ for all ℓ, where

$$\|\gamma_\ell - \tilde{\gamma}_\ell\|_\omega := \sup_{x \in \mathcal{N},\ \|x\| \leq 1} \sqrt{\omega\Big((\gamma_\ell(x) - \tilde{\gamma}_\ell(x))^*(\gamma_\ell(x) - \tilde{\gamma}_\ell(x))\Big)}\ . \tag{3.21}$$

  Then [19], there exists $\delta(\epsilon) > 0$ depending on the dimension of the finite dimensional algebra $\mathcal{N}$ and vanishing when $\epsilon \to 0$, such that

$$\Big|H_\omega(\gamma_0, \gamma_1 \ldots, \gamma_{n-1}) - H_\omega(\tilde{\gamma}_0, \tilde{\gamma}_1 \ldots, \tilde{\gamma}_{n-1})\Big| \leq n\, \delta(\epsilon). \tag{3.22}$$

On the basis of these properties, one proves the existence of the limit

$$h_\omega^{\mathsf{CNT}}(\theta, \gamma) := \lim_n \frac{1}{n} H_\omega(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}) \tag{3.23}$$

and defines [19]:

Definition 3.2.1

The CNT-entropy of a quantum dynamical system $(\mathcal{M}, \Theta, \omega)$ is

$$h_\omega^{\mathsf{CNT}}(\Theta) := \sup_\gamma h_\omega^{\mathsf{CNT}}(\Theta, \gamma)\ .$$

3.2.2. Partitions of unit and ALF–Entropy

The quantum dynamical entropy proposed in [20] by Alicki and Fannes, ALF–entropy[5] for short, is based on the idea that, in analogy with what one does for the metric entropy, one can model symbolically the evolution of quantum systems by means of the right shift along a spin chain. In the quantum case the finite–dimensional matrix algebras at the various sites are not diagonal, but, typically, full matrix algebras, that is the spin at each site is a quantum spin.

This is done by means of p.u. $\mathcal{Y} = \{y_0, y_1, \ldots, y_{D-1}\} \subset \mathcal{M}_0 \subset \mathcal{M}$, already defined in

[5] L – stands for Lindblad

Section 3.2; here $\mathcal{M}_0$ denotes a Θ–invariant subalgebra. With $\mathcal{Y}$ and the state ω one constructs the $D \times D$ matrix with entries $\omega(y_j^* y_i)$; such a matrix is a density matrix $\rho[\mathcal{Y}]$:

$$\rho[\mathcal{Y}]_{i,j} := \omega(y_j^* y_i) \cdot \tag{3.24}$$

It is thus possible to define the entropy of a partition of unit as (compare (3.12)):

$$H_\omega[\mathcal{Y}] := -\operatorname{Tr}\Big(\rho[\mathcal{Y}] \log \rho[\mathcal{Y}]\Big) \cdot \tag{3.25}$$

Further, given two partitions of unit $\mathcal{Y} = \Big(y_0, y_1, \ldots, y_D\Big)$, $\mathcal{Z} = \Big(z_0, z_1, \ldots, z_B\Big)$, of size D, respectively B, one gets a finer partition of unit of size BD as the set

$$\mathcal{Y} \circ \mathcal{Z} := \Big(y_0 z_0, \cdots, y_0 z_B; y_1 z_0, \cdots, y_1 z_B; \cdots; y_D z_0, \cdots, y_D z_B\Big) \cdot \tag{3.26}$$

After j time–steps, $\mathcal{Y}$ evolves into $\Theta^j(\mathcal{Y}) := \Big\{\Theta^j(y_1), \Theta^j(y_2), \cdots, \Theta^j(y_D)\Big\}$. Since Θ is an automorphism, $\Theta^j(\mathcal{Y})$ is a partition of unit; then, one refines $\Theta^j(\mathcal{Y})$, $0 \leqslant j \leqslant n-1$, into a larger partition of unit

$$\mathcal{Y}^{[0,n-1]} := \Theta^{n-1}(\mathcal{Y}) \circ \Theta^{n-2}(\mathcal{Y}) \circ \cdots \circ \Theta(\mathcal{Y}) \circ \mathcal{Y} \cdot \tag{3.27}$$

We shall denote the typical element of $\Big[\mathcal{Y}^{[0,n-1]}\Big]$ by

$$\Big[\mathcal{Y}^{[0,n-1]}\Big]_{\boldsymbol{i}} = \Theta^{n-1}\Big(y_{i_{n-1}}\Big)\,\Theta^{n-2}\Big(y_{i_{n-2}}\Big) \cdots \quad \Theta(y_{i_1}) \quad y_{i_0} \cdot \tag{3.28}$$

Each refinement is in turn associated with a density matrix $\rho_{\mathcal{Y}}^{[0,n-1]} := \rho\left[\mathcal{Y}^{[0,n-1]}\right]$ which is a state on the algebra $\mathbf{M}_D^{[0,n-1]} := \otimes_{\ell=0}^{n-1}(\mathbf{M}_D)_\ell$, with entries

$$\Big[\rho\Big[\mathcal{Y}^{[0,n-1]}\Big]\Big]_{\boldsymbol{i},\boldsymbol{j}} := \omega\Big(y_{j_0}^* \Theta\left(y_{j_1}^*\right) \cdots \Theta^{n-1}\Big(y_{j_{n-1}}^* y_{i_{n-1}}\Big) \cdots \Theta\left(y_{i_1}\right) y_{i_0}\Big) \cdot \tag{3.29}$$

Moreover each refinement has an entropy

$$H_\omega\Big[\mathcal{Y}^{[0,n-1]}\Big] = -\operatorname{Tr}\Big(\rho\Big[\mathcal{Y}^{[0,n-1]}\Big] \log \rho\Big[\mathcal{Y}^{[0,n-1]}\Big]\Big) \cdot \tag{3.30}$$

The states $\rho_{\mathcal{Y}}^{[0,n-1]}$ are compatible: $\rho_{\mathcal{Y}}^{[0,n-1]} \upharpoonright \mathbf{M}_D^{[0,n-2]} = \rho_{\mathcal{Y}}^{[0,n-2]}$, and define a global state $\rho_{\mathcal{Y}}$ on the quantum spin chain $\mathbf{M}_D^\infty := \otimes_{\ell=0}^\infty (\mathbf{M}_D)_\ell$.

Then, as in the previous Section, it is possible to associate with the quantum dynamical system $(\mathcal{M}, \omega, \Theta)$ a symbolic dynamics which amounts to the right–shift, $\sigma : (\mathbf{M}_D)_\ell \mapsto (\mathbf{M}_D)_{\ell+1}$, along the quantum spin half–chain (compare (3.9)).

Non-commutativity becomes evident when we check whether $\rho_{\mathcal{Y}}$ is shift-invariant. This requires[6] $\omega\Big(\sum_\ell y_\ell^* x\, y_\ell\Big) = \omega(x)$ for all $x \in \mathcal{M}$. Note that this is the case in which $\rho \mapsto \Gamma_{\mathcal{Y}}^*(\rho) := \rho$ (see. equation (3.14)). For a comparison between classical and quantum spin chain properties, see table 3.1.

Classical System $(\mathcal{X}, \mu, T)$	**Quantum System** $(\mathcal{M}, \omega, \Theta)$
$\sum_{i_{n-1}} \mu_{i_0\, i_1 \cdots i_{n-1}} = \mu_{i_0\, i_1 \cdots i_{n-2}}$ $\Longrightarrow$ the local states $\rho_{\mathcal{E}}^{[0,n-1]}$ form a compatible family	$\sum_{\ell=1}^{D} y_\ell^* y_\ell = \mathbb{1}$ and $\Theta(\mathbb{1}) = \mathbb{1}$ $\Longrightarrow$ the local states $\rho_{\mathcal{Y}}^{[0,n-1]}$ form a compatible family
the local states $\rho_{\mathcal{E}(\mathcal{Y})}^{[0,n-1]}$ define a global state $\rho_{\mathcal{E}(\mathcal{Y})}$ on the infinite tensor product	
$\sum_{i_0} \mu_{i_0\, i_1 \cdots i_{n-1}} = \mu_{i_1\, i_2 \cdots i_{n-1}}$ $\Longrightarrow$ the global state $\rho_{\mathcal{E}}$ is translation invariant	Non abelian structure. of the algebra $\mathcal{M}$ $\Longrightarrow$ absence of translation invariance for the global state $\rho_{\mathcal{Y}}$

Table 3.1: Comparison between Classical and Quantum System

In this case, the existence of a limit as in (3.3) is not guaranteed and one has to define the ALF–entropy of $(\mathcal{M}, \omega, \Theta)$ as

[6]Moreover it is required that $\Big[\rho\Big[\mathcal{Y}^{[0,n-1]}\Big]\Big]_{\boldsymbol{i},\boldsymbol{j}} = \Big[\rho\Big[\Theta\Big(\mathcal{Y}^{[0,n-1]}\Big)\Big]\Big]_{\boldsymbol{i},\boldsymbol{j}}$, but this condition is clearly satisfied from he fact that ω is invariant for the *morphism Θ. Then we derive:

$$\begin{aligned}
\Big[\rho\Big[\mathcal{Y}^{[0,n-1]}\Big]\Big]_{\boldsymbol{i},\boldsymbol{j}} =& \omega\Big(y_{j_0}^*\Theta\left(y_{j_2}^*\right)\cdots\Theta^{n-1}\left(y_{j_{n-1}}^* y_{i_{n-1}}\right)\cdots\Theta\left(y_{i_1}\right)y_{i_0}\Big)\\
& (\omega\circ\Theta)\Big(y_{j_0}^*\Theta\left(y_{j_2}^*\right)\cdots\Theta^{n-1}\left(y_{j_{n-1}}^* y_{i_{n-1}}\right)\cdots\Theta\left(y_{i_1}\right)y_{i_0}\Big)\\
& \omega\left(\Theta\left(y_{j_0}^*\Theta\left(y_{j_2}^*\right)\cdots\Theta^{n-1}\left(y_{j_{n-1}}^* y_{i_{n-1}}\right)\cdots\Theta\left(y_{i_1}\right)y_{i_0}\right)\right)\\
& \omega\left(\Theta\left(y_{j_0}^*\right)\Theta^2\left(y_{j_2}^*\right)\cdots\Theta^{n}\left(y_{j_{n-1}}^* y_{i_{n-1}}\right)\cdots\Theta^2\left(y_{i_1}\right)\Theta\left(y_{i_0}\right)\right)\\
=& \Big[\rho\Big[\Theta\Big(\mathcal{Y}^{[0,n-1]}\Big)\Big]\Big]_{\boldsymbol{i},\boldsymbol{j}}\,.
\end{aligned}$$

Definition 3.2.2

$$h^{\mathsf{ALF}}_{\omega,\mathcal{M}_0}(\Theta) := \sup_{\mathcal{Y}\subset\mathcal{M}_0} h^{\mathsf{ALF}}_{\omega,\mathcal{M}_0}(\Theta,\mathcal{Y}) \;, \tag{3.31a}$$

where
$$h^{\mathsf{ALF}}_{\omega,\mathcal{M}_0}(\Theta,\mathcal{Y}) := \limsup_n \frac{1}{n} H_\omega\left[\mathcal{Y}^{[0,n-1]}\right] \;. \tag{3.31b}$$

Like the metric entropy of a partition $\mathcal{E}$, also the ALF–entropy of a partition of unit $\mathcal{Y}$ can be physically interpreted as an asymptotic *entropy production* relative to a specific coarse–graining.

3.3. Comparison of dynamical entropies

In this section we outline some of the main features of both quantum dynamical entropies. The complete proofs of the above facts can be found in [19] for the CNT and [20, 49] for the ALF-entropy. Here, we just sketch them, emphasizing those parts that are important to the study of their classical limit.

3.3.1. Entropy production in classical systems

Given a dynamical system $\big(\mathcal{M},\omega,\Theta\big)$, we will prove now that the CNT- and the ALF-entropy coincide with the Kolmogorov metric entropy when $\mathcal{M} = \mathcal{A}_\mathcal{X}$ is the Abelian von Neumann algebra $L^\infty_\mu(\mathcal{X})$ and Θ is a *automorphism of the same kind of the ones defined in (2.6–2.8), that is $\Theta^j(f)(\boldsymbol{x}) = f(T^j(\boldsymbol{x}))$.

Proposition 3.3.1

Let $\big(\mathcal{A}_\mathcal{X},\omega_\mu,\Theta\big)$ represent a classical dynamical system. Then, with the notations of the previous sections

$$h^{\mathsf{CNT}}_{\omega_\mu}(\Theta) = h_\mu(T) = h^{\mathsf{ALF}}_{\omega_\mu,\mathcal{A}_\mathcal{X}}(\Theta).$$

Proof of Proposition 3.3.1:

<u>CNT-Entropy.</u> In this case, $h^{\mathsf{CNT}}_{\omega_\mu}(\Theta)$ is computable by using natural embedding of finite dimensional subalgebras of $\mathcal{A}_\mathcal{X}$ rather than generic cpu maps γ. Partitions $\mathcal{C} = \{C_0, C_1, \ldots, C_{D-1}\}$ of $\mathcal{X}$ can be identified with the finite dimensional subalgebras $\mathcal{N}_\mathcal{C} \in \mathcal{M}_\mu$ generated by the characteristic functions χ_{C_j} of the atoms of the partition, with

$\omega_\mu(\chi_C) = \mu(C)$. Also, the refinements $\mathcal{C}^{[0,n-1]}$ of the evolving partitions $T^{-j}(\mathcal{C})$ correspond to the subalgebras $\mathcal{N}_{\mathcal{C}}^{[0,n-1]}$ generated by $\chi_{C_{\boldsymbol{i}}} = \prod_{j=0}^{n-1} \chi_{T^{-j}(C_{i_j})}$.

Thus, if $\imath_{\mathcal{N}_{\mathcal{C}}}$ embeds $\mathcal{N}_{\mathcal{C}}$ into $\mathcal{A}_{\mathcal{X}}$, then $\omega_\mu \circ \imath_{\mathcal{N}_{\mathcal{C}}}$ corresponds to the state $\omega_\mu \upharpoonright \mathcal{N}_{\mathcal{C}}$, which is obtained by restriction of ω_μ to $\mathcal{N}_{\mathcal{C}}$ and is completely determined by the expectation values $\omega_\mu(\chi_{C_j})$, $1 \leq j \leq n-1$.

Further, identifying the cpu maps $\gamma_\ell = \Theta^\ell \circ \imath_{\mathcal{N}_{\mathcal{C}}}$ with the corresponding subalgebras $\Theta^\ell(\mathcal{N}_{\mathcal{C}})$, $h^{\mathsf{CNT}}_{\omega_\mu}(\Theta) = h_\mu(T)$ follows from

$$H_\omega(\mathcal{N}_{\mathcal{C}}, \Theta(\mathcal{N}_{\mathcal{C}}), \ldots, \Theta^{n-1}(\mathcal{N}_{\mathcal{C}})) = S_\mu(\mathcal{C}^{[0,n-1]}), \quad \forall\, \mathcal{C}, \tag{3.32}$$

see (3.3). In order to prove (3.32), we decompose the reference state as

$$\omega_\mu = \sum_{\boldsymbol{i}} \mu_{\boldsymbol{i}}\, \omega_{\boldsymbol{i}} \qquad \text{with} \qquad \omega_{\boldsymbol{i}}(f) := \frac{1}{\mu_{\boldsymbol{i}}} \int_{\mathcal{X}} \mu(dx)\, \chi_{C_{\boldsymbol{i}}}(x)\, f(x)$$

where $\mu_{\boldsymbol{i}} = \mu(C_{\boldsymbol{i}})$, see (3.17). Then, $\sum_{\boldsymbol{i}} \eta(\mu_{\boldsymbol{i}}) = S_\mu(\mathcal{C}^{[0,n-1]})$.

On the other hand,

$$\omega^\ell_{i_\ell}(f) = \frac{1}{\mu^\ell_{i_\ell}} \int_{\mathcal{X}} \mu(dx)\, \chi_{T^{-\ell}(C_{i_\ell})}(x)\, f(x) \qquad \text{and} \qquad \mu^\ell_{i_\ell} = \mu(C_{i_\ell}).$$

It follows that $\omega_\mu \circ \imath_{\mathcal{N}_{\mathcal{C}}} = \omega_\mu \upharpoonright \mathcal{N}_{\mathcal{C}}$ is the discrete measure $\{\mu^\ell_0, \mu^\ell_1, \ldots, \mu^\ell_{n-1}\}$ for all $\ell = 0, 1, \ldots, n-1$ and, finally, that $S(\omega^\ell_{i_\ell} \circ \gamma_\ell) = 0$ as $\omega^\ell_{i_\ell} \circ j_\ell = \omega^\ell_{i_\ell} \upharpoonright \Theta^\ell(\mathcal{N}_{\mathcal{C}})$ is a discrete measure with values 0 and 1.

ALF-Entropy. The characteristic functions of measurable subsets of $\mathcal{X}$ constitute a *subalgebra $\mathcal{N}_0 \subseteq \mathcal{A}_{\mathcal{X}}$; moreover, given a partition $\mathcal{C}$ of $\mathcal{X}$, the characteristic functions χ_{C_ℓ} of its atoms C_ℓ, $\mathcal{N}_{\mathcal{C}} = \{\chi_{C_1}, \chi_{C_2}, \ldots, \chi_{C_D}\}$ is a partition of unit in $\mathcal{N}_0$. From the definition of Θ it follows that $\Theta^j(\chi_{C_\ell}) = \chi_{T^{-j}(C_\ell)}$ and from (1.3) that $\left[\rho\big[\mathcal{N}_{\mathcal{C}}^{[0,n-1]}\big]\right]_{\boldsymbol{i},\boldsymbol{j}} = \delta_{\boldsymbol{i},\boldsymbol{j}}\, \mu_{\boldsymbol{i}}$ (see (3.1)), whence $H_\omega\big[\mathcal{N}_{\mathcal{C}}^{[0,n-1]}\big] = S_\mu\big(\mathcal{C}^{[0,n-1]}\big)$ (see (3.2) and (3.25)). In such a case, the lim sup in (3.31b) is actually a true limit and yields (3.3). ■

Remark 3.3.1

In the particular case of the hyperbolic automorphisms of the torus, we may restrict our attention to *p.u.* whose elements belong to the $*$-algebra $\mathcal{W}_{\rm exp}$ of complex functions f on $\mathbb{T}^2$ such that the support of $\hat{f}$ is bounded [49]:

$$h_\mu(T) = h^{\mathsf{ALF}}_{(\omega_\mu,\mathcal{N}_0)}(\Theta) = h^{\mathsf{ALF}}_{(\omega_\mu,\mathcal{W}_{\rm exp})}(\Theta).$$

Remarkably, the computation of the classical Kolmogorov entropy via the quantum mechanical ALF-entropy yields a proof of (3.6) that is much simpler than the standard ones [26, 27].

3.3.2. Entropy production in finite dimensional systems

The next case we are dealing with is characterized by finite-dimensional algebra $\mathcal{M}$, as for the quantized hyperbolic automorphisms of the torus considered in Proposition 2.4.2; in this case both the CNT- and the ALF-entropy are zero, see [19, 20]. Consequently, if we decide to take the strict positivity of quantum dynamical entropies as a signature of quantum chaos, quantized hyperbolic automorphisms of the torus cannot be called chaotic.

Remark 3.3.2

Of course the latter observation depends on the quantum dynamical entropy we are dealing with. There exist many alternative definitions (different from ALF and CNT), and some of them need no to be equal to zero for all quantum systems defined on a finite dimensional Hilbert space: an interesting example is represented by the *Coherent States Entropy* introduced in [23].

Proposition 3.3.2

Let $(\mathcal{M},\Theta,\omega)$ be a quantum dynamical system with $\mathcal{M}$, a finite dimensional C*-algebra, then,

$$h^{\mathsf{CNT}}_\omega(\Theta) = 0 \qquad \text{and} \qquad h^{\mathsf{ALF}}_{\omega,\mathcal{M}}(\Theta) = 0.$$

Proof of Proposition 3.3.2:

<u>CNT-Entropy:</u> as in the commutative case, $h^{\mathsf{CNT}}_\omega(\Theta)$ is computable by means of cpu maps γ that are the natural embedding $\imath_{\mathcal{N}}$ of subalgebras $\mathcal{N} \subseteq \mathcal{M}$ into $\mathcal{M}$. Since each $\Theta^\ell(\mathcal{N})$ is obviously contained in the algebra $\mathcal{N}^{[0,n-1]} \subseteq \mathcal{M}$ generated by the subalgebras $\Theta^j(\mathcal{N})$, $j = 0, 1, \ldots, n-1$, from the properties of the n-subalgebra functionals H and identifying

again the natural embedding $\gamma_\ell := \Theta^\ell \circ \imath_\mathcal{N}$ with the subalgebras $\Theta^\ell(\mathcal{N}) \subseteq \mathcal{M}$, we derive

$$\begin{aligned} H_\omega(\gamma_0,\gamma_1,\ldots,\gamma_{n-1}) &= H_\omega(\mathcal{N},\Theta(\mathcal{N}),\ldots,\Theta^{n-1}(\mathcal{N})) \\ &\leq H_\omega(\mathcal{N}^{[0,n-1]},\mathcal{N}^{[0,n-1]},\ldots,\mathcal{N}^{[0,n-1]}) \quad \text{by monotonicity}^7 \\ &\leq H_\omega(\mathcal{N}^{[0,n-1]}) \quad \text{by (3.19)} \\ &\leq S\big(\omega \restriction \mathcal{N}^{[0,n-1]}\big) \quad \text{by boundedness} \\ &\leq \log N, \end{aligned}$$

where $\mathcal{M} \subseteq \mathcal{M}_N$. In fact, $\omega \restriction \mathcal{N}$ amounts to a density matrix with eigenvalues λ_ℓ and von Neumann entropy $S(\omega \restriction \mathcal{N}) = -\sum_{\ell=1}^d \lambda_\ell \log \lambda_\ell \leq \log d$. Therefore, for all $\mathcal{N} \subseteq \mathcal{M}$, $h^{\mathsf{CNT}}_\omega(\Theta,\mathcal{N}) = 0$.

<u>ALF-Entropy:</u> Let the state ω on $\mathcal{M}_N$ be given by $\omega(x) = \mathrm{Tr}(\rho\, x)$, where ρ is a density matrix in $\mathcal{M}_N$. Given a partition of unity $\mathcal{Y} = \{y_i\}_{i=1,2,\ldots,D}$, the following cpu map $\Phi_\mathcal{Y}$

$$\mathcal{M}_D \otimes \mathcal{M}_N \ni M \otimes x \xrightarrow{\ \Phi_\mathcal{Y}\ } \Phi_\mathcal{Y}(M \otimes x) := \sum_{i,j} y_i^* x\, y_j\, M_{ij} \in \mathcal{M}_N \tag{3.33}$$

can be used to define a state $\Phi^*_\mathcal{Y}(\rho)$ on $\mathcal{M}_D \otimes \mathcal{M}_N$ which is dual to ω:

$$\Phi^*_\mathcal{Y}(\rho)(M \otimes x) = \mathrm{Tr}\Big(\rho\, \Phi_\mathcal{Y}(M \otimes x)\Big), \quad M \in \mathcal{M}_D,\ x \in \mathcal{M}_N.$$

Since $\sum_{j=0}^D y_j^* y_j = \mathbb{1}$, it follows[8] that $\Phi^*_\mathcal{Y}(\rho^k) = \big(\Phi^*_\mathcal{Y}(\rho)\big)^k$. Therefore, ρ and $\Phi^*_\mathcal{Y}(\rho)$ have the same spectrum, apart possibly from the eigenvalue zero, and thus the same von Neumann entropy. Moreover, $\Phi^*_\mathcal{Y}(\rho) \restriction \mathcal{M}_D = \rho[\mathcal{Y}]$ and $\Phi^*_\mathcal{Y}(\rho) \restriction \mathcal{M}_N = \Gamma^*_\mathcal{Y}(\rho)$ as in (3.14). Applying the triangle inequality [50, 51]

$$\begin{aligned} S\big(\Phi^*_\mathcal{Y}(\rho)\big) &\geq \Big| S\big(\Phi^*_\mathcal{Y}(\rho) \restriction \mathcal{M}_D\big) - S\big(\Phi^*_\mathcal{Y}(\rho) \restriction \mathcal{M}_N\big) \Big| \quad \text{, that is} \\ S(\rho) &\geq \Big| S\big(\rho[\mathcal{Y}]\big) - S\big(\Gamma^*_\mathcal{Y}(\rho)\big) \Big| \ , \end{aligned}$$

that leads to $S(\rho[\mathcal{Y}]) \leq 2 \log d$. Finally, as evolving *p.u.* $\Theta^j(\mathcal{Y})$ and their ordered refinements (3.27–3.28) remain in $\mathcal{M}_N$, one gets

$$\limsup_k \frac{1}{k} H_\omega[\mathcal{Y}^{[0,n-1]}] = 0, \quad \mathcal{Y} \subset \mathcal{M}_N \cdot \qquad \blacksquare$$

[7] In order to match the notation of (3.20), all cpu maps i_ℓ comparing there can now be thought as the natural embedding of $\mathcal{N}$ into $\mathcal{N}^{[0,n-1]}$.

[8] In the $\mathcal{M}_D \otimes \mathcal{M}_N$ space, $\Phi^*_\mathcal{Y}(\rho)$ is represented by the enlarged density matrix $\overrightarrow{\mathcal{Y}}\rho\overrightarrow{\mathcal{Y}}^{\mathrm{H}}$, where $\overrightarrow{\mathcal{Y}}$ denotes the D-dimensional vector of $N \times N$ matrices $(y_1, y_2, \ldots, y_D)$ and the superscript "H" stays for "Hermitian conjugate".

From the considerations of above, it is clear that the main field of application of the CNT- and ALF-entropies are infinite quantum systems, where the differences between the two come to the fore [52]. The former has been proved to be useful to connect randomness with clustering properties and asymptotic commutativity. A rather strong form of clustering and asymptotic Abelianness is necessary to have a non-vanishing CNT-entropy [48, 53, 54]. In particular, the infinite dimensional quantization of the automorphisms of the torus has vanishing CNT-entropy for most of irrational values of the deformation parameter ϕ, whereas, independently of the value of ϕ, the ALF-entropy is always equal to the positive Lyapounov exponent. These results reflect the different perspectives upon which the two constructions are based.

3.4. An explicit construction: ALF–Entropy of Sawtooth Maps

In the following we develop a technique suited to compute and to simplify the Von Neumann entropy $H_\omega\left[\mathcal{Y}^{[0,n-1]}\right]$ of (3.30) for the class of discrete classical systems $\left(\mathcal{D}_N, \tau_N, \widetilde{\Theta}_{N,\alpha}\right)$, whose continuous limit in $\left(L^\infty_\mu\left(\mathcal{X}\right), \omega_\mu, \Theta_\alpha\right)$ has been shown in Section 2.4.3.

For the class of discrete systems we are dealing with, one can not define a metric entropy, being the measure a discrete one, instead we can profitably use quantum dynamical entropies, although we are in a commutative case. Indeed, the only necessary ingredient to construct such kind of entropies, is the algebraic description $\left(\mathcal{D}_N, \tau_N, \widetilde{\Theta}_{N,\alpha}\right)$ and, in the ALF entropy computation, the use of a partition of unit.

The reason to choose the ALF entropy instead of the CNT is the numerical compatibility of the former; indeed the variational problem in (3.18) is apparently very hard to be attached numerically.

By remember Proposition 3.3.2, we know that we cannot go to compute neither $h^{\mathsf{ALF}}_{\omega,\mathcal{M}_0}(\Theta)$ nor $h^{\mathsf{ALF}}_{\omega,\mathcal{M}_0}(\Theta, \mathcal{Y})$ of Definition 3.2.2, because these quantity are expected to be zero. The analysis of entropy production will be performed in the next Chapter, now we only set up the framework to compute it.

A useful partition of unit in $\left(L^\infty_\mu\left(\mathcal{X}\right), \omega_\mu, \Theta_\alpha\right)$ is constructed by collecting a finite number D of Weyl operators $\widetilde{W}(\boldsymbol{r}_j)$ defined in (1.32), indexed by their labels $\boldsymbol{r}_j$, as in the following

Definition 3.4.1

Given a subset Λ of the lattice consisting of points

$$\left\{\boldsymbol{r}_j\right\}_{j=1}^{D} =: \Lambda \subset (\mathbb{Z}/N\mathbb{Z})^2 \cdot \;, \tag{3.34}$$

we shall denote by $\widetilde{\mathcal{Y}}$ the partition of unit in $\left(L^\infty_\mu(\mathcal{X}), \omega_\mu, \Theta_\alpha\right)$ given by:

$$\widetilde{\mathcal{Y}} = \left\{\widetilde{y}_j\right\}_{j=1}^{D} := \left\{\frac{1}{\sqrt{D}}\,\widetilde{W}(\boldsymbol{r}_j)\right\}_{j=1}^{D} \cdot \tag{3.35}$$

From the above definition, the elements of the refined partitions in (3.28) take the form:

$$\left[\widetilde{\mathcal{Y}}^{[0,n-1]}\right]_{\boldsymbol{i}} = \frac{1}{N}\frac{1}{D^{\frac{n}{2}}} \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} e^{\frac{2\pi i}{N}\left[\boldsymbol{r}_{i_{n-1}}\cdot V_\alpha^{n-1}(\boldsymbol{\ell})+\cdots+\boldsymbol{r}_{i_1}\cdot V_\alpha(\boldsymbol{\ell})+\boldsymbol{r}_{i_0}\cdot\boldsymbol{\ell}\right]} \left|\boldsymbol{\ell}\right\rangle\left\langle\boldsymbol{\ell}\right| \;\cdot \tag{3.36}$$

Then, the multitime correlation matrix $\rho^{[0,n-1]}_{\widetilde{\mathcal{Y}}}$ in (3.29) has entries:

$$\left[\rho\left[\widetilde{\mathcal{Y}}^{[0,n-1]}\right]\right]_{\boldsymbol{i},\boldsymbol{j}} = \frac{1}{N^2}\frac{1}{D^n} \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} e^{\frac{2\pi i}{N}\sum_{p=0}^{n-1}\left(\boldsymbol{r}_{i_p}-\boldsymbol{r}_{j_p}\right)\cdot V_\alpha^p(\boldsymbol{\ell})} \;,\quad V_\alpha^0(\boldsymbol{\ell}) = \mathbb{1} \tag{3.37}$$

$$= \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} \left\langle \boldsymbol{i}\right| g_{\boldsymbol{\ell}}(n)\rangle \left\langle g_{\boldsymbol{\ell}}(n)\right|\boldsymbol{j}\rangle \;, \tag{3.38}$$

$$\text{with}\quad \left\langle \boldsymbol{i}\right| g_{\boldsymbol{\ell}}(n)\rangle := \frac{1}{N}\frac{1}{D^{\frac{n}{2}}} e^{\frac{2\pi i}{N}\sum_{p=0}^{n-1}\boldsymbol{r}_{i_p}\cdot V_\alpha^p(\boldsymbol{\ell})} \in \mathbb{C}^{D^n} \;. \tag{3.39}$$

The density matrix $\rho^{[0,n-1]}_{\widetilde{\mathcal{Y}}}$ can now be used to numerically compute the Von Neumann entropy $H_{\tau_N}\left[\mathcal{Y}^{[0,n-1]}\right]$ of (3.30); however, the large dimension ($D^n \times D^n$) makes the computational problem very hard, a part for small numbers of iterations. Our goal is to prove that another matrix (of fixed dimension $N^2 \times N^2$) can be used instead of $\rho^{[0,n-1]}_{\widetilde{\mathcal{Y}}}$. In particular, the next proposition can be seen as an extension of the strategy that led us to prove Proposition 3.3.2

Proposition 3.4.1

Let $\mathcal{G}(n)$ be the $N^2 \times N^2$ matrix with entries

$$\mathcal{G}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2}(n) := \left\langle g_{\boldsymbol{\ell}_2}(n)\right| g_{\boldsymbol{\ell}_1}(n)\rangle \tag{3.40}$$

given by the scalar products of the vectors $|g_{\boldsymbol{\ell}}(n)\rangle \in \mathcal{H}_{D^n} = \mathbb{C}^{D^n}$ in (3.39). Then, the entropy of the partition of unit $\widetilde{\mathcal{Y}}^{[0,n-1]}$ with elements (3.35) is given by:

$$H_{\tau_N}\left[\widetilde{\mathcal{Y}}^{[0,n-1]}\right] = -\mathrm{Tr}_{\mathcal{H}_N^D}\Big(\mathcal{G}(n)\log\mathcal{G}(n)\Big) \tag{3.41}$$

Proof of Proposition 3.4.1:

$\mathcal{G}(n)$ is hermitian and from (3.39) it follows that $\mathrm{Tr}_{\mathcal{H}_{N^2}}\mathcal{G}(n) = 1$.
Let $\mathcal{H} := \mathcal{H}_{D^n}\otimes\mathcal{H}_N^D$ and consider the projection $\rho_\psi = |\psi\rangle\langle\psi|$ onto

$$\mathcal{H} \ni |\psi\rangle := \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} |g_{\boldsymbol{\ell}}(n)\rangle\otimes|\boldsymbol{\ell}\rangle\,. \tag{3.42}$$

We denote by Σ_1 the restriction of ρ_ψ to the full matrix algebra $M_1 := M_{D^n}(\mathbb{C})$ and by Σ_2 the restriction to $M_2 := M_{N^2}(\mathbb{C})$. It follows that:

$$\mathop{\mathrm{Tr}}_{\mathcal{H}_{D^n}}(\Sigma_1\cdot m_1) = \langle\psi|\,m_1\otimes\mathbb{1}_2\,|\psi\rangle = \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2}\langle g_{\boldsymbol{\ell}}|\,m_1\,|g_{\boldsymbol{\ell}}\rangle\ ,\ \forall m_1\in M_1\ \cdot$$

Thus, from (3.38),

$$\Sigma_1 = \rho_{\widetilde{\mathcal{Y}}}^{[0,n-1]} = \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2}|g_{\boldsymbol{\ell}}(n)\rangle\langle g_{\boldsymbol{\ell}}(n)|\ \cdot \tag{3.43}$$

On the other hand, from

$$\begin{aligned}\mathop{\mathrm{Tr}}_{\mathcal{H}_N^D}(\Sigma_2\cdot m_2) &= \langle\psi|\,\mathbb{1}_1\otimes m_2\,|\psi\rangle\\ &= \sum_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2\in(\mathbb{Z}/N\mathbb{Z})^2}\langle g_{\boldsymbol{\ell}_2}(n)|\,g_{\boldsymbol{\ell}_1}(n)\rangle\langle\boldsymbol{\ell}_2|m_2|\boldsymbol{\ell}_1\rangle\ ,\ \forall m_2\in M_2\ ,\end{aligned}$$

it turns out that $\Sigma_2 = \mathcal{G}(n)$, whence the result follows from Araki–Lieb's inequality [50]■

3.4.1. A simpler form for the $\{T_\alpha\}$ subfamily of the UMG

We now return to the explicit computation of the density matrix $\mathcal{G}(n)$ in Proposition 3.4.1. By using the transposed matrix T_α^{tr}, the vectors (3.39) now read

$$\langle\boldsymbol{i}|\,g_{\boldsymbol{\ell}}(n)\rangle = \frac{1}{N\,D^{\frac{n}{2}}}e^{\frac{2\pi i}{N}\,\boldsymbol{\ell}\cdot\boldsymbol{f}_{\Lambda,\alpha}^{(n),N}(\boldsymbol{i})} \tag{3.44}$$

$$\boldsymbol{f}_{\Lambda,\alpha}^{(n),N}(\boldsymbol{i}) := \sum_{p=0}^{n-1}\left(T_\alpha^{\mathrm{tr}}\right)^p\boldsymbol{r}_{i_p}\quad(\mathrm{mod}\ N) \tag{3.45}$$

where we made explicit the various dependencies of (3.45) on n the time–step, N the inverse lattice–spacing, the chosen set Λ of $\boldsymbol{r}_j$'s and the α parameter of the dynamics in $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})^2$.

In the following we shall use the equivalence classes

$$[\boldsymbol{r}] := \left\{ \boldsymbol{i} \in \Omega_D^{(n)} \;\middle|\; \boldsymbol{f}_{\Lambda,\alpha}^{(n),N}(\boldsymbol{i}) \equiv \boldsymbol{r} \in (\mathbb{Z}/N\mathbb{Z})^2 \pmod N \right\} , \tag{3.46}$$

their cardinalities $\#[\boldsymbol{r}]$ and, in particular, the frequency function $\nu_{\Lambda,\alpha}^{(n),N}$

$$(\mathbb{Z}/N\mathbb{Z})^2 \ni \boldsymbol{r} \longmapsto \nu_{\Lambda,\alpha}^{(n),N}(\boldsymbol{r}) := \frac{\#[\boldsymbol{r}]}{D^n} . \tag{3.47}$$

Proposition 3.4.2

The Von Neumann entropy of the refined (exponential) partition of unit up to time $n-1$ is given by:

$$H_{\tau_N}\left[\widetilde{\mathcal{Y}}^{[0,n-1]}\right] = - \sum_{\boldsymbol{r}\in(\mathbb{Z}/N\mathbb{Z})^2} \nu_{\Lambda,\alpha}^{(n),N}(\boldsymbol{r}) \log \nu_{\Lambda,\alpha}^{(n),N}(\boldsymbol{r}) \tag{3.48}$$

Proof of Proposition 3.4.2:

Using (3.44), the matrix $\mathcal{G}(n)$ in Proposition 3.4.1 can be written as:

$$\mathcal{G}(n) = \frac{1}{D^n} \sum_{\boldsymbol{i}\in\Omega_D^n} |\, f_{\boldsymbol{i}}(n)\rangle \langle f_{\boldsymbol{i}}(n)| , \tag{3.49}$$

$$\langle \boldsymbol{\ell} \,|\, f_{\boldsymbol{i}}(n)\rangle = \frac{1}{N} e^{\frac{2\pi i}{N} \boldsymbol{f}_{\Lambda,\alpha}^{(n),N}(\boldsymbol{i})\cdot\boldsymbol{\ell}} \tag{3.50}$$

The vectors $|\, f_{\boldsymbol{i}}(n)\rangle \in \mathcal{H}_N^D = \mathbb{C}^{N^2}$ are such that $\langle f_{\boldsymbol{i}}(n)|\, f_{\boldsymbol{j}}(n)\rangle = \delta^{(N)}_{\boldsymbol{f}_{\Lambda,\alpha}^{(n),N}(\boldsymbol{i}) ,\, \boldsymbol{f}_{\Lambda,\alpha}^{(n),N}(\boldsymbol{j})}$, where with $\delta^{(N)}$ is the N–periodic Kronecker delta. For sake of simplicity, we say that $|\, f_{\boldsymbol{i}}(n)\rangle$ belongs to the equivalence class $[\boldsymbol{r}]$ in (3.46) if $\boldsymbol{i} \in [\boldsymbol{r}]$; vectors in different equivalence classes are thus orthogonal, whereas those in a same equivalence class $[\boldsymbol{r}]$ are such that

$$\begin{aligned}
\langle \boldsymbol{\ell}_1 | \left(\sum_{\boldsymbol{i}\in[\boldsymbol{r}]} |\, f_{\boldsymbol{i}}(n)\rangle \langle f_{\boldsymbol{i}}(n)| \right) |\, \boldsymbol{\ell}_2\rangle &= \frac{1}{N^2} \sum_{\boldsymbol{i}\in[\boldsymbol{r}]} e^{\frac{2\pi i}{N} \boldsymbol{f}_{\Lambda,\alpha}^{(n),N}(\boldsymbol{i})\cdot(\boldsymbol{\ell}_1-\boldsymbol{\ell}_2)} \\
&= D^n \, \nu_{\Lambda,\alpha}^{(n),N}(\boldsymbol{r}) \;\; \langle \boldsymbol{\ell}_1 |\, \boldsymbol{e}(\boldsymbol{r})\rangle \, \langle \boldsymbol{e}(\boldsymbol{r}) |\, \boldsymbol{\ell}_2\rangle \\
\langle \boldsymbol{\ell} |\, \boldsymbol{e}(\boldsymbol{r})\rangle &= \frac{e^{\frac{2\pi i}{N} \boldsymbol{r}\cdot\boldsymbol{\ell}}}{N} \in \mathcal{H}_N^D
\end{aligned}$$

Therefore, the result follows from the spectral decomposition

$$\mathcal{G}(n) = \sum_{\boldsymbol{r}\in(\mathbb{Z}/N\mathbb{Z})^2} \nu_{\Lambda,\alpha}^{(n),N}(\boldsymbol{r}) \, |\boldsymbol{e}(\boldsymbol{r})\rangle \langle \boldsymbol{e}(\boldsymbol{r})| \cdot$$

■

Chapter 4

Classical/Continuous Limit of Quantum Dynamical Entropies

Proposition 3.3.2 confirms the intuition that finite dimensional, discrete time, quantum dynamical systems, however complicated the distribution of their quasi-energies might be, cannot produce enough information over large times to generate a non-vanishing entropy per unit time. This is due to the fact that, despite the presence of almost random features over finite intervals, the time evolution cannot bear random signatures if watched long enough, because almost periodicity would always prevail asymptotically.

However, this does not mean that the dynamics may not be able to show a significant entropy rate over finite interval of times, these being typical of the underlying dynamics; all this Chapter is devoted to explore this phenomenon.

As already observed in the Introduction, in quantum chaos one deals with quantized classically chaotic systems; there, one finds that classical and quantum mechanics are both correct descriptions over times scaling with $\log \hbar^{-1}$. Therefore, the classical–quantum correspondence occurs over times much smaller than the Heisenberg recursion time that typically scales as $\hbar^{-\alpha}$, $\alpha > 0$. In other words, for quantized classically chaotic systems, the classical description has to be replaced by the quantum one much sooner than for integrable systems.

4.1. CNT and ALF Entropies on $\big(\mathcal{M}_N, \tau_N, \Theta_N\big)$

In this section we take the the CNT and the ALF-entropy as good indicators of the degree of randomness of a quantum dynamical system. Then, we show that underlying classical chaos plus Hilbert space finiteness make a characteristic logarithmic time scale emerge over which these systems can be called chaotic.

4.1.1. CNT-entropy

Theorem 8 : Let $\big(\mathcal{X}, \mu, T\big)$ be a classical dynamical system which is the classical limit of a sequence of finite dimensional quantum dynamical systems $\big(\mathcal{M}_N, \tau_N, \Theta_N\big)$. We also assume that the dynamical localization condition 2.4.1 holds. If

1. $\mathcal{C} = \{C_0, C_1, \ldots, C_{D-1}\}$ is a finite measurable partition of $\mathcal{X}$,
2. $\mathcal{N}_\mathcal{C} \subset L^\infty_\mu(\mathcal{X})$ is the finite dimensional subalgebra generated by the characteristic functions χ_{C_j} of the atoms of $\mathcal{C}$,
3. $\imath_{\mathcal{N}_\mathcal{C}}$ is the natural embedding of $\mathcal{N}_\mathcal{C}$ into $L^\infty_\mu(\mathcal{X})$, $\mathcal{J}_{N\infty}$ the anti-Wick quantization map and

$$\gamma^\ell_\mathcal{C} := \Theta^\ell_N \circ \mathcal{J}_{N\infty} \circ \imath_{\mathcal{N}_\mathcal{C}}, \quad \ell = 0, 1, \ldots, k-1,$$

then there exists an α such that

$$\lim_{\substack{k,\ N\to\infty \\ k\le\alpha\log N}} \frac{1}{k}\Big|H_{\tau_N}(\gamma^0_\mathcal{C}, \gamma^1_\mathcal{C}, \ldots, \gamma^{k-1}_\mathcal{C}) - S_\mu\big(\mathcal{C}^{[0,k-1]}\big)\Big| = 0.$$

Proof of Theorem 8:

We split the proof in two parts:

1. We relate the quantal evolution $\gamma^\ell_\mathcal{C} = \Theta^\ell_N \circ \mathcal{J}_{N\infty} \circ \imath_{\mathcal{N}_\mathcal{C}}$ to the classical evolution $\tilde{\gamma}^\ell_\mathcal{C} := \mathcal{J}_{N\infty} \circ \Theta^\ell \circ \imath_{\mathcal{N}_\mathcal{C}}$ using the continuity property of the entropy functional.

2. We find an upper and a lower bound to the entropy functional that converge to the KS-entropy in the long time limit.

We define for convenience the algebra $\mathcal{N}^\ell_\mathcal{C} := \Theta^\ell(\mathcal{N}_\mathcal{C})$ and the algebra $\mathcal{N}^{[0,k-1]}_\mathcal{C}$ corresponding to the refinements $\mathcal{C}^{[0,k-1]} = \bigvee_{\ell=0}^{k-1} T^{-\ell}(\mathcal{C})$ which consist of atoms $\mathcal{C}_{\boldsymbol{i}} := \bigcap_{\ell=0}^{k-1} T^{-\ell}(C_{i_\ell})$

labeled by the multi-indices $\boldsymbol{i} = (i_0, i_1, \dots, i_{k-1})$. Thus the algebra $\mathcal{N}_{\mathcal{C}}^{[0,k-1]}$ is generated by the characteristic functions $\chi_{C_{\boldsymbol{i}}}$.

Step 1

The maps $\gamma_{\mathcal{C}}^{\ell}$ and $\tilde{\gamma}_{\mathcal{C}}^{\ell}$ connect the quantum and classical time evolution. Indeed, using Proposition 2.4.1

$$k \le \alpha \log N \Rightarrow \|\Theta_N^k \circ \mathcal{J}_{N\infty} \circ \imath_{\mathcal{N}_{\mathcal{C}}}(f) - \mathcal{J}_{N\infty} \circ \Theta^k \circ \imath_{\mathcal{N}_{\mathcal{C}}}(f)\|_2 \le \varepsilon,$$

or

$$k \le \alpha \log N \Rightarrow \|\gamma_{\mathcal{C}}^k - \tilde{\gamma}_{\mathcal{C}}^k\|_2 \le \varepsilon$$

This in turn implies, due to strong continuity,

$$\left| H_{\tau_N}(\gamma_{\mathcal{C}}^0, \gamma_{\mathcal{C}}^1, \dots, \gamma_{\mathcal{C}}^{k-1}) - H_{\tau_N}(\tilde{\gamma}_{\mathcal{C}}^0, \tilde{\gamma}_{\mathcal{C}}^1, \dots, \tilde{\gamma}_{\mathcal{C}}^{k-1}) \right| \le k\delta(\varepsilon)$$

with $\delta(\varepsilon) > 0$ depending on the dimension of the space $\mathcal{N}_{\mathcal{C}}$ and vanishing when $\varepsilon \to 0$. From now on we can concentrate on the classical evolution and benefit from its properties.

Step 2, upper bound

We now show that

$$H_{\tau_N}(\tilde{\gamma}_{\mathcal{C}}^0, \tilde{\gamma}_{\mathcal{C}}^1, \dots, \tilde{\gamma}_{\mathcal{C}}^{k-1}) \le S_\mu(\mathcal{C}^{[0,k-1]}).$$

Notice that we can embed $\mathcal{N}_{\mathcal{C}}^{\ell}$ into $L_\mu^\infty(\mathcal{X})$ by first embedding it into $\mathcal{N}_{\mathcal{C}}^{[0,k-1]}$ with $\imath_{\mathcal{N}_{\mathcal{C}}^{[0,k-1]}\mathcal{N}_{\mathcal{C}}^{\ell}}$ and then embedding $\mathcal{N}_{\mathcal{C}}^{[0,k-1]}$ into $L_\mu^\infty(\mathcal{X})$ with $\imath_{\mathcal{N}_{\mathcal{C}}^{[0,k-1]}}$:

$$\imath_{\mathcal{N}_{\mathcal{C}}^{\ell}} = \imath_{\mathcal{N}_{\mathcal{C}}^{[0,k-1]}} \circ \imath_{\mathcal{N}_{\mathcal{C}}^{[0,k-1]}\mathcal{N}_{\mathcal{C}}^{\ell}}.$$

We now estimate:

$$\begin{aligned}
& H_{\tau_N}(\tilde{\gamma}_{\mathcal{C}}^0, \tilde{\gamma}_{\mathcal{C}}^1, \cdots, \tilde{\gamma}_{\mathcal{C}}^{k-1}) = \\
& \quad = H_{\tau_N}(\mathcal{J}_{N\infty} \circ \Theta^0 \circ \imath_{\mathcal{N}_{\mathcal{C}}}, \cdots, \mathcal{J}_{N\infty} \circ \Theta^{k-1} \circ \imath_{\mathcal{N}_{\mathcal{C}}}) = \\
& \quad = H_{\tau_N}(\mathcal{J}_{N\infty} \circ \imath_{\mathcal{N}_{\mathcal{C}}^0}, \cdots, \mathcal{J}_{N\infty} \circ \imath_{\mathcal{N}_{\mathcal{C}}^{k-1}}) = \\
& \quad = H_{\tau_N}(\mathcal{J}_{N\infty} \circ \imath_{\mathcal{N}_{\mathcal{C}}^{[0,k-1]}} \circ \imath_{\mathcal{N}_{\mathcal{C}}^{[0,k-1]}\mathcal{N}_{\mathcal{C}}^0}, \cdots, \mathcal{J}_{N\infty} \circ \imath_{\mathcal{N}_{\mathcal{C}}^{[0,k-1]}} \circ \imath_{\mathcal{N}_{\mathcal{C}}^{[0,k-1]}\mathcal{N}_{\mathcal{C}}^{k-1}}) \le \\
& \quad \le H_{\tau_N}(\mathcal{J}_{N\infty} \circ \imath_{\mathcal{N}_{\mathcal{C}}^{[0,k-1]}}, \cdots, \mathcal{J}_{N\infty} \circ \imath_{\mathcal{N}_{\mathcal{C}}^{[0,k-1]}}) \le
\end{aligned}$$

$$\leq H_{\tau_N}(\mathcal{J}_{N\infty} \circ \imath_{\mathcal{N}_{\mathcal{C}}^{[0,k-1]}}) \leq$$
$$\leq S\left(\tau_N \circ \mathcal{J}_{N\infty} \circ \imath_{\mathcal{N}_{\mathcal{C}}^{[0,k-1]}}\right) . \tag{4.1}$$

The first inequality follows from monotonicity of the entropy functional, the second from invariance under repetitions (see (3.19)) and the third from boundedness in terms of von Neumann entropies. The state $\tau_N \circ \mathcal{J}_{N\infty} \circ \imath_{\mathcal{N}_{\mathcal{C}}^{[0,k-1]}}$ takes the values

$$\tau_N\big(\mathcal{J}_{N\infty}(\chi_{C_{\boldsymbol{i}}})\big) = \tau_N\Big(N \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, \chi_{C_{\boldsymbol{i}}}(\boldsymbol{x})\, |C_N^1(\boldsymbol{x})\rangle\langle C_N^1(\boldsymbol{x})|\Big)$$
$$= \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, \chi_{C_{\boldsymbol{i}}}(\boldsymbol{x})\, \langle C_N^1(\boldsymbol{x}), C_N^1(\boldsymbol{x})\rangle = \omega_\mu(\chi_{C_{\boldsymbol{i}}}) = \mu(C_{\boldsymbol{i}}).$$

This gives, together with $S(\mu(C_{\boldsymbol{i}})) = S_\mu(\mathcal{C}^{[0,k-1]})$, the desired upper bound.

Step 2, lower bound

We show that $\forall\, \varepsilon > 0$ there exists an N' such that

$$H_{\tau_N}(\tilde{\gamma}_{\mathcal{C}}^0, \tilde{\gamma}_{\mathcal{C}}^1, \ldots, \tilde{\gamma}_{\mathcal{C}}^{k-1}) \geq S_\mu(\mathcal{C}^{[0,k-1]}) - k\varepsilon$$

will holds eventually for $N > N'$.
As $H_{\tau_N}(\tilde{\gamma}_{\mathcal{C}}^0, \tilde{\gamma}_{\mathcal{C}}^1, \ldots, \tilde{\gamma}_{\mathcal{C}}^{k-1})$ is defined as a supremum over decompositions of the state τ_N, we can construct a lower bound by picking a good decomposition. Consider the decomposition $\tau_N = \sum_{\boldsymbol{i}} \mu_{\boldsymbol{i}}\, \omega_{\boldsymbol{i}}$ with

$$\begin{cases} \omega_{\boldsymbol{i}} : \quad \mathcal{M}_N \ni x \mapsto \omega_{\boldsymbol{i}}(x) := \dfrac{\tau_N\left[(\mathcal{J}_{N\infty}(\chi_{C_{\boldsymbol{i}}}))(x)\right]}{\tau_N\left[\mathcal{J}_{N\infty}(\chi_{C_{\boldsymbol{i}}})\right]} \\ \mu_{\boldsymbol{i}} := \tau_N\left[\mathcal{J}_{N\infty}(\chi_{C_{\boldsymbol{i}}})\right] \qquad = \mu\left(C_{\boldsymbol{i}}\right) \end{cases}$$

and the subdecompositions $\tau_N = \sum_{j_\ell} \mu_{j_\ell}^\ell\, \omega_{j_\ell}^\ell, \quad \ell = 0, 1, \ldots, k-1$, with[1]

$$\begin{cases} \omega_{j_\ell}^\ell : \quad \mathcal{M}_N \ni x \mapsto \omega_{j_\ell}^\ell(x) := \dfrac{\tau_N\left[(\mathcal{J}_{N\infty}(\chi_{T^{-\ell}(C_{j_\ell})}))(x)\right]}{\tau_N\left[\mathcal{J}_{N\infty}(\chi_{C_{j_l}})\right]} \\ \mu_{j_\ell}^\ell := \tau_N\left[\mathcal{J}_{N\infty}(\chi_{C_{j_\ell}})\right] \qquad = \mu\left(C_{j_\ell}\right) \end{cases}$$

In comparison with (3.16), it is not necessary to go to the commutant for one can use the ciclicity property of the trace[2]. We then have:

$$H_{\tau_N}(\tilde{\gamma}^0_{\mathcal{C}}, \tilde{\gamma}^1_{\mathcal{C}}, \ldots, \tilde{\gamma}^{k-1}_{\mathcal{C}}) \geq S_\mu(\mathcal{C}^{[0,k-1]}) - \sum_{\ell=0}^{k-1} \sum_{i_\ell \in I_\ell} \mu^\ell_{i_\ell} \, S(\omega^\ell_{i_\ell} \circ \tilde{\gamma}^\ell_{\mathcal{C}}).$$

The inequality stems from the fact that $H_{\tau_N}(\tilde{\gamma}^0_{\mathcal{C}}, \tilde{\gamma}^1_{\mathcal{C}}, \ldots, \tilde{\gamma}^{k-1}_{\mathcal{C}})$ is a supremum, whereas the middle terms in the original definition of the entropy functional in (3.18) drop out because they are equal in magnitude but opposite in sign[3]. For $s = 0, 1, \ldots, D-1$, $\omega^\ell_{i_\ell} \circ \tilde{\gamma}^\ell_{\mathcal{C}}$ takes on the values

$$\omega^\ell_{i_\ell} \left[\mathcal{J}_{N\infty}(\chi_{T^{-\ell}(C_s)}) \right] = \left(\mu^\ell_{i_\ell} \right)^{-1} \tau_N \left[\mathcal{J}_{N\infty}(\chi_{T^{-\ell}(C_{i_\ell})}) \, \mathcal{J}_{N\infty}(\chi_{T^{-\ell}(C_s)}) \right] .$$

[1]In the derivation of following formulae we extensively use the relation

$$\sum_{\substack{\boldsymbol{i} \\ i_\ell \text{ fixed}}} \chi_{C_{\boldsymbol{i}}}(\boldsymbol{y}) = \chi_{T^{-\ell}(C_{i_\ell})}(\boldsymbol{y}) \ \cdot$$

[2]$\mathcal{M}_N \ni x \geqslant 0 \Longrightarrow x = zz^*$ for some $z \in \mathcal{M}_N$; then it follows $\omega_{\boldsymbol{i}}(x) = \tau_N(y_{\boldsymbol{i}} z z^*) = \tau_N(z^* y_{\boldsymbol{i}} z) \geqslant 0$, for $\mathcal{M}_N \ni y_{\boldsymbol{i}} := \mathcal{J}_{N\infty}(\chi_{C_{\boldsymbol{i}}})$ that is obviously greater than zero; indeed for all $|\psi\rangle \in \mathcal{H}_N$ we have that $\langle \psi \,| \mathcal{J}_{N\infty}(\chi_{C_{\boldsymbol{i}}}) | \, \psi \rangle = N \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x}) \, \chi_{C_{\boldsymbol{i}}}(\boldsymbol{x}) \left| \langle \psi | \, C^1_N(\boldsymbol{x}) \rangle \right|^2 \geqslant 0$.

[3]Indeed $S(\tau_N \circ \tilde{\gamma}^\ell_{\mathcal{C}}) = \sum_{i_\ell \in I_\ell} \eta(\mu^\ell_{i_\ell}) \ \cdot$

Due to Proposition 1.5.2, these converge to $(\mu(C_{i_\ell}))^{-1}\omega_\mu(\chi_{T^{-\ell}(C_{i_\ell})}\,\chi_{T^{-\ell}(C_s)}) = \delta_{s,i_\ell}$. This means that in the limit the von Neumann entropy will be zero. Or stated more carefully:

$$N' := \max_{s\in\{0,1,\cdots,D-1\}} \{N_s\}$$

↑ We can determine N' by:

$$\forall\, s \in \{0,1,\cdots,D-1\}\ \exists\, N_s \in \mathbb{N} \text{ s.t. } N > N_s \Longrightarrow \left|\mu^\ell_{i_\ell}\left(\omega^\ell_{i_\ell}\circ\tilde{\gamma}^\ell_{\mathcal{C}}\right)(s) - \mu^\ell_{i_\ell}\delta_{s,i_\ell}\right| < \delta_{\varepsilon'}\mu^\ell_{i_\ell}$$

↑ In correspondence to that $\delta_{\varepsilon'}$

Uniform continuity of $\eta(x)$ function on $[0,1]$ guarantees:

$$\forall\,\varepsilon' > 0,\ \exists\,\delta_{\varepsilon'} > 0 \text{ s.t. } \left|\left(\omega^\ell_{i_\ell}\circ\tilde{\gamma}^\ell_{\mathcal{C}}\right)(s) - \delta_{s,i_\ell}\right| < \delta_{\varepsilon'} \Longrightarrow$$

$$\Longrightarrow \left|\eta\left[\left(\omega^\ell_{i_\ell}\circ\tilde{\gamma}^\ell_{\mathcal{C}}\right)(s)\right] - \eta\left[\delta_{s,i_\ell}\right]\right| < \frac{\varepsilon'}{D}\ ,\ \forall\, s \in \{0,1,\cdots,D-1\}$$

↓ summing over $s\in\{0,1,\cdots,D-1\}$

$$S(\omega^\ell_{i_\ell}\circ\tilde{\gamma}^\ell_{\mathcal{C}}) \leqslant \varepsilon'$$

↓ that is

$$\sum_{\ell=0}^{k-1}\sum_{i_\ell\in I_\ell}\mu^\ell_{i_\ell}\,S(\omega^\ell_{i_\ell}\circ\tilde{\gamma}^\ell_{\mathcal{C}}) \leqslant k\varepsilon' \tag{4.1}$$

We thus obtain a lower bound.

Combining our results and choosing $\tilde{N} := \max(N, N')$, we conclude

$$S_\mu(\mathcal{C}^{[0,k-1]}) - k\varepsilon' - k\delta(\varepsilon) \le H_{\tau_N}(\gamma^0_{\mathcal{C}}, \gamma^1_{\mathcal{C}}, \ldots, \gamma^{k-1}_{\mathcal{C}}) \le S_\mu(\mathcal{C}^{[0,k-1]}) + k\delta(\varepsilon)\ \cdot \qquad \blacksquare$$

4.1.2. ALF-entropy

Theorem 9 : Let $(\mathcal{X}, \mu, T)$ be a classical dynamical system which is the classical limit of a sequence of finite dimensional quantum dynamical systems $(\mathcal{M}_N, \tau_N, \Theta_N)$. We also assume that the dynamical localization condition 2.4.1 holds. If

1. $\mathcal{C} = \{C_0, C_1, \ldots, C_{D-1}\}$ is a finite measurable partition of $\mathcal{X}$,
2. $\mathcal{Y}_N = \{y_0, y_1, \ldots, y_D\}$ is a bistochastic partition of unity, which is the quantization of the previous partition, namely $y_i = \mathcal{J}_{N\infty}(\chi_{C_i})$ for $i \in \{0, 1, \cdots, D-1\}$ and $y_D := \sqrt{\mathbb{1} - \sum_{i=0}^{D-1} y_i^* y_i}$,

then there exists an α such that

$$\lim_{\substack{k,N\to\infty \\ k\leq\alpha\log N}} \frac{1}{k} \left| H_{\tau_N}[\mathcal{Y}^{[0,k-1]}] - S_\mu(\mathcal{C}^{[0,k-1]}) \right| = 0.$$

Proof of Theorem 9:

First notice that $\mathcal{Y}_N = \{y_0, y_1, \ldots, y_D\}$ is indeed a bistochastic partition. We have

$$\begin{aligned} y_i^* &= \mathcal{J}_{N\infty}(\chi_{C_i})^* = \mathcal{J}_{N\infty}(\overline{\chi}_{C_i}) = \mathcal{J}_{N\infty}(\chi_{C_i}) = y_i \\ 0 &\leq \mathcal{J}_{N\infty}(\chi_{C_i})^2 = y_i^2 \leq \gamma_{N\infty}(\chi_{C_i}^2) = \gamma_{N\infty}(\chi_{C_i}) \end{aligned}$$

Summing the last line over i from 0 to $D-1$, we see that $\sum_{i=0}^{D-1} y_i^2 \leq \mathbb{1}$, This means that $\{y_0, y_1, \ldots, y_{D-1}\}$ is not a partition of unity, but we can use this property to define an extra element y_D which completes it to a bistochastic partition of unity, $\mathcal{Y}_N = \{y_0, y_1, \ldots, y_D\}$:

$$y_D := \sqrt{\mathbb{1} - \sum_{i=0}^{D-1} y_i^* y_i}$$

The bistochasticity is a useful property because it implies translation invariance of the state on the quantum spin chain, state which arises during the construction of the ALF-entropy.

The density matrix $\rho[\mathcal{Y}^{[0,k-1]}]$ of the refined partition reads (see (3.29))

$$\begin{aligned} \rho\left[\mathcal{Y}^{[0,k-1]}\right] &= \sum_{\boldsymbol{i},\boldsymbol{j}} \rho\left[\mathcal{Y}^{[0,k-1]}\right]_{\boldsymbol{i},\boldsymbol{j}} |e_{\boldsymbol{i}}\rangle\langle e_{\boldsymbol{j}}| \\ &= \sum_{\boldsymbol{i},\boldsymbol{j}} \tau_N\Big(y_{j_1}^* \Theta_N(y_{j_1}^*) \cdots \Theta_N^{k-1}(y_{j_k}^*) \Theta_N^{k-1}(y_{i_k}) \cdots \Theta_N(y_{i_1}) y_{i_1}\Big) |e_{\boldsymbol{i}}\rangle\langle e_{\boldsymbol{j}}| \qquad (4.2) \end{aligned}$$

Now we will expand this formula using the operators y_i defined above, the quantities $K_\ell(\boldsymbol{x},\boldsymbol{y})$ defined in (2.38) and controlling the element y_D as follows:

$$\begin{aligned}\|y_D\|_2^2 &= \Big\|\sqrt{\mathbb{1}-\sum_{i=0}^{D-1} y_i^* y_i}\Big\|_2^2 = \tau_N\Big(\mathbb{1}-\sum_{i=0}^{D-1} y_i^* y_i\Big)\\ &= \tau_N\Big(\sum_{i=0}^{D-1} y_i^*\,(\mathbb{1}-y_i)\Big)\\ &= \int \mu\,(\mathrm{d}\boldsymbol{y})\,\mu\,(\mathrm{d}\boldsymbol{z}) \sum_{\substack{i,j=0\\ i\neq j}}^{D-1} \chi_i(\boldsymbol{y})\,\chi_j(\boldsymbol{z})\,N|K_0(\boldsymbol{y},\boldsymbol{z})|^2. \end{aligned} \tag{4.3}$$

Thus, in the limit of large N, $N|K_0(\boldsymbol{y},\boldsymbol{z})|^2$ is just $\delta(\boldsymbol{y}-\boldsymbol{z})$ (see (2.38)) so that (4.3) tends to $\int \mu\,(\mathrm{d}\boldsymbol{z}) \sum_{i\neq j} \chi_i(\boldsymbol{z})\,\chi_j(\boldsymbol{z}) = 0$ and we can consistently neglect those entries of $\rho[\mathcal{Y}^{[0,k-1]}]$ containing y_D.

By means of the properties of coherent states, we write out explicitly[4] the elements of the density matrix in (4.2)

$$\begin{aligned}\rho\left[\mathcal{Y}^{[0,k-1]}\right]_{\boldsymbol{i},\boldsymbol{j}} &= \tau_N\Big(y_{j_1}^*\Theta_N(y_{j_1}^*)\cdots\Theta_N^{k-1}(y_{j_k}^*)\Theta_N^{k-1}(y_{i_k})\cdots\Theta_N(y_{i_1})y_{i_1}\Big)\\ &= \tau_N\Big(\,y_{j_1}^*\,U_T\,y_{j_1}^*\,U_T\,\cdots\,U_T\,y_{j_k}^*\,y_{i_k}\,U_T^*\,\cdots\,U_T^*\,y_{i_1}\,U_T^*\,y_{i_1}\Big)\\ &= N^{2k-1}\int\left(\prod_{\ell=1}^{k}\mu\,(\mathrm{d}\boldsymbol{y}_\ell)\,\mu\,(\mathrm{d}\boldsymbol{z}_\ell)\;\chi_{C_{j_\ell}}(\boldsymbol{y}_\ell)\,\chi_{C_{i_\ell}}(\boldsymbol{z}_\ell)\right)\;\times\\ &\times\,K_0(\boldsymbol{z}_1,\boldsymbol{y}_1)\left(\prod_{p=1}^{k-1}K_1(\boldsymbol{y}_p,\boldsymbol{y}_{p+1})\right)\,K_0(\boldsymbol{y}_k,\boldsymbol{z}_k)\left(\prod_{q=1}^{k-1}K_{-1}(\boldsymbol{z}_{k-q+1},\boldsymbol{z}_{k-q})\right)\,. \end{aligned}\tag{4.4}$$

We now use that for N large enough,

$$\Big|N\int\mu\,(\mathrm{d}\boldsymbol{y})\;\chi_C(\boldsymbol{y})\,K_m(\boldsymbol{x},\boldsymbol{y})\,K_n(\boldsymbol{y},\boldsymbol{z})-\chi_{T^{-m}C}(\boldsymbol{x})\,K_{m+n}(\boldsymbol{x},\boldsymbol{z})\Big|\leq\varepsilon_m(N)\,, \tag{4.5}$$

where $\varepsilon_m(N)\to 0$ with $N\to\infty$ uniformly in $\boldsymbol{x},\boldsymbol{z}\in\mathcal{X}$. This is a consequence of the dynamical localization condition 2.4.1 and can be rigorously proven in the same way as Proposition 1.5.1. However, the rough idea is the following: from the property 3.1.3 of

[4]Every elements of the p.u. is written in terms of C.S. as $y_{j_\ell} = \int \mu\,(\mathrm{d}\boldsymbol{x})\,\chi_{C_{j_\ell}}(\boldsymbol{x})\,\big|\,C_N^1\,(\boldsymbol{x})\,\big\rangle\,\big\langle\,C_N^1\,(\boldsymbol{x})\,\big|$: we make use of $\boldsymbol{y}_\ell$ and $\boldsymbol{z}_\ell$ as variables in the integral representation of $y_{j_\ell}^*$, respectively y_{i_ℓ}.

coherent states, one derives

$$
\begin{aligned}
N \int \mu(\mathrm{d}\boldsymbol{y})\, \chi_C(\boldsymbol{y})\, K_m(\boldsymbol{x},\boldsymbol{y})\, K_n(\boldsymbol{y},\boldsymbol{z}) &= \\
&= N \int \mu(\mathrm{d}\boldsymbol{y}) \left[1 + \Big(\chi_C(\boldsymbol{y}) - 1\Big)\right] K_m(\boldsymbol{x},\boldsymbol{y})\, K_n(\boldsymbol{y},\boldsymbol{z}) = \\
&= K_{m+n}(\boldsymbol{x},\boldsymbol{z}) + N \int \mu(\mathrm{d}\boldsymbol{y}) \Big(\chi_C(\boldsymbol{y}) - 1\Big) K_m(\boldsymbol{x},\boldsymbol{y})\, K_n(\boldsymbol{y},\boldsymbol{z}) \cdot
\end{aligned}
\tag{4.6}
$$

For large N we look two cases:

- $\boldsymbol{x} \notin T^{-m}(C)$ – then the condition 2.4.1 makes the integral in (4.5) negligible small, whereas the second term in the l.h.s of the same equation is exactly zero;

- $\boldsymbol{x} \in T^{-m}(C)$ – in this case it is the second integral in formula (4.6) which can be neglected, and using (4.6) in (4.5) we find negligibility.

By applying (4.5) to the couples of products in (4.4) one after the other, noting that every single integral in (4.4) is less or equal to one, and using triangle inequality for $|\cdot|$, we finally arrive at the upper bound

$$
\left| \rho\left[\mathcal{Y}^{[0,k-1]}\right]_{\boldsymbol{i},\boldsymbol{j}} - \delta_{\boldsymbol{i},\boldsymbol{j}}\, \mu(C_{\boldsymbol{i}}) \right| \le \Big(2 \sum_{m=1}^{k-1} \varepsilon_m(N) + \varepsilon_0(N)\Big) =: \epsilon(N),
$$

where $C_{\boldsymbol{i}} := \bigcap_{\ell=1}^{k} T^{-\ell+1} C_{i_\ell}$ is an element of the partition $\mathcal{C}^{[0,k-1]}$.

We now set $\sigma\left[\mathcal{C}^{[0,k-1]}\right] := \sum_{\boldsymbol{i}} \mu(C_{\boldsymbol{i}}) |e_{\boldsymbol{i}}\rangle\langle e_{\boldsymbol{i}}|$ and use the following estimate: let A be an arbitrary matrix of dimension d and let $\{e_1, e_2, \dots, e_d\}$ and $\{f_1, f_2, \dots, f_d\}$ be two orthonormal bases of $\mathbb{C}^d$, then $\|A\|_1 := \mathrm{Tr}\,|A| \le \sum_{i,j} |\langle e_i, A\, f_j\rangle|$. This yields

$$
\Delta(k) := \left\| \rho\left[\mathcal{Y}^{[0,k-1]}\right] - \sigma\left[\mathcal{C}^{[0,k-1]}\right] \right\|_1 = \mathrm{Tr}\left| \rho\left[\mathcal{Y}^{[0,k-1]}\right] - \sigma\left[\mathcal{C}^{[0,k-1]}\right] \right| \le D^{2k} \epsilon(N).
$$

Finally, by the continuity of the von Neumann entropy [55], we get

$$
\left| S\left(\rho\left[\mathcal{Y}^{[0,k-1]}\right]\right) - S\left(\sigma\left[\mathcal{C}^{[0,k-1]}\right]\right) \right| \leqslant \Delta(k) \log D^k + \eta(\Delta(k)) \; .
$$

Since, from $k \le \alpha \log N$, $D^{2k} \leqslant N^{2\alpha \log D}$, if we want the bound $D^{2k}\epsilon(N)$ to converge to zero with $N \to \infty$, the parameter α has to be chosen accordingly. Then, the result follows because the von Neumann entropy of σ reduces to the Shannon entropy of the refinements of the classical partition. ■

4.2. Numerical analysis of ALF Entropies in Discrete Classical Chaos

Here in the following we are considering not the quantization of classical systems, but their discretization; nevertheless, we have seen that, under certain respects, quantization and discretization are like procedures with the inverse of the number of states N playing the role of $\hbar$ in the latter case.

We are then interested to study how the classical continuous behaviour emerges from the discretized one when $N \to \infty$; in particular, we want to investigate the presence of characteristic time scales and of "breaking–times" τ_B, namely those times beyond which the discretized systems cease to produce entropy because their granularity takes over and the dynamics reveals in full its regularity.

Propositions 3.4.1 and 3.4.2 afford useful means to attack such a problem numerically. In the following we shall be concerned with the time behavior of the entropy of partition of units as in Definition 3.4.1, the presence of breaking–times $\tau_B(\Lambda, N, \alpha)$, and their dependence on the set Λ, on the number of states N and on the dynamical parameter α.

As we shall see, in many cases τ_B depends quite heavily on the chosen partition of unit; we shall then try to cook up a strategy to find a τ_B as stable as possible upon variation of partitions, being led by the idea that the "true" τ_B has to be strongly related to the Lyapounov exponent of the underlying continuous dynamical system.

Equations (3.41) and (3.48) allow us to compute the Von Neumann entropy of the state $\rho^{[0,n-1]}_{\widetilde{\mathcal{Y}}}$; if we were to compute the ALF–entropy according to the definitions (3.31), the result would be zero, in agreement with fact that the Lyapounov exponent for a system with a finite number of states vanishes. Indeed, it is sufficient to notice that the entropy $H_{\tau_N}\left[\widetilde{\mathcal{Y}}^{[0,n-1]}\right]$ is bounded from above by the entropy of the tracial state $\dfrac{1}{N^2}\mathbb{1}_{N^2}$, that is by $2\log N$; therefore the expression

$$h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n) := \frac{1}{n} H_{\tau_N}\left[\widetilde{\mathcal{Y}}^{[0,n-1]}\right], \tag{4.7}$$

goes to zero with $n \longrightarrow 0$. It is for this reason that, in the following, we will focus upon the temporal evolution of the function $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n)$ instead of taking its lim sup over the number of iterations n.

In the same spirit, we will not take the supremum of (4.7) over all possible partitions $\widetilde{\mathcal{Y}}$ (originated by different Λ); instead, we will study the dependence of $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n)$ on different choices of partitions. In fact, if we vary over all possible choices of partitions of unit, we could choose $\Lambda = (\mathbb{Z}/N\mathbb{Z})^2$ in (3.34), that is $D = N^2$; then summation over all possible $\boldsymbol{r} \in (\mathbb{Z}/N\mathbb{Z})^2$ would make the matrix elements $\mathcal{G}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2}(n)$ in (3.40) equal to $\frac{\delta_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2}}{N^2}$, whence $H_{\tau_N}\left[\widetilde{\mathcal{Y}}^{[0,n-1]}\right] = 2\log N$.

4.2.1. The case of the $\{T_\alpha\}$ subfamily of the UMG

The maximum of H_{τ_N} is reached when the frequencies (3.47)

$$\nu^{(n),N}_{\Lambda,\alpha} : (\mathbb{Z}/N\mathbb{Z})^2 \mapsto [0,1]$$

become equal to $1/N^2$ over the torus: we will see that this is indeed what happens to the frequencies $\nu^{(n),N}_{\Lambda,\alpha}$ with $n \longrightarrow \infty$. The latter behaviour can be reached in various ways depending on:

- hyperbolic or elliptic regimes, namely on the dynamical parameter α;
- number of elements (D) in the partition Λ;
- mutual location of the D elements $\boldsymbol{r}_i$ in Λ.

For later use we introduce the set of grid points with non–zero frequencies

$$\Gamma^{(n),N}_{\Lambda,\alpha} := \left\{ \frac{\boldsymbol{\ell}}{N} \;\middle|\; \boldsymbol{\ell} \in (\mathbb{Z}/N\mathbb{Z})^2 \;,\; \nu^{(n),N}_{\Lambda,\alpha}(\boldsymbol{\ell}) \neq 0 \right\} \cdot \tag{4.8}$$

4.2.1.1. Hyperbolic regime with D randomly chosen points $\boldsymbol{r}_i$ in $\boldsymbol{\Lambda}$

In the hyperbolic regime corresponding to $\alpha \in \mathbb{Z} \setminus \{-4,-3,-2,-1,0\}$, $\Gamma^{(n),N}_{\Lambda,\alpha}$ tends to increase its cardinality with the number of time–steps n. Roughly speaking, there appear to be two distinct temporal patterns: a first one, during which $\#\left(\Gamma^{(n),N}_{\Lambda,\alpha}\right) \simeq D^n \leqslant N^2$ and almost every $\nu^{(n),N}_{\Lambda,\alpha} \simeq D^{-n}$, followed by a second one characterized by frequencies frozen to $\nu^{(n),N}_{\Lambda,\alpha}(\boldsymbol{\ell}) = \frac{1}{N^2}$, $\forall \boldsymbol{\ell} \in (\mathbb{Z}/N\mathbb{Z})^2$. The second temporal pattern is reached when, during the first one, $\Gamma^{(n),N}_{\Lambda,\alpha}$ has covered the whole lattice and $D^n \simeq N^2$.

From the point of view of the entropies, the first temporal regime is characterized by

$$H_{\tau_N}(\alpha,\Lambda,n) \sim n \cdot \log D \qquad , \qquad h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n) \sim \log D \;,$$

while the second one by

$$H_{\tau_N}(\alpha,\Lambda,n) \sim 2\log N \qquad , \qquad h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n) \sim \frac{2\log N}{n} \;.$$

The transition between these two regimes occurs at $\bar{n} = \log_D N^2$. However this time cannot be considered a realistic breaking–time, as it too strongly depends on the chosen partition.

Figure 4.2 (columns a and c) shows the mechanism clearly in a temperature–like plot: hot points correspond to points of $\Gamma^{(n),N}_{\Lambda,\alpha}$ and their number increases for small numbers of iterations until the plot assume a uniform green color for large n.

The linear and stationary behaviors of $H_{\tau_N}(\alpha,\Lambda,n)$ are apparent in fig. 4.4, where four different plateaus ($2\log N$) are reached for four different N, and in fig. 4.5, in which four different slopes are showed for four different number of elements in the partition. With the same parameters as in fig. 4.5, fig. 4.6 shows the corresponding entropy production $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n)$.

4.2.1.2. Hyperbolic regime with D nearest neighbors r_i in Λ

In the following, we will consider a set of points $\Lambda = \{\boldsymbol{r}_i\}_{i=1\ldots D}$ very close to each other, instances of which are as below:

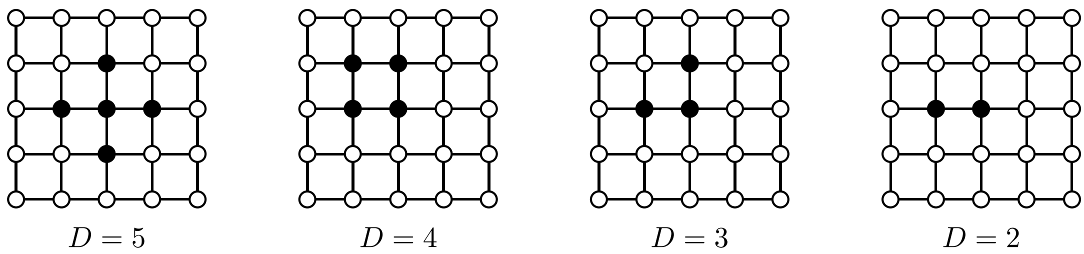

Figure 4.1: Several combinations of D nearest neighbors in Λ for different values D.

From eqs. (3.46–3.47), the frequencies $\nu^{(n),N}_{\Lambda,\alpha}(\boldsymbol{\ell})$ result proportional to how many strings have equal images $\boldsymbol{\ell}$, through the function $\boldsymbol{f}^{(n),N}_{\Lambda,\alpha}$ in (3.45). Due to the fact that $[T_\alpha]_{11} = [T_\alpha]_{21} = 1$, non–injectivity of $\boldsymbol{f}^{(n),N}_{\Lambda,\alpha}$ occurs very frequently when $\{\boldsymbol{r}_i\}$ are very close to each other. This is a dynamical effect that, in continuous systems [49], leads to an entropy production approaching the Lyapounov exponent. Even in the discrete case, during a finite time interval though, $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n)$ exhibits the same behavior until H_{τ_N} reaches the upper bound $2\log N$. From then on, the system behaves as described in subsection 4.2.1.1, and the entropy production goes to zero as:

$$h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n) \sim \frac{1}{n} \quad \text{(see fig. 4.7).}$$

Concerning figure 4.3 (column d), whose corresponding graph for $h_{\tau_N,\mathcal{W}_\infty}(1,\Lambda,n)$ is labeled by $\triangleright$ in fig. 4.7, we make the following consideration:

- for $n = 1$ the red spot corresponds to five $\boldsymbol{r}_i$ grouped as in fig. 4.1.
  In this case $h_{\tau_N,\mathcal{W}_\infty}(1,\Lambda,1) = \log D = \log 5$;

- for $n \in [2,5]$ the red spot begins to stretch along the stretching direction of T_1. In this case, the frequencies $\nu_{\Lambda,\alpha}^{(n),N}$ are not constant on the warm region: this leads to a decrease of $h_{\tau_N,\mathcal{W}_\infty}(1,\Lambda,n)$;

- for $n \in [6,10]$ the warm region becomes so elongated that it starts feeling the folding condition so that, with increasing time–steps, it eventually fully covers the originally pale–blue space. In this case, the behavior of $h_{\tau_N,\mathcal{W}_\infty}(1,\Lambda,n)$ remains the same as before up to $n = 10$;

- for $n = 11$, $\Gamma_{\Lambda,\alpha}^{(n),N}$ coincides with the whole lattice;

- for larger times, the frequencies $\nu_{\Lambda,1}^{(n),N}$ tend to the constant value $\frac{1}{N^2}$ on almost every point of the grid. In this case, the behaviour of the entropy production undergoes a critical change (the crossover occurring at $n = 11$) as showed in fig. 4.7.

Again, we cannot conclude that $n = 11$ is a realistic breaking–time, because once more we have strong dependence on the chosen partition (namely from the number D of its elements). For instance, in fig. 4.7, one can see that partitions with 3 points reach their corresponding "breaking–times" faster than that with $D = 5$; also they do it in an N–dependent way.

For a chosen set Λ consisting of D elements very close to each other and N very large, $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n) \simeq \log\lambda$ (which is the asymptote in the continuous case) from a certain $\bar{n}$ up to a time τ_B. Since this latter is now partition independent, it can properly be considered as the breaking–time of the system; it is given by

$$\tau_B = \log_\lambda N^2. \tag{4.9}$$

> *It is evident from equation* (4.9) *that if one knows* τ_B *then also* $\log\lambda$ *is known. Usually, one is interested in the latter which is a sign of the instability of the continuous classical system. In the following we develop an algorithm which allows us to extract* $\log\lambda$ *from studying the corresponding discretized classical system and its ALF–entropy.*

In working conditions, N is not large enough to allow for $\bar{n}$ being smaller than τ_B; what happens in such a case is that $H_{\tau_N}(\alpha,\Lambda,n)\simeq 2\log N$ before the asymptote for $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n)$ is reached. Given $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n)$ for $n<\tau_B$, it is thus necessary to seek means how to estimate the long time behaviour that one would have if the system were continuous.

Remarks 4.2.1

When estimating Lyapounov exponents from discretized hyperbolic classical systems, by using partitions consisting of nearest neighbors, we have to take into account some facts:

a. $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n)$ does not increase with n; therefore, if $D<\lambda$, $h_{\tau_N,\mathcal{W}_\infty}$ cannot reach the Lyapounov exponent. Denoted by $\boxed{\log\lambda(D)}$ the asymptote that we extrapolate from the data[5], in general we have $\lambda(D)\leqslant\log D<\lambda$. For instance, for $\alpha=1$, $\lambda=2.618\ldots>2$ and partitions with $D=2$ cannot produce an entropy greater then $\log 2$; this is the case for the entropies below the dotted line in figs. 4.5 e 4.6;

b. partitions with D small but greater than λ allow $\log\lambda$ to be reached in a very short time and $\lambda(D)$ is very close to λ in this case;

c. partitions with $D\gg\lambda$ require very long time to converge to $\log\lambda$ (and so very large N) and, moreover, it is not a trivial task to deal with them from a computational point of view. On the contrary the entropy behaviour for such partitions offers very good estimates of λ (compare, in fig. 4.7, ▹ with ◇, △, ○ and □) ;

d. in order to compute λ (and then τ_B, by (4.9)), one can calculate $\lambda(D)$ for increasing D, until it converges to a stable value λ;

e. due to number theoretical reasons, the UMG on $(\mathbb{Z}/N\mathbb{Z})^2$ present several anomalies. An instance of them is showed in fig. 4.3 (col. f), where a partition

[5] $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n)$ may even equal $\log\lambda(D)$ from the start.

with five nearest neighbors on a lattice of 200×200 points confines the image of $\boldsymbol{f}_{\Lambda,\alpha}^{(n),N}$ (under the action of a T_α map with $\alpha = 17$) on a subgrid of the torus. In this and analogous cases, there occurs an anomalous depletion of the entropy production and no significant information is obtainable from it. To avoid this difficulties, in Section 4.2.2 we will go beyond the UMG subclass considered so far and we will include in our analysis the full family of Sawtooth Maps.

4.2.1.3. Elliptic regime ($\alpha \in \{-1, -2, -3\}$)

One can show that all evolution matrices T_α are characterized by the following property:

$$T_\alpha^2 = \bar{\alpha}\, T_\alpha - \mathbb{1} \qquad , \qquad \bar{\alpha} := (\alpha + 2) \ \cdot \tag{4.10}$$

In the elliptic regime $\alpha \in \{-1, -2, -3\}$, therefore $\bar{\alpha} \in \{-1, 0, 1\}$ and relation (4.10) determines a periodic evolution with periods:

$$T_{-1}^{\ 3} = -\mathbb{1} \qquad\qquad (T_{-1}^{\ 6} = \mathbb{1}) \tag{4.11a}$$

$$T_{-2}^{\ 2} = -\mathbb{1} \qquad\qquad (T_{-2}^{\ 4} = \mathbb{1}) \tag{4.11b}$$

$$T_{-3}^{\ 3} = +\mathbb{1} \ \cdot \tag{4.11c}$$

It has to be stressed that, in the elliptic regime, the relations (4.11) <u>do not hold</u> "modulo N", instead they are completely independent from N.

Due to the high degree of symmetry in relations (4.10–4.11), the frequencies $\nu_{\Lambda,\alpha}^{(n),N}$ are different from zero only on a small subset of the whole lattice.

This behavior is apparent in fig. 4.2 : col. b, in which we consider five randomly distributed $\boldsymbol{r}_i$ in Λ, and in fig. 4.3 : col. e, in which the five $\boldsymbol{r}_i$ are grouped as in fig. 4.1. In both cases, the Von Neumann entropy $H_{\tau_N}(n)$ is not linearly increasing with n, instead it assumes a log–shaped profile (up to the breaking–time, see fig. 4.4).

Remark 4.2.2

The last observation indicates how the entropy production analysis can be used to recognize whether a dynamical systems is hyperbolic or not. If we use randomly distributed points as a partition, we observe that hyperbolic systems show constant entropy production (up to the breaking–time), whereas the others do not.

Moreover, unlike hyperbolic ones, elliptic systems do not change their behaviour with N (for reasonably large N) as clearly showed in fig. 4.4, in which elliptic systems ($\alpha = -2$) with four different values of N give the same plot. On the contrary, we have dependence on how rich is the chosen partition, similarly to what we have for hyperbolic systems, as showed in fig. 4.7.

4.2.1.4. Parabolic regime ($\alpha \in \{0, 4\}$)

This regime is characterized by $\lambda = \lambda^{-1} = \pm 1$, that is $\log |\lambda| = 0$ (see Remark 2.1.1, c.). These systems behave as the hyperbolic ones (see subsections 4.2.1.1 and 4.2.1.2) and this is true also for the the general behavior of the entropy production, apart from the fact that we never fall in the condition (a.) of Remark 4.2.1. Then, for sufficiently large N, every partition consisting of D grouped $\boldsymbol{r}_i$ will reach the asymptote $\log |\lambda| = 0$.

4.2.2. The case of Sawtooth Maps

From a computational point of view, the study of the entropy production in the case of Sawtooth Maps S_α is more complicated than for the T_α's. The reason to study numerically these dynamical systems is twofold:

- to avoid the difficulties described in Remark 4.2.1 (e.);
- to deal, in a way compatible with numerical computation limits, with the largest possible spectrum of accessible Lyapounov exponent. We know that

$$\alpha \in \mathbb{Z} \bigcap \{\text{non elliptic domain}\} \Longrightarrow \lambda^{\pm}(T_\alpha) = \lambda^{\pm}(S_\alpha) = \frac{\alpha + 2 \pm \sqrt{(\alpha+2)^2 - 4}}{2} .$$

  In order to fit $\log \lambda_\alpha$ ($\log \lambda_\alpha$ being the Lyapounov exponent corresponding to a given α) via entropy production analysis, we need D elements in the partition (see points b. and c. of Remark 4.2.1) with $D \geqslant \lambda_\alpha$. Moreover, if we were to study the power of our method for <u>different integer</u> values of α we would be forced forced to use very large D, in which case we would need very long computing times in order to evaluate numerically the entropy production $h_{\tau_N, \mathcal{W}_\infty}(\alpha, \Lambda, n)$ in a reasonable interval of times n. Instead, for Sawtooth Maps, we can fix the parameters (N, D, Λ) and study λ_α for α confined in a small domain, but free to assume every real value in that domain.

In the following, we investigate the case of α in the hyperbolic regime with D nearest neighbors $\boldsymbol{r}_i$ in Λ, as done in subsection 4.2.1.2. In particular, figures (4.9–4.12) refer to the following fixed parameters:

$$N = 38 \quad ; \quad n_{\max} = 5 \quad ; \quad D = 5 \quad ;$$

$$\Lambda \; : \; \boldsymbol{r}_1 = \begin{pmatrix} 7 \\ 8 \end{pmatrix} , \; \boldsymbol{r}_2 = \begin{pmatrix} 7 \\ 9 \end{pmatrix} , \; \boldsymbol{r}_3 = \begin{pmatrix} 6 \\ 8 \end{pmatrix} , \; \boldsymbol{r}_4 = \begin{pmatrix} 7 \\ 7 \end{pmatrix} , \; \boldsymbol{r}_5 = \begin{pmatrix} 8 \\ 8 \end{pmatrix} \; ;$$

α : from 0.00 to 1.00 with an incremental step of 0.05.

First, we compute the Von Neumann entropy (3.41) using the (hermitian) matrix $\mathcal{G}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2}(n)$ defined in (3.40). This is actually a diagonalization problem: once that the N^2 eigenvalues $\{\eta_i\}_{i=1}^{N^2}$ are found, then

$$H_{\tau_N}\left[\widetilde{\mathcal{Y}}^{[0,n-1]}\right] = -\sum_{i=1}^{N^2} \eta_i \log \eta_i \; \cdot \tag{4.12}$$

Then, from (4.7), we can determine $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n)$. In the numerical example, the (Λ–dependent) breaking–time occurs after $n = 5$; for this reason we have chosen $n_{\max} = 5$. In fact, we are interested in the region where the discrete system behaves almost as a continuous one.

In figure 4.9, the entropy production is plotted for the chosen set of α's: for very large N (that is close to the continuum limit, in which no breaking–time occurs) all curves (characterized by different α's) would tend to $\log \lambda_\alpha$ with n.

One way to determine the asymptote $\log \lambda_\alpha$ is to fit the decreasing function $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n)$ over the range of data and extrapolate the fit for $n \to \infty$. Of course we can not perform the fit with polynomials, because every polynomial diverges in the $n \to \infty$ limit.

A better strategy is to compactify the time evolution by means of a isomorphic positive function s with bounded range, for instance:

$$\mathbb{N} \ni n \longmapsto s_n := \frac{2}{\pi} \arctan(n-1) \in [0,1] \; \cdot \tag{4.13}$$

Then, for fixed α, in fig. 4.10 we consider $n_{\max}$ points $(s_n \, , \, h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n))$ and extract the asymptotic value of $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n)$ for $n \to \infty$, that is the value of $h_{\tau_N,\mathcal{W}_\infty}\left(\alpha,\Lambda,s^{-1}(t)\right)$ for $t \to 1^-$, as follows.

Given a graph consisting of $m \in \{2,3,\cdots,n_{\max}\}$ points, in our case the first m points of curves as in fig. 4.10, namely

$$\{(s_1 \, , \, h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,1)) \, , (s_2 \, , \, h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,2)) \, , \cdots , (s_m \, , \, h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,m))\} \, ,$$

the data are fit by a Lagrange polynomial $\mathcal{P}^m\,(t)$ (of degree $m-1$)

$$\mathcal{P}^m\,(t) = \sum_{i=1}^{m} P_i\,(t) \tag{4.14a}$$

$$\text{where} \qquad P_i\,(t) = \prod_{\substack{j=1 \\ j\neq i}}^{m} \frac{t-s_j}{s_i-s_j}\, h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,i) \ \cdot \tag{4.14b}$$

The value assumed by this polynomial when $t \to 1^-$ (corresponding to $n \to \infty$) will be the estimate (of degree m) of the Lyapounov exponent, denoted by l^m_α: the higher the value of m, the more accurate the estimate. From (4.14) we get:

$$l^m_\alpha := \mathcal{P}^m\,(t)\Big|_{t=1} = \sum_{i=1}^{m} h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,i) \prod_{\substack{j=1 \\ j\neq i}}^{m} \frac{1-s_j}{s_i-s_j} \ . \tag{4.15}$$

The various l^m_α are plotted in figure 4.11 as functions of m for all considered α. The convergence of l^m_α with m is showed in figure 4.12, together with the theoretical Lyapounov exponent $\log \lambda_\alpha$; as expected, we find that the latter is the asymptote of $\{l^m_\alpha\}_m$ with respect to the polynomial degree m.

The dotted line in fig. 4.10 extrapolates 21 α–curves in compactified time up to $t=1$ using five points in the Lagrange polynomial approximation.

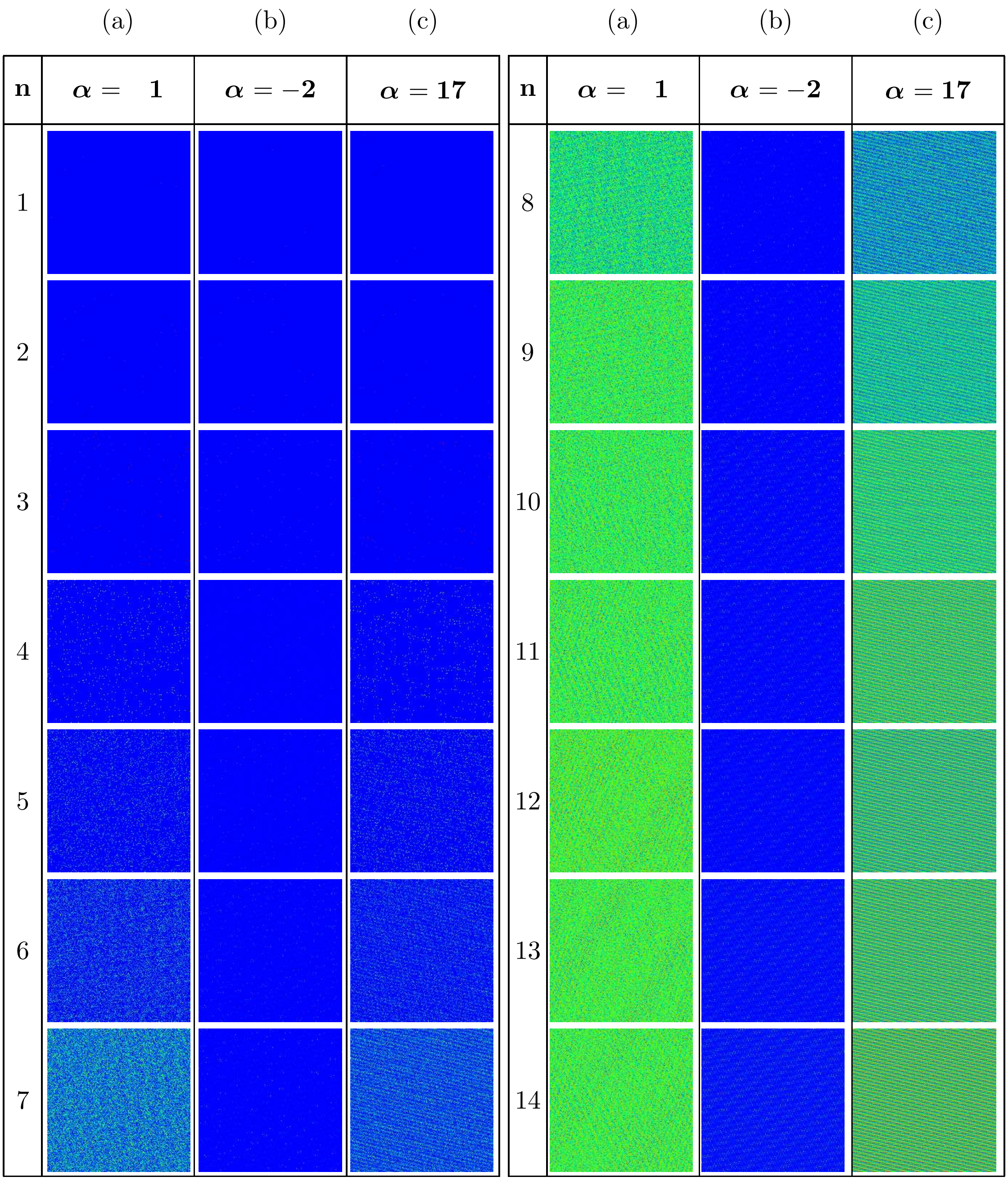

Figure 4.2: Temperature–like plots showing the frequencies $\nu^{(n),N}_{\Lambda,\alpha}$ in two hyperbolic regimes (columns a and c) and an elliptic one (col. b), for five randomly distributed $\boldsymbol{r}_i$ in Λ with $N = 200$. Pale–blue corresponds to $\nu^{(n),N}_{\Lambda,\alpha} = 0$. In the hyperbolic cases, $\nu^{(n),N}_{\Lambda,\alpha}$ tends to equidistribute on $(\mathbb{Z}/N\mathbb{Z})^2$ with increasing n and becomes constant when the breaking–time is reached.

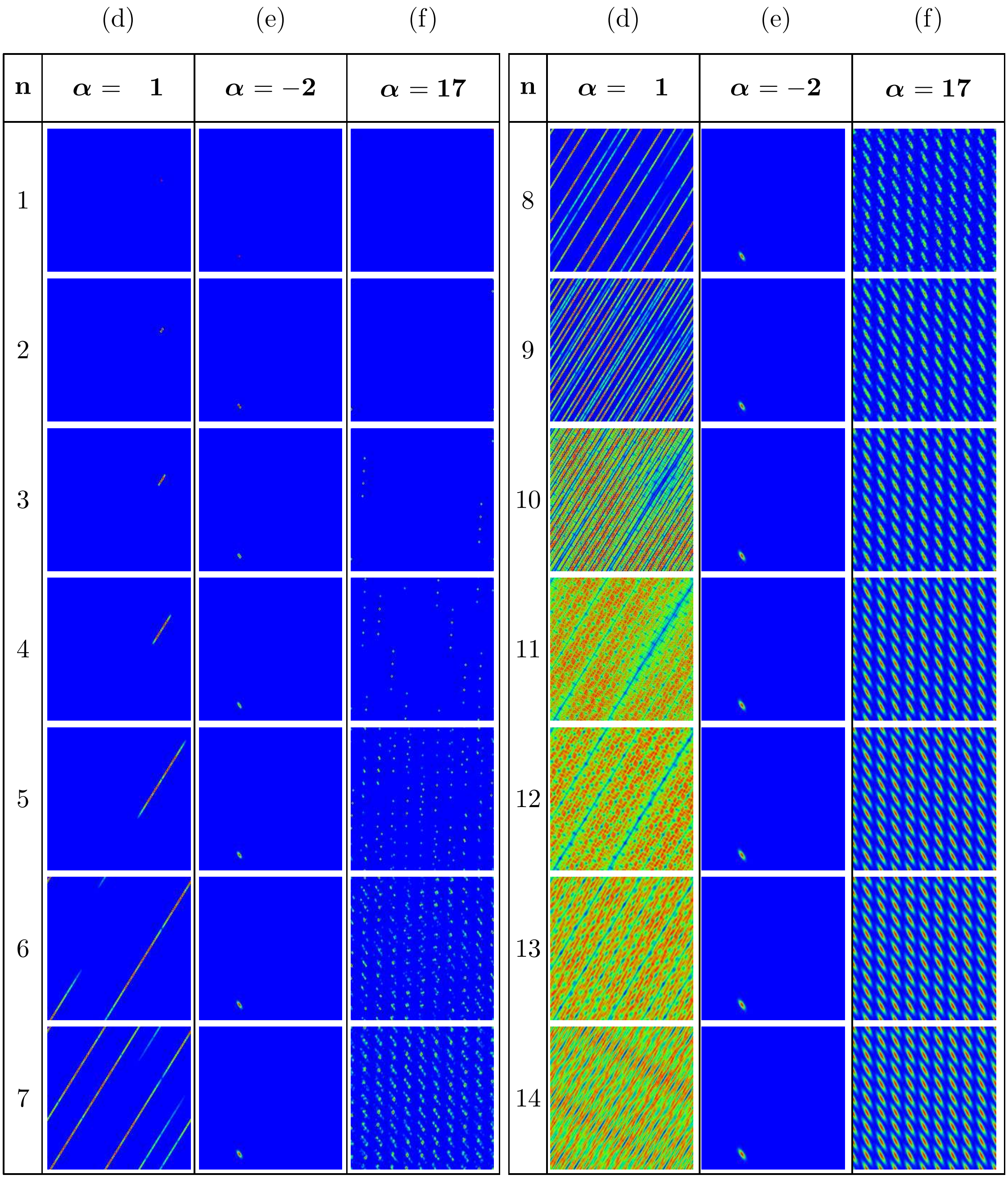

Figure 4.3: Temperature–like plots showing $\nu^{(n),N}_{\Lambda,\alpha}$ in two hyperbolic (columns d and f) and one elliptic (col. e) regime, for five nearest neighboring $\boldsymbol{r}_i$ in Λ ($N = 200$). Pale–blue corresponds to $\nu^{(n),N}_{\Lambda,\alpha} = 0$. When the system is chaotic, the frequencies tend to equidistribute on $(\mathbb{Z}/N\mathbb{Z})^2$ with increasing n and to approach, when the breaking–time is reached, the constant value $\frac{1}{N^2}$. Col. (f) shows how the dynamics can be confined on a sublattice by a particular combination (α, N, Λ) with a corresponding entropy decrease.

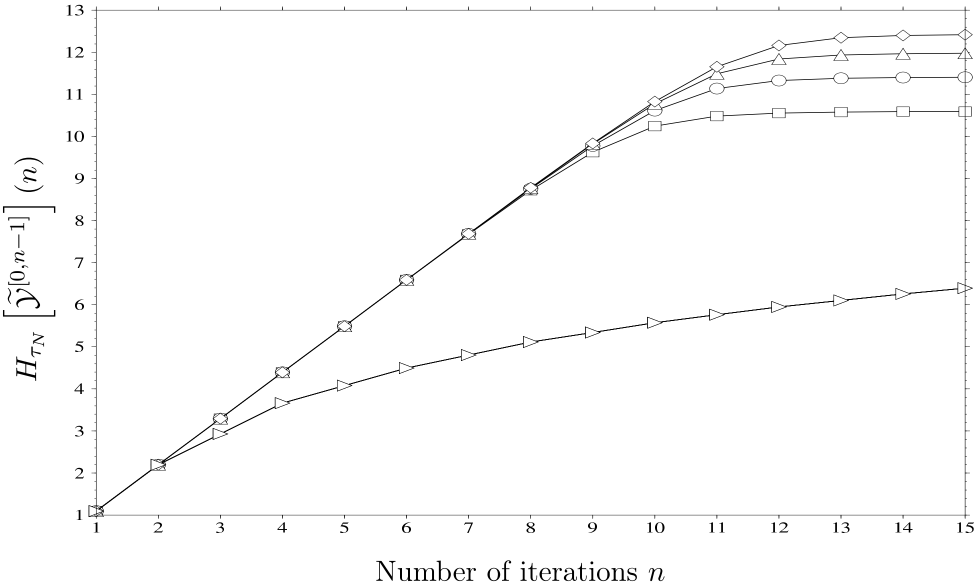

Figure 4.4: Von Neumann entropy $H_{\tau_N}(n)$ in four hyperbolic ($\alpha = 1$ for $\diamond$, $\triangle$, $\circ$, $\square$) and four elliptic ($\alpha = -2$ for $\triangleright$) cases, for three randomly distributed $\boldsymbol{r}_i$ in Λ. Values for N are: $\diamond = 500$, $\triangle = 400$, $\circ = 300$ and $\square = 200$, whereas the curve labeled by $\triangleright$ represents four elliptic systems with $N \in \{200, 300, 400, 500\}$.

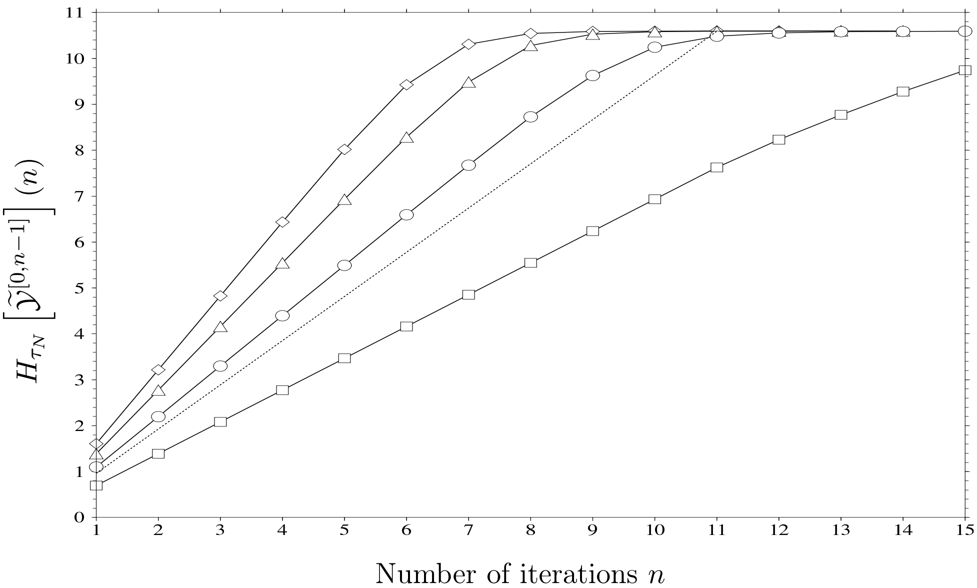

Figure 4.5: Von Neumann entropy $H_{\tau_N}(n)$ in four hyperbolic ($\alpha = 1$) cases, for D randomly distributed $\boldsymbol{r}_i$ in Λ, with $N = 200$. Value for D are: $\diamond = 5$, $\triangle = 4$, $\circ = 3$ and $\square = 2$. The dotted line represents $H_{\tau_N}(n) = \log\lambda \cdot n$ where $\log\lambda = 0.962\ldots$ is the Lyapounov exponent at$\alpha = 1$.

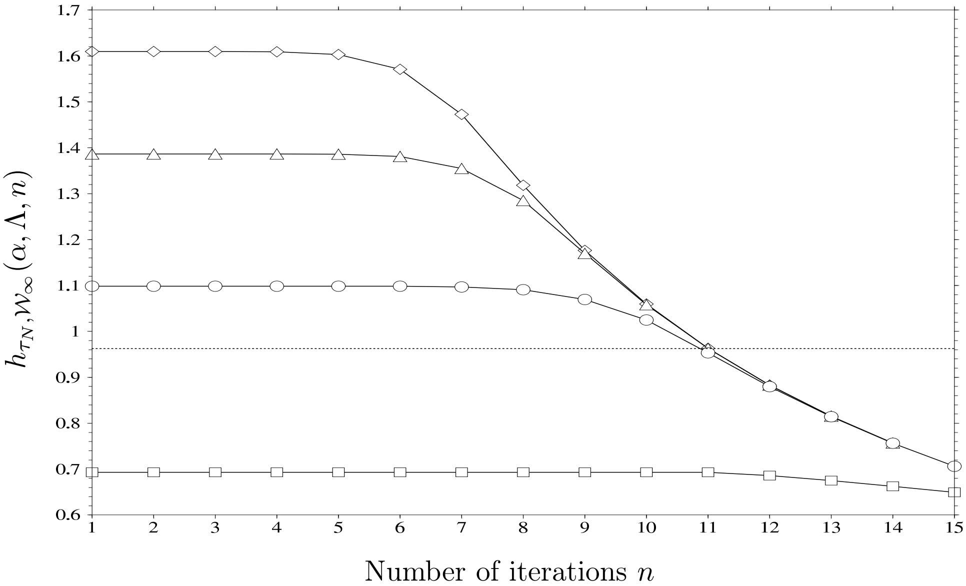

Figure 4.6: Entropy production $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n)$ in four hyperbolic ($\alpha = 1$) cases, for D randomly distributed $\boldsymbol{r}_i$ in Λ, with $N = 200$. Values for D are: $\diamond = 5$, $\triangle = 4$, $\circ = 3$ and $\square = 2$. The dotted line corresponds to the Lyapounov exponent $\log\lambda = 0.962\ldots$ at $\alpha = 1$.

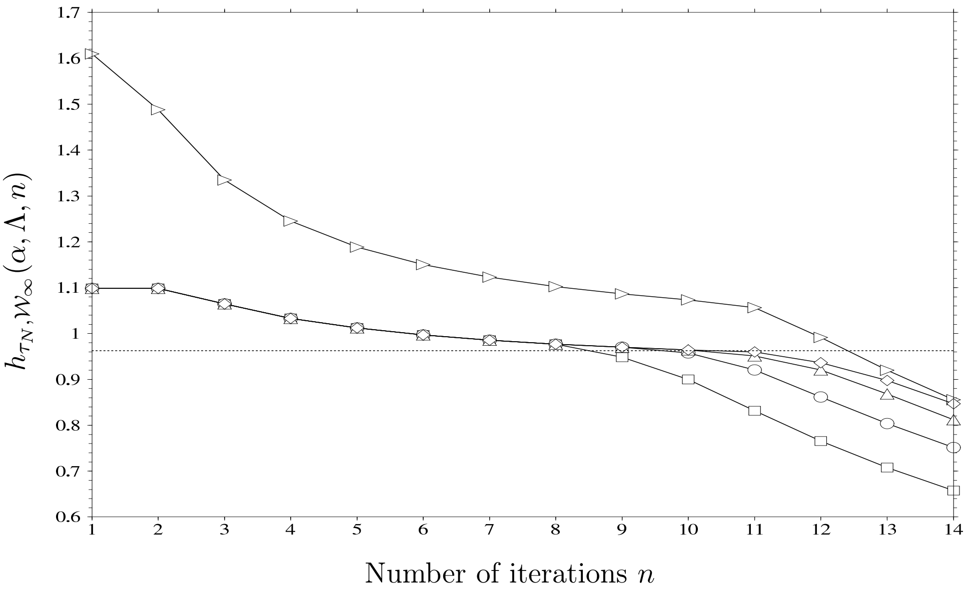

Figure 4.7: Entropy production $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n)$ in five hyperbolic ($\alpha = 1$) cases, for D nearest neighboring points $\boldsymbol{r}_i$ in Λ. Values for (N, D) are: $\triangleright = (200, 5)$, $\diamond = (500, 3)$, $\triangle = (400, 3)$, $\circ = (300, 3)$ and $\square = (200, 3)$. The dotted line corresponds to the Lyapounov exponent $\log\lambda = 0.962\ldots$ at $\alpha = 1$ and represents the natural asymptote for all these curves in absence of breaking–time.

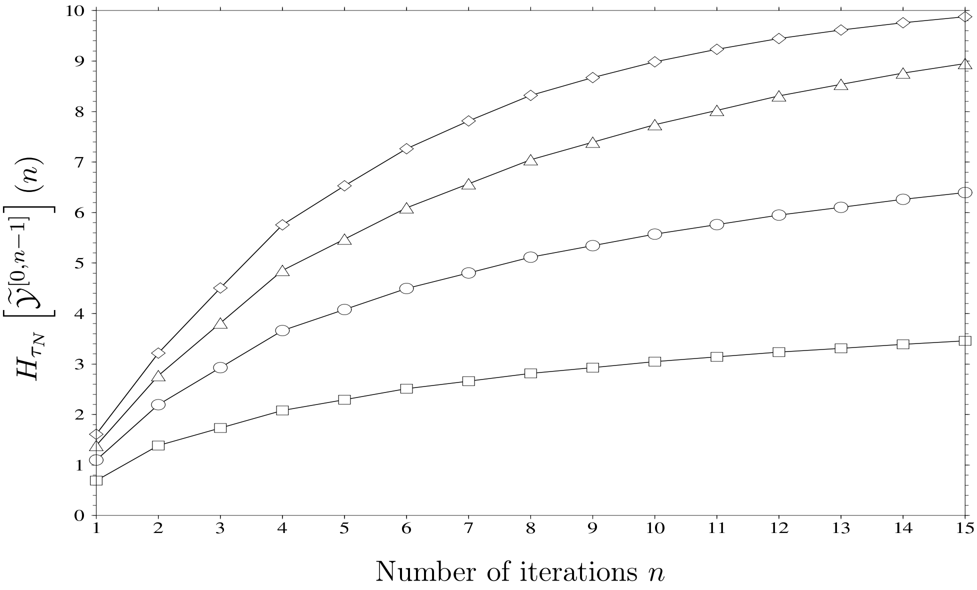

Figure 4.8: Von Neumann entropy $H_{\tau_N}(n)$ in four elliptic ($\alpha = -2$) cases, for D randomly distributed $\boldsymbol{r}_i$ in Λ, with $N = 200$. Value for D are: $\diamond = 5$, $\triangle = 4$, $\circ = 3$ and $\square = 2$.

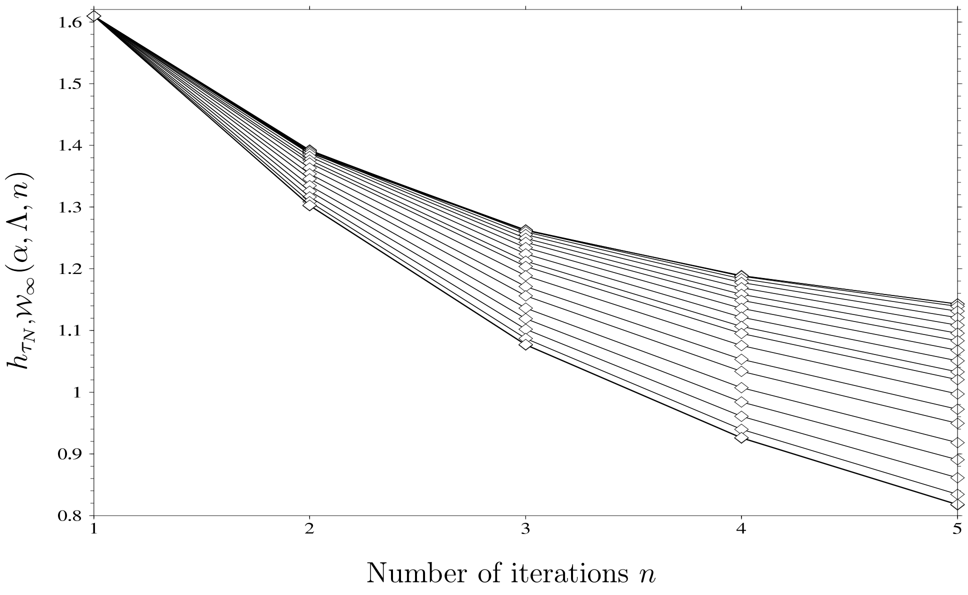

Figure 4.9: Entropy production $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n)$ for 21 hyperbolic Sawtooth maps, relative to a for a cluster of 5 nearest neighborings points $\boldsymbol{r}_i$ in Λ, with $N = 38$. The parameter α decreases from $\alpha = 1.00$ (corresponding to the upper curve) to $\alpha = 0.00$ (lower curve) through 21 equispaced steps.

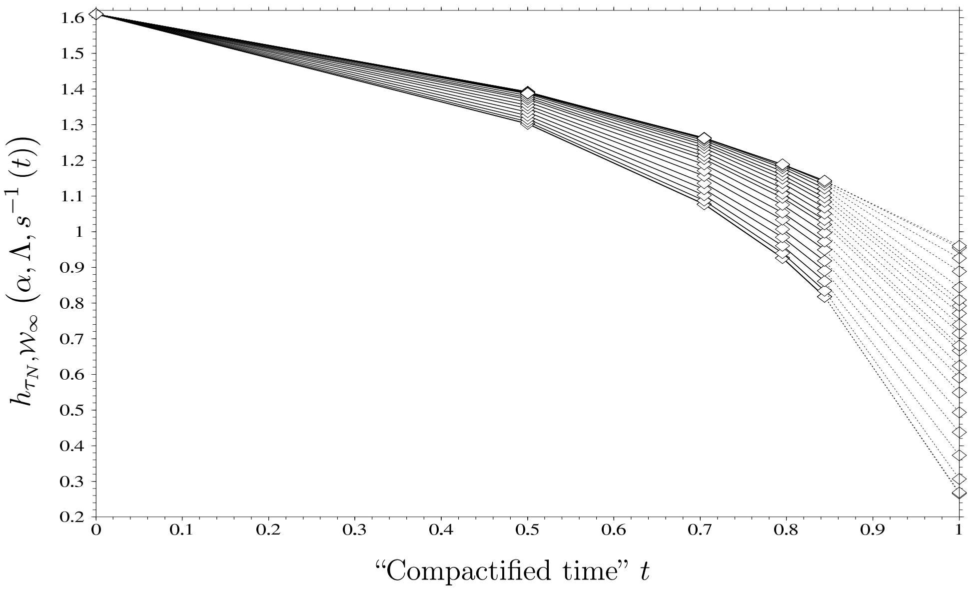

Figure 4.10: The solid lines correspond to $(s_n\,,\,h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n))$, with $n\in\{1,2,3,4,5\}$, for the values of α considered in figure 4.9. Every α–curve is continued as a dotted line up to $\left(1,l^5_\alpha\right)$, where l^5_α is the Lyapounov exponent extracted from the curve by fitting all the five points via a Lagrange polynomial $\mathcal{P}^m\,(t)$.

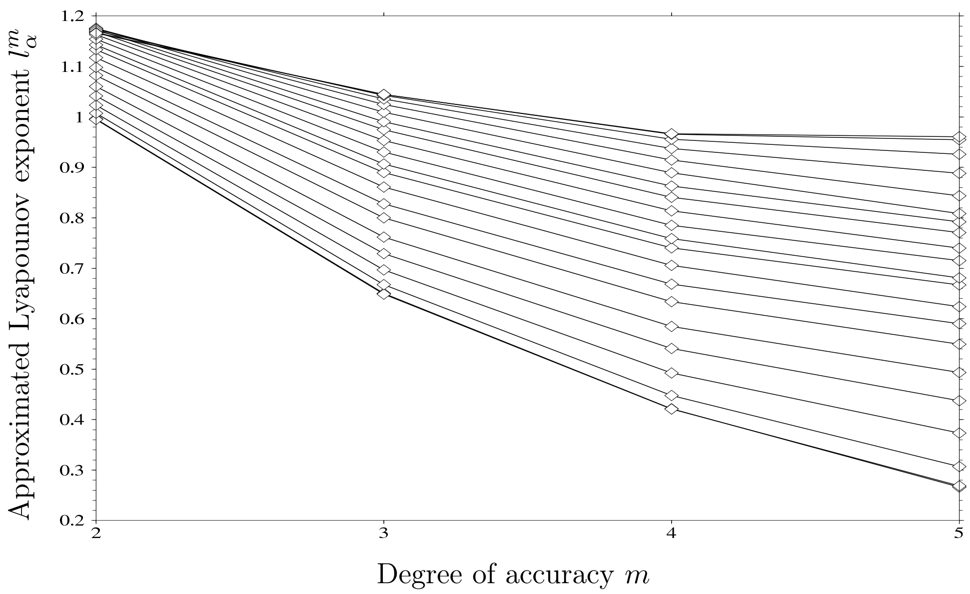

Figure 4.11: Four estimated Lyapounov exponents l^m_α plotted vs. their degree of accuracy m for the values of α considered in figures 4.9 and 4.10.

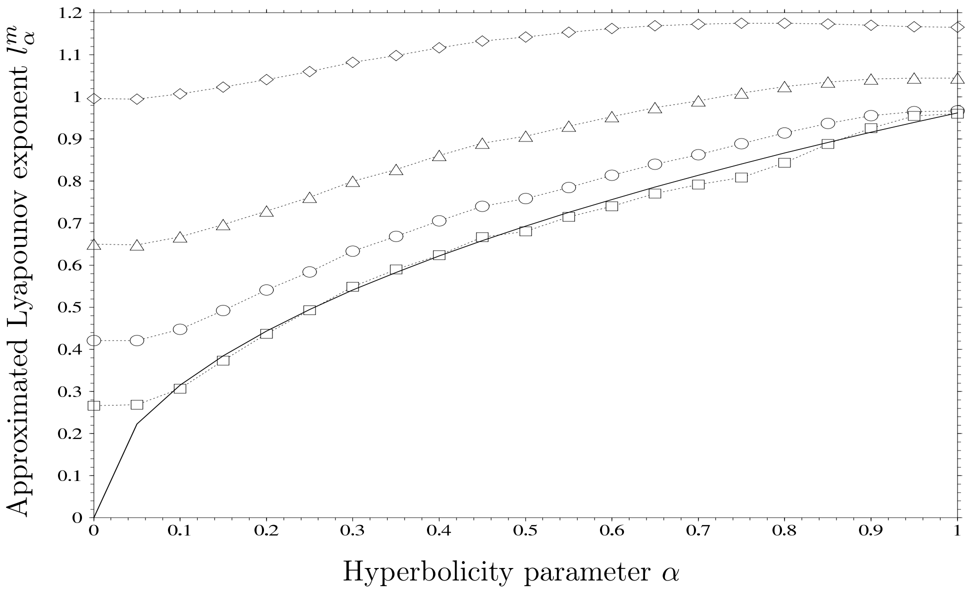

Figure 4.12: Plots of the four estimated of Lyapounov exponents l_α^m of figure 4.11 vs. the considered values of α. The polynomial degree m is as follows: $\diamond = 2$, $\triangle = 3$, $\circ = 4$ and $\square = 5$. The solid line corresponds to the theoretical Lyapounov exponent $\log \lambda_\alpha = \log\left(\alpha + 2 + \sqrt{\alpha\,(\alpha+4)}\ \right) - \log 2$.

Appendix A

Non Overcompleteness of the set of states of Section 1.4.2

The coherent state (1.50) can be rewritten in the shorter form:

$$|\beta_N(\boldsymbol{x})\rangle = \sum_{(\mu,\nu)\in\{0,1\}^2} \lambda_{\mu\nu}(\boldsymbol{x})\,|\lfloor Nx_1\rfloor + \mu, \lfloor Nx_2\rfloor + \nu\rangle \;, \tag{A.1}$$

whereas for the λ–coefficient we can write

$$\lambda_{\mu\nu}(\boldsymbol{x}) = \cos\left[\frac{\pi}{2}(\mu - \langle Nx_1\rangle)\right]\cos\left[\frac{\pi}{2}(\nu - \langle Nx_2\rangle)\right] \tag{A.2}$$

Overcompleteness property of Definition 1.4.1 can be expressed as

$$N^2\int_{\mathcal{X}}\mu(\mathrm{d}\boldsymbol{x})\,\langle\boldsymbol{\ell}\mid\beta_N(\boldsymbol{x})\rangle\langle\beta_N(\boldsymbol{x})\mid\boldsymbol{m}\rangle = \delta^{(N)}_{\boldsymbol{\ell},\boldsymbol{m}}, \qquad \forall\boldsymbol{\ell},\boldsymbol{m}\in(\mathbb{Z}/N\mathbb{Z})^2 \tag{A.3}$$

and this is exactly what we are going to check. Let us define with $I_{\boldsymbol{\ell},\boldsymbol{m}}$ the l.h.s. of (A.3) then, using (A.1–A.2), we can write

$$\begin{aligned} I_{\boldsymbol{\ell},\boldsymbol{m}} \coloneqq N^2 \sum_{(\mu,\nu,\rho,\sigma)\in\{0,1\}^4} \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2\, \cos\left[\frac{\pi}{2}(\mu - \langle Nx_1\rangle)\right]\cos\left[\frac{\pi}{2}(\nu - \langle Nx_2\rangle)\right] \times \\ \times\, \cos\left[\frac{\pi}{2}(\rho - \langle Nx_1\rangle)\right]\cos\left[\frac{\pi}{2}(\sigma - \langle Nx_2\rangle)\right] \times \\ \times \left\langle \ell_1,\ell_2 \,\middle|\, \lfloor Nx_1\rfloor + \rho, \lfloor Nx_2\rfloor + \sigma\right\rangle\left\langle \lfloor Nx_1\rfloor + \mu, \lfloor Nx_2\rfloor + \nu \,\middle|\, m_1,m_2\right\rangle \;, \end{aligned} \tag{A.4}$$

$$\text{that is } I_{\boldsymbol{\ell},\boldsymbol{m}} := N^2 \sum_{(\mu,\nu,\rho,\sigma)\in\{0,1\}^4} \times$$
$$\times \int_0^1 \mathrm{d}x_1 \cos\left[\frac{\pi}{2}(\mu - \langle Nx_1\rangle)\right] \cos\left[\frac{\pi}{2}(\rho - \langle Nx_1\rangle)\right] \delta^{(N)}_{\ell_1\,,\,\lfloor Nx_1\rfloor+\rho}\, \delta^{(N)}_{\lfloor Nx_1\rfloor+\mu\,,\,m_1} \times$$
$$\times \int_0^1 \mathrm{d}x_2 \cos\left[\frac{\pi}{2}(\nu - \langle Nx_2\rangle)\right] \cos\left[\frac{\pi}{2}(\sigma - \langle Nx_2\rangle)\right] \delta^{(N)}_{\ell_2\,,\,\lfloor Nx_2\rfloor+\sigma}\, \delta^{(N)}_{\lfloor Nx_2\rfloor+\nu\,,\,m_2},$$

or $I_{\boldsymbol{\ell},\boldsymbol{m}} = \Gamma_{\ell_1,m_1} \times \Gamma_{\ell_2,m_2}$, with $\Gamma_{p,q}$ defined by

$$\Gamma_{p,q} := N \sum_{(\mu,\rho)\in\{0,1\}^2} \times$$
$$\times \int_0^1 \mathrm{d}y \cos\left[\frac{\pi}{2}(\mu - \langle Ny\rangle)\right] \cos\left[\frac{\pi}{2}(\rho - \langle Ny\rangle)\right] \delta^{(N)}_{p\,,\,\lfloor Ny\rfloor+\rho}\, \delta^{(N)}_{q\,,\,\lfloor Ny\rfloor+\mu} =$$
$$= N \sum_{(\mu,\rho)\in\{0,1\}^2} \times$$
$$\times \int_0^1 \mathrm{d}y \cos\left[\frac{\pi}{2}(\mu - \langle Ny\rangle)\right] \cos\left[\frac{\pi}{2}(\rho - \langle Ny\rangle)\right] \delta^{(N)}_{p-\rho\,,\,\lfloor Ny\rfloor}\, \delta^{(N)}_{q-\mu\,,\,\lfloor Ny\rfloor} =$$
$$= N \sum_{(\mu,\rho)\in\{0,1\}^2} \times$$
$$\times \left\{\int_0^1 \mathrm{d}y \cos\left[\frac{\pi}{2}(\mu - \langle Ny\rangle)\right] \cos\left[\frac{\pi}{2}(\rho - \langle Ny\rangle)\right] \delta^{(N)}_{p-\rho\,,\,\lfloor Ny\rfloor}\right\} \delta^{(N)}_{q-\mu\,,\,p-\rho}. \tag{A.5}$$

Defining the symbol $((s)) := \{t \in (\mathbb{Z}/N\mathbb{Z}) : t = s\}$ (the element in the residual class (mod N) representing s), in order to have the integrand of (A.5) different from zero we must have $((p-\rho)) \leqslant Ny < ((p-\rho)) + 1$ (note that in that range $\langle Ny\rangle = Ny - ((p-\rho))$), and (A.5) reads:

$$\Gamma_{p,q} = N \sum_{(\mu,\rho)\in\{0,1\}^2} \int_{\frac{((p-\rho))}{N}}^{\frac{((p-\rho))+1}{N}} \mathrm{d}y$$
$$\cos\left[\frac{\pi}{2}\Big(\mu - Ny + ((p-\rho))\Big)\right] \cos\left[\frac{\pi}{2}\Big(\rho - Ny + ((p-\rho))\Big)\right] \delta^{(N)}_{q-\mu\,,\,p-\rho}.$$

Using now Werner trigonometric formula, we get

$$\begin{aligned}\Gamma_{p,q} = \quad & \frac{N}{2} \sum_{(\mu,\rho)\in\{0,1\}^2} \int_{\frac{((p-\rho))}{N}}^{\frac{((p-\rho))+1}{N}} \mathrm{d}y \cos\left[\frac{\pi}{2}(\mu+\rho) - \pi Ny + \pi((p-\rho))\right] \delta^{(N)}_{q-\mu\,,\,p-\rho} + \\ & + \frac{N}{2} \sum_{(\mu,\rho)\in\{0,1\}^2} \int_{\frac{((p-\rho))}{N}}^{\frac{((p-\rho))+1}{N}} \mathrm{d}y \cos\left[\frac{\pi}{2}(\mu-\rho)\right] \delta^{(N)}_{q-\mu\,,\,p-\rho} =\end{aligned}$$

(A.6)

$$
\begin{aligned}
= \quad & \frac{N}{2} \sum_{(\mu,\rho)\in\{0,1\}^2} (-1)^{((p-\rho))} \cos\left[\frac{\pi}{2}(\mu+\rho)\right] \int_{\frac{((p-\rho))}{N}}^{\frac{((p-\rho))+1}{N}} \mathrm{d}y \, \cos(\pi N y) \; \delta^{(N)}_{q-\mu\,,\,p-\rho} + \\
& + \frac{N}{2} \sum_{(\mu,\rho)\in\{0,1\}^2} (-1)^{((p-\rho))} \sin\left[\frac{\pi}{2}(\mu+\rho)\right] \int_{\frac{((p-\rho))}{N}}^{\frac{((p-\rho))+1}{N}} \mathrm{d}y \, \sin(\pi N y) \; \delta^{(N)}_{q-\mu\,,\,p-\rho} + \\
& + \frac{1}{2} \sum_{(\mu,\rho)\in\{0,1\}^2} \cos\left[\frac{\pi}{2}(\mu-\rho)\right] \delta^{(N)}_{q-\mu\,,\,p-\rho} = \\
= 0 + & \frac{1}{2} \sum_{(\mu,\rho)\in\{0,1\}^2} \cos\left[\frac{\pi}{2}(\mu-\rho)\right] \delta^{(N)}_{q-\mu\,,\,p-\rho} - \frac{1}{2\pi} \sum_{(\mu,\rho)\in\{0,1\}^2} (-1)^{((p-\rho))} \times \\
& \times \sin\left[\frac{\pi}{2}(\mu+\rho)\right] \left\{ \cos\left[\pi((p-\rho))+\pi\right] - \cos\left[\pi((p-\rho))\right] \right\} \delta^{(N)}_{q-\mu\,,\,p-\rho} = \\
= & \frac{1}{2} \sum_{(\mu,\rho)\in\{0,1\}^2} \delta^{(N)}_{q-\mu\,,\,p-\rho} \left\{ \frac{2}{\pi} \sin\left[\frac{\pi}{2}(\mu+\rho)\right] + \cos\left[\frac{\pi}{2}(\mu-\rho)\right] \right\} . \qquad \text{(A.7)}
\end{aligned}
$$

For $(\mu, \rho) \in \{0,1\}^2$, we have: $\begin{cases} \cos\left[\frac{\pi}{2}(\mu-\rho)\right] & = \delta_{\mu,\rho} \\ \sin\left[\frac{\pi}{2}(\mu+\rho)\right] & = 1-\delta_{\mu,\rho} \end{cases}$ thus from (A.7) we get:

$$
\begin{aligned}
\Gamma_{p,q} &= \frac{1}{2} \sum_{(\mu,\rho)\in\{0,1\}^2} \delta^{(N)}_{q-\mu\,,\,p-\rho} \left\{ \frac{2}{\pi}(1-\delta_{\mu,\rho}) + \delta_{\mu,\rho} \right\} = \\
&= \frac{1}{\pi} \sum_{(\mu,\rho)\in\{0,1\}^2} \delta^{(N)}_{q-\mu\,,\,p-\rho} + \frac{1}{2} \sum_{\rho\in\{0,1\}} \left(1-\frac{2}{\pi}\right) \left[\sum_{\mu\in\{0,1\}} \delta^{(N)}_{q-\mu\,,\,p-\rho} \, \delta_{\mu\,,\,\rho} \right] = \\
&= \frac{1}{\pi} \sum_{(\mu,\rho)\in\{(0,0),(0,1),(1,0),(1,1)\}} \delta^{(N)}_{q-p\,,\,\mu-\rho} + \frac{1}{2} \sum_{\rho\in\{0,1\}} \left(1-\frac{2}{\pi}\right) \delta^{(N)}_{q\,,\,p} = \\
&= \frac{1}{\pi} \left(\delta^{(N)}_{q-p\,,\,0} + \delta^{(N)}_{q-p\,,\,-1} + \delta^{(N)}_{q-p\,,\,1} + \delta^{(N)}_{q-p\,,\,0} \right) + \delta^{(N)}_{q\,,\,p} - \frac{2}{\pi} \delta^{(N)}_{q\,,\,p} = \\
&= \delta^{(N)}_{q\,,\,p} + \frac{1}{\pi} \left(\delta^{(N)}_{q\,,\,p+1} + \delta^{(N)}_{q+1\,,\,p} \right) . \qquad \text{(A.8)}
\end{aligned}
$$

Then we can compute $I_{\boldsymbol{\ell},\boldsymbol{m}} = \Gamma_{\ell_1,m_1} \times \Gamma_{\ell_2,m_2}$ that is different from $\delta^{(N)}_{\boldsymbol{\ell},\boldsymbol{m}}$, as expected from equation (A.3). Thus we conclude that the set $\{|\beta_N(\boldsymbol{x})\rangle \mid \boldsymbol{x} \in \mathbb{T}^2\}$ does not satisfy the overcompleteness property.

Appendix B

Proofs of Lemmas 2.4.2 and 2.4.3

Proof of lemma 2.4.2:

1) (2.50) follow from

$$\|S_\alpha \cdot \boldsymbol{v}\|^2_{\mathbb{R}^2} = \left\langle \boldsymbol{v} \left| S^\dagger_\alpha S_\alpha \right| \boldsymbol{v} \right\rangle \tag{B.1}$$

Indeed the matrix $S^\dagger_\alpha S_\alpha$ is real, symmetric, positive, with determinant equal to one; thus it has two orthogonal eigenvectors, corresponding to two different positive eigenvalues, η^2 and η^{-2}, depending only on α, with $\eta^2 > 1 \ \forall\, \alpha \in \mathbb{R}$.

The same argument can be used for the matrix S^{-1}_α: the matrix $\left(S_\alpha S^\dagger_\alpha\right)^{-1}$ has the same eigenvalues η^2 and η^{-2}.

2) In order to prove (2.51), it is convenient to unfold $\mathbb{T}^2$ and the discontinuity of S_α on the plane $\mathbb{R}^2$. This is most easily done as follows. Points $A \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ are equivalence classes $[\boldsymbol{a}]$ of points in $\mathbb{R}^2$ such that

$$[\boldsymbol{a}] := \left\{\boldsymbol{a} + \boldsymbol{n}\ ,\ \boldsymbol{n} \in \mathbb{Z}^2\right\}\ ,\ \boldsymbol{a} \in [0,1)^2\ \cdot \tag{B.2}$$

Given $A, B \in \mathbb{T}^2$, let $A^b \in [\boldsymbol{a}]$ be the closest vector to $\boldsymbol{b}$ in the Euclidean norm $\|\cdot\|_{\mathbb{R}^2}$, namely that vector such that

$$d_{\mathbb{T}^2}\left([\boldsymbol{a}], [\boldsymbol{b}]\right) = \left\| A^b - \boldsymbol{b} \right\|_{\mathbb{R}^2}\ \cdot \tag{B.3}$$

Notice that

$$d_{\mathbb{T}^2}\left([\boldsymbol{a}], [\boldsymbol{b}]\right) = \|\boldsymbol{a} - \boldsymbol{b}\|_{\mathbb{R}^2} \quad \text{iff} \quad \|\boldsymbol{a} - \boldsymbol{b}\|_{\mathbb{R}^2} \leqslant \frac{1}{2} \tag{B.4}$$

2a) (A,B) not crossing γ_{-1} means that the segment $\left(A^b,\boldsymbol{b}\right)$ does not intersect γ_{-1}. Periodically covering the plane–$\mathbb{R}^2$ by squares $[0,1)^2$, the γ_{-1}-lines form a set of (parallel) straight lines $x_1-x_2=n\in\mathbb{Z}$; it follows that $\left(A^b,\boldsymbol{b}\right)$ does not cross γ_{-1} iff

$$\left\lfloor A_1^b-A_2^b\right\rfloor=\lfloor b_1-b_2\rfloor\ , \tag{B.5}$$

where the integral part on the r.h.s. takes values $0,-1$, depending on which side of the diagonal γ_{-1} the point $\boldsymbol{b}$ lies within. Indeed, one can check that if any two points $\boldsymbol{x},\boldsymbol{y}\in\mathbb{T}^2$ lie on opposite sides with respect to γ_{-1} then they must violate the above condition (B.5) on the integer part of the differences of their components.
As $S_\alpha^{\pm}$ are not sensitive to the integer part of their arguments, their actions are the same on all elements of the equivalence classes (B.2). Therefore,

$$d_{\mathbb{T}^2}\left(S_\alpha^{-1}(A),S_\alpha^{-1}(B)\right)=d_{\mathbb{T}^2}\left(S_\alpha^{-1}([\boldsymbol{a}]),S_\alpha^{-1}([\boldsymbol{b}])\right)=d_{\mathbb{T}^2}\left(S_\alpha^{-1}\left(A^b\right),S_\alpha^{-1}(\boldsymbol{b})\right)=$$

(by (2.4–2.5) and (1.35))

$$=\min_{\boldsymbol{m}\in\mathbb{Z}^2}\left\|\begin{pmatrix}\left\langle A_1^b-A_2^b\right\rangle\\ \left\langle-\alpha\left\langle A_1^b-A_2^b\right\rangle+A_2^b\right\rangle\end{pmatrix}-\begin{pmatrix}\langle b_1-b_2\rangle\\ \langle-\alpha\langle b_1-b_2\rangle+b_2\rangle\end{pmatrix}+\boldsymbol{m}\right\|_{\mathbb{R}^2}$$

(by using $\langle x\rangle=x-\lfloor x\rfloor$)

$$\begin{aligned}=\min_{\boldsymbol{m}\in\mathbb{Z}^2}&\left\|\begin{pmatrix}\left(A_1^b-A_2^b\right)-(b_1-b_2)\\ -\alpha\left\langle A_1^b-A_2^b\right\rangle+A_2^b+\alpha\langle b_1-b_2\rangle-b_2\end{pmatrix}+\right.\\&\left.+\begin{pmatrix}\lfloor b_1-b_2\rfloor-\left\lfloor A_1^b-A_2^b\right\rfloor+m_1\\ \lfloor-\alpha\langle b_1-b_2\rangle+b_2\rfloor-\left\lfloor-\alpha\left\langle A_1^b-A_2^b\right\rangle+A_2^b\right\rfloor+m_2\end{pmatrix}\right\|_{\mathbb{R}^2}\\=\min_{\boldsymbol{m}'\in\mathbb{Z}^2}&\left\|\begin{pmatrix}\left(A_1^b-b_1\right)-\left(A_2^b-b_2\right)\\ -\alpha\left(A_1^b-A_2^b\right)+\alpha(b_1-b_2)+\left(A_2^b-b_2\right)\end{pmatrix}+\right.\\&\left.+\alpha\begin{pmatrix}0\\ \left\lfloor A_1^b-A_2^b\right\rfloor-\lfloor b_1-b_2\rfloor\end{pmatrix}+\boldsymbol{m}'\right\|_{\mathbb{R}^2}\end{aligned}$$

(because of (B.5))

$$=\min_{\boldsymbol{m}'\in\mathbb{Z}^2}\left\|\begin{pmatrix}1&-1\\-\alpha&1+\alpha\end{pmatrix}\begin{pmatrix}A_1^b-b_1\\A_2^b-b_2\end{pmatrix}+\boldsymbol{m}'\right\|_{\mathbb{R}^2}=d_{\mathbb{T}^2}\left(S_\alpha^{-1}\cdot\left(A^b-\boldsymbol{b}\right),0\right) \tag{B.6}$$

Applying point (1) of this lemma and using the hypothesis, we can estimate

$$\left\| S_\alpha^{-1} \cdot \left(A^b - \boldsymbol{b} \right) \right\|_{\mathbb{R}^2} \leqslant \eta \left\| A^b - \boldsymbol{b} \right\|_{\mathbb{R}^2} = \eta \, d_{\mathbb{T}^2}(A, B) < \frac{1}{2} \cdot \tag{B.7}$$

In particular, using (B.4), the previous inequalities imply

$$d_{\mathbb{T}^2}\left(S_\alpha^{-1} \cdot \left(A^b - \boldsymbol{b} \right), 0 \right) = \left\| S_\alpha^{-1} \cdot \left(A^b - \boldsymbol{b} \right) \right\|_{\mathbb{R}^2} \leqslant \eta \, d_{\mathbb{T}^2}(A, B) \ \cdot \tag{B.8}$$

2b) Analogously, the union of γ_0-lines constitute a set of straight lines $x_1 = n \in \mathbb{Z}$; Therefore the segment $\left(A^b, \boldsymbol{b} \right)$ does not cross γ_0 iff

$$\left\lfloor A_1^b \right\rfloor = \lfloor b_1 \rfloor \ \cdot \tag{B.9}$$

Explicitly

$$d_{\mathbb{T}^2}(S_\alpha(A), S_\alpha(B)) = d_{\mathbb{T}^2}(S_\alpha([\boldsymbol{a}]), S_\alpha([\boldsymbol{b}])) = d_{\mathbb{T}^2}\left(S_\alpha\left(A^b \right), S_\alpha(\boldsymbol{b}) \right)$$

(by (2.1b) and (1.35))

$$= \min_{\boldsymbol{m} \in \mathbb{Z}^2} \left\| \left\langle S_\alpha \cdot \begin{pmatrix} \langle A_1^b \rangle \\ A_2^b \end{pmatrix} \right\rangle - \left\langle S_\alpha \cdot \begin{pmatrix} \langle b_1 \rangle \\ b_2 \end{pmatrix} \right\rangle + \boldsymbol{m} \right\|_{\mathbb{R}^2}$$

(since $\langle x \rangle = x - \lfloor x \rfloor$)

$$\begin{aligned} &= \min_{\boldsymbol{m} \in \mathbb{Z}^2} \left\| S_\alpha \cdot \begin{pmatrix} \langle A_1^b \rangle - \langle b_1 \rangle \\ A_2^b - b_2 \end{pmatrix} - \left\lfloor S_\alpha \cdot \begin{pmatrix} \langle A_1^b \rangle \\ A_2^b \end{pmatrix} \right\rfloor + \left\lfloor S_\alpha \cdot \begin{pmatrix} \langle b_1 \rangle \\ b_2 \end{pmatrix} \right\rfloor + \boldsymbol{m} \right\|_{\mathbb{R}^2} \\ &= \min_{\boldsymbol{m'} \in \mathbb{Z}^2} \left\| S_\alpha \cdot \left(A^b - \boldsymbol{b} \right) - S_\alpha \cdot \begin{pmatrix} \lfloor A_1^b \rfloor - \lfloor b_1 \rfloor \\ 0 \end{pmatrix} + \boldsymbol{m'} \right\|_{\mathbb{R}^2} \end{aligned}$$

Condition (B.9) makes the second vector vanish and we obtain

$$d_{\mathbb{T}^2}(S_\alpha(A), S_\alpha(B)) = d_{\mathbb{T}^2}\left(S_\alpha \cdot \left(A^b - \boldsymbol{b} \right), 0 \right) \tag{B.10}$$

The proof thus is exactly completed as before.

3) We denote by $d_{\mathbb{T}^2}(\boldsymbol{x}, \gamma) = \inf_{\boldsymbol{y} \in \gamma} d_{\mathbb{T}^2}(\boldsymbol{x}, \boldsymbol{y})$ the distance of the point $\boldsymbol{x} \in \mathbb{T}^2$ from a curve $\gamma \in \mathbb{T}^2$. Then, from Definition 2.47 we have:

$$\boldsymbol{x} \in \overline{\gamma}_{p-1}(\varepsilon) \Longrightarrow \varepsilon \geqslant d_{\mathbb{T}^2}(\boldsymbol{x}, \gamma_{p-1}) = d_{\mathbb{T}^2}(\boldsymbol{x}, \boldsymbol{y}^\star) \ , \tag{B.11}$$

where $\boldsymbol{y}^\star$ is the nearest point to $\boldsymbol{x}$ belonging to γ_{p-1}.
We distinguish two cases:

3′) The segment $(\boldsymbol{x}, \boldsymbol{y}^\star)$ does not cross γ_{-1}

(even if $\boldsymbol{y}^\star \in \gamma_{-1}$ or $\boldsymbol{x} \in \gamma_{-1}$, we are in a non–crossing condition).
From (B.11) and point (2a), since $S_\alpha^{-1}(\boldsymbol{y}^\star) \in \gamma_p$ (see (2.44)), we get

$$d_{\mathbb{T}^2}\left(S_\alpha^{-1}(\boldsymbol{x}), \gamma_p\right) \leqslant d_{\mathbb{T}^2}\left(S_\alpha^{-1}(\boldsymbol{x}), S_\alpha^{-1}(\boldsymbol{y}^\star)\right) \leqslant \eta\, d_{\mathbb{T}^2}(\boldsymbol{x}, \boldsymbol{y}^\star) \leqslant \eta\, \varepsilon \cdot \tag{B.12}$$

Therefore $S_\alpha^{-1}(\boldsymbol{x}) \in \overline{\gamma}_p(\eta\, \varepsilon)$

3″) The segment $(\boldsymbol{x}, \boldsymbol{y}^\star)$ crosses γ_{-1}.

In this case, there exists $\boldsymbol{z} \in \gamma_{-1} \cap (\boldsymbol{x}, \boldsymbol{y}^\star)$ such that

$$d_{\mathbb{T}^2}(\boldsymbol{x}, \boldsymbol{y}^\star) = d_{\mathbb{T}^2}(\boldsymbol{x}, \boldsymbol{z}) + d_{\mathbb{T}^2}(\boldsymbol{z}, \boldsymbol{y}^\star) \cdot \tag{B.13}$$

Then, from (B.11) and (B.13),

$$\varepsilon \geqslant d_{\mathbb{T}^2}(\boldsymbol{x}, \boldsymbol{y}^\star) \geqslant d_{\mathbb{T}^2}(\boldsymbol{x}, \boldsymbol{z}) \cdot \tag{B.14}$$

Since, according to (2.44), $S_\alpha^{-1}(\boldsymbol{z}) \in \gamma_0$, from point (2a), we get

$$d_{\mathbb{T}^2}\left(S_\alpha^{-1}(\boldsymbol{x}), \gamma_0\right) \leqslant d_{\mathbb{T}^2}\left(S_\alpha^{-1}(\boldsymbol{x}), S_\alpha^{-1}(\boldsymbol{z})\right) \leqslant \eta\, \varepsilon\ , \tag{B.15}$$

that is $S_\alpha^{-1}(\boldsymbol{x}) \in \overline{\gamma}_0(\eta\, \varepsilon)$.

4) From point (3), it follows that, when $0 \leqslant \varepsilon \leqslant \frac{1}{2}$, for $p \in \mathbb{N}^+$,

$$\boldsymbol{x} \notin \left(\overline{\gamma}_p(\varepsilon) \cup \overline{\gamma}_0(\varepsilon)\right) \Longrightarrow S_\alpha(\boldsymbol{x}) \notin \overline{\gamma}_{p-1}\left(\eta^{-1}\varepsilon\right) \cdot \tag{B.16}$$

We prove by induction that, when $0 \leqslant \varepsilon \leqslant \frac{1}{2}$, for $m \in \mathbb{N}^+$,

$$\boldsymbol{x} \notin \bigcup_{p=0}^{m} \overline{\gamma}_p(\varepsilon) \Longrightarrow S_\alpha(\boldsymbol{x}) \notin \bigcup_{p=0}^{m-1} \overline{\gamma}_p\left(\eta^{-1}\varepsilon\right) \cdot . \tag{B.17}$$

For $m = 1$, (B.17) follows from (B.16); suppose (B.17) holds for $m = r$, then take

$$\boldsymbol{x} \notin \bigcup_{p=0}^{r+1} \overline{\gamma}_p(\varepsilon)\ . \text{ This means that } \quad \boldsymbol{x} \notin \bigcup_{p=0}^{r} \overline{\gamma}_p(\varepsilon) \quad \text{and} \quad \boldsymbol{x} \notin \left(\overline{\gamma}_{r+1}(\varepsilon) \cup \overline{\gamma}_0(\varepsilon)\right) \cdot$$

Now, using the hypothesis and (B.16), we get

$$\boldsymbol{x} \notin \bigcup_{p=0}^{r+1} \overline{\gamma}_p(\varepsilon) \Longrightarrow S_\alpha(\boldsymbol{x}) \notin \bigcup_{p=0}^{r-1} \overline{\gamma}_p\left(\eta^{-1}\varepsilon\right) \quad \text{and} \quad S_\alpha(\boldsymbol{x}) \notin \overline{\gamma}_r\left(\eta^{-1}\varepsilon\right) \cdot \tag{B.18}$$

Then (B.17) is proved for all $m \in \mathbb{N}^+$. Now observe the following: applying (B.17) to $S_\alpha(\boldsymbol{x})$ instead of $\boldsymbol{x}$ one gets

$$S_\alpha(\boldsymbol{x}) \notin \bigcup_{p=0}^{m-1} \overline{\gamma}_p\left(\eta^{-1}\varepsilon\right) \Longrightarrow S_\alpha^2(\boldsymbol{x}) \notin \bigcup_{p=0}^{m-2} \overline{\gamma}_p\left(\eta^{-2}\varepsilon\right) \cdot \tag{B.19}$$

By iterating (B.17), with $m = n-1$, we deduce

$$\boldsymbol{x} \notin \bigcup_{p=0}^{n-1} \overline{\gamma}_p(\varepsilon) \Longrightarrow S_\alpha^q(\boldsymbol{x}) \notin \bigcup_{p=0}^{n-1-q} \overline{\gamma}_p\left(\eta^{-q}\varepsilon\right) \ , \ \forall\ 0 \leqslant q < n \cdot \tag{B.20}$$

In particular $S_\alpha^q(\boldsymbol{x}) \notin \overline{\gamma}_0\left(\eta^{-q}\varepsilon\right)$, which leads to the lower bound

$$d_{\mathbb{T}^2}\left(S_\alpha^q(\boldsymbol{x}), \gamma_0\right) > \eta^{-q}\varepsilon \ , \ \forall\ 0 \leqslant q < n \ , \tag{B.21}$$

whence the result follows in view of Definitions (2.27) and (2.48).

5) The prove is by induction; we fix n and choose $N > \widetilde{N} + 3 = 2\sqrt{2}n\eta^{2n} + 3$.

$\boldsymbol{p=0}$) from Definitions (2.27) and (2.34), it follows

$$d_{\mathbb{T}^2}\left(\frac{U_\alpha^0(N\boldsymbol{x})}{N}, \frac{V_\alpha^0(\hat{\boldsymbol{x}}_N)}{N}\right) = d_{\mathbb{T}^2}\left(\boldsymbol{x}, \frac{\hat{\boldsymbol{x}}_N}{N}\right) < \frac{1}{\sqrt{2}N} < \frac{\sqrt{2}}{N} \ , \tag{B.22}$$

Where the first inequality follows from (B.37) in (2.4.3), thus relation (2.54) holds for $p = 0$.

$\boldsymbol{p=1}$)

$$\left\| \frac{U_\alpha(N\boldsymbol{x})}{N} - \frac{V_\alpha(\hat{\boldsymbol{x}}_N)}{N} \right\|_{\mathbb{R}^2} \leqslant \left\| \frac{U_\alpha(N\boldsymbol{x})}{N} - \frac{U_\alpha(\hat{\boldsymbol{x}}_N)}{N} \right\|_{\mathbb{R}^2} + \left\| \frac{U_\alpha(\hat{\boldsymbol{x}}_N)}{N} - \frac{V_\alpha(\hat{\boldsymbol{x}}_N)}{N} \right\|_{\mathbb{R}^2} \cdot \tag{B.23}$$

Now, by definitions (2.27) and (2.34), we have for the second term of (B.23)

$$\left\| \frac{U_\alpha(\hat{\boldsymbol{x}}_N)}{N} - \frac{V_\alpha(\hat{\boldsymbol{x}}_N)}{N} \right\|_{\mathbb{R}^2} = \left\| \frac{1}{N} \left\langle N\, S_\alpha\left(\frac{\hat{\boldsymbol{x}}_N}{N}\right) \right\rangle \right\|_{\mathbb{R}^2} < \frac{\sqrt{2}}{N} \ , \tag{B.24}$$

whereas for the first term in (B.23), (B.22) together with the non–crossing condition with respect to γ_0, which is apparently fulfilled by $\left(\boldsymbol{x}\,,\,\frac{\hat{\boldsymbol{x}}_N}{N}\right)$, allow us to use point (2b) of this Lemma and to get

$$\left\| \frac{U_\alpha\left(N\boldsymbol{x}\right)}{N} - \frac{U_\alpha\left(\hat{\boldsymbol{x}}_N\right)}{N} \right\|_{\mathbb{R}^2} = \left\| S_\alpha\left(\boldsymbol{x}\right) - S_\alpha\left(\frac{\hat{\boldsymbol{x}}_N}{N}\right) \right\|_{\mathbb{R}^2} \leqslant \eta \frac{\sqrt{2}}{N}\,. \tag{B.25}$$

Thus, inserting (B.24) and (B.25) in (B.23), we obtain

$$\left\| \frac{U_\alpha\left(N\boldsymbol{x}\right)}{N} - \frac{V_\alpha\left(\hat{\boldsymbol{x}}_N\right)}{N} \right\|_{\mathbb{R}^2} \leqslant \left(\eta+1\right)\frac{\sqrt{2}}{N} \tag{B.26}$$

$$(\text{for } N > \tilde{N}+3) \qquad < \frac{\eta+1}{2\,n\,\eta^{2n}} < \frac{1}{2}\,. \tag{B.27}$$

But then the Euclidean norm equals the distance $d_{\mathbb{T}^2}\left(\frac{U_\alpha\left(N\boldsymbol{x}\right)}{N}\,,\,\frac{V_\alpha\left(\hat{\boldsymbol{x}}_N\right)}{N}\right)$; thus relation (2.54) holds for $p=1$.

$\boldsymbol{p = q - 1\,,\; q \leqslant n}$) Since

$$d_{\mathbb{T}^2}\left(\frac{U_\alpha^q\left(N\boldsymbol{x}\right)}{N}, \frac{V_\alpha^q\left(\hat{\boldsymbol{x}}_N\right)}{N}\right) \leqslant d_{\mathbb{T}^2}\left(\frac{U_\alpha\left(U_\alpha^{q-1}\left(N\boldsymbol{x}\right)\right)}{N}, \frac{U_\alpha\left(V_\alpha^{q-1}\left(\hat{\boldsymbol{x}}_N\right)\right)}{N}\right) + $$
$$+\, d_{\mathbb{T}^2}\left(\frac{U_\alpha\left(V_\alpha^{q-1}\left(\hat{\boldsymbol{x}}_N\right)\right)}{N}, \frac{V_\alpha\left(V_\alpha^{q-1}\left(\hat{\boldsymbol{x}}_N\right)\right)}{N}\right)\,, \tag{B.28}$$

using (2.27) in the first term and noting that, from definitions (2.27) and (2.34), the second term is less or equal to $\frac{\sqrt{2}}{N}$, we get

$$d_{\mathbb{T}^2}\left(\frac{U_\alpha^q\left(N\boldsymbol{x}\right)}{N}, \frac{V_\alpha^q\left(\hat{\boldsymbol{x}}_N\right)}{N}\right) \leqslant d_{\mathbb{T}^2}\left(S_\alpha\left(\frac{U_\alpha^{q-1}\left(N\boldsymbol{x}\right)}{N}\right), S_\alpha\left(\frac{V_\alpha^{q-1}\left(\hat{\boldsymbol{x}}_N\right)}{N}\right)\right) + \frac{\sqrt{2}}{N}\,. \tag{B.29}$$

By the induction hypothesis we have:

$$d_{\mathbb{T}^2}\left(\frac{U_\alpha^{q-1}\left(N\boldsymbol{x}\right)}{N}, \frac{V_\alpha^{q-1}\left(\hat{\boldsymbol{x}}_N\right)}{N}\right) \leqslant \frac{\sqrt{2}}{N}\left(\frac{\eta^q-1}{\eta-1}\right) \tag{B.30}$$

$$\leqslant \frac{\sqrt{2}}{N}\, q\, \eta^{q-1} \tag{B.31}$$

$$(\eta > 1,\; q \leqslant n \quad \Longrightarrow) \qquad < \eta^{q-2n}\,\frac{q}{n}\,\left(\frac{1}{2}\eta^{-1}\right) < \frac{1}{2}\eta^{-1}\,. \tag{B.32}$$

Moreover, from point (4) above with $\varepsilon = \frac{\widetilde{N}}{2N}$, $0 \leqslant q \leqslant n$ and $\eta > 1$ we derive

$$d_{\mathbb{T}^2}\left(\frac{U_\alpha^{q-1}(N\boldsymbol{x})}{N}, \gamma_0\right) > \frac{n\sqrt{2}\,\eta^{2n-q+1}}{N} > \frac{\sqrt{2}}{N}\,q\,\eta^{q-1}\ . \tag{B.33}$$

Therefore, comparing (B.33) with (B.31)

$$d_{\mathbb{T}^2}\left(\frac{U_\alpha^{q-1}(N\boldsymbol{x})}{N}, \frac{V_\alpha^{q-1}(\hat{\boldsymbol{x}}_N)}{N}\right) < d_{\mathbb{T}^2}\left(\frac{U_\alpha^{q-1}(N\boldsymbol{x})}{N}, \gamma_0\right) \quad , \quad \forall q \leqslant n\ \cdot \tag{B.34}$$

Therefore, we deduce that the segment $\left(\frac{U_\alpha^{q-1}(N\boldsymbol{x})}{N}, \frac{V_\alpha^{q-1}(\hat{\boldsymbol{x}}_N)}{N}\right)$ cannot cross the line γ_0. This condition, together with (B.32), allows us to use point (2b) in (B.29) to finally estimate

$$d_{\mathbb{T}^2}\left(\frac{U_\alpha^{q}(N\boldsymbol{x})}{N}, \frac{V_\alpha^{q}(\hat{\boldsymbol{x}}_N)}{N}\right) \leqslant \eta\frac{\sqrt{2}}{N}\left(\frac{\eta^q - 1}{\eta - 1}\right) + \frac{\sqrt{2}}{N} = \frac{\sqrt{2}}{N}\left(\frac{\eta^{q+1} - 1}{\eta - 1}\right) \tag{B.35}$$

and this concludes the proof. ■

Proof of lemma 2.4.3:

a) In (2.44), we have defined $\gamma_p = S_\alpha^{-p}(\gamma_0)$ where $S_\alpha^{-1}(\boldsymbol{x})$ (and then also $S_\alpha^{-p}(\boldsymbol{x})$) is a piecewise continuous mapping onto $\mathbb{T}^2$ with jump–discontinuities across the γ_p lines due to the presence of the function $\langle\cdot\rangle$ in (2.4). Away from the discontinuities, $S_\alpha^{-p}(\boldsymbol{x})$ behaves as the matrix action $S_\alpha^{-p}\cdot\boldsymbol{x}$, that is nothing but the action of the Sawtooth Map in the tangent space. By integrating the evolution given by $S_\alpha^{-p}\cdot\boldsymbol{x}$ on the tangent space along γ_0, it follows that $l(\gamma_p)$ is the length of the $\mathbb{R}^2$–segment $\left\{\boldsymbol{x} \in \mathbb{R}^2 \,\middle|\, \boldsymbol{x} = S_\alpha^{-p}\cdot\left(\begin{smallmatrix}0\\ y\end{smallmatrix}\right)\ ,\ y \in [0,1)\right\}$, which, in its turn, is the image of the (length one) segment γ_0 under the matrix action given by $S_\alpha^{-p}\cdot\boldsymbol{x}$.
Finally, using (2.50a) we get the result.

b) Let $\overline{L}(\varepsilon)$ denote the set of points having distance from a segment of length L smaller or equal than ε: it has a volume (given by the Lebesgue measure μ) given by

$$\mu\left(\overline{L}(\varepsilon)\right) = 2\,L\,\varepsilon + \pi\varepsilon^2\ ,$$

where the last term on the r.h.s. takes into account a small set of points close to the extremity of the segment. Then (2.55b) follows from (2.55a).

c) This follows from Definition (2.48):

$$\mu\left(\overline{\Gamma}_n(\varepsilon)\right) = \mu\left(\bigcup_{p=0}^{n-1} \overline{\gamma}_p(\varepsilon)\right) \qquad \leqslant \sum_{p=0}^{n-1} \mu\left(\overline{\gamma}_p(\varepsilon)\right) \ \cdot$$

Using (2.55b), we can write:

$$\mu\left(\overline{\Gamma}_n(\varepsilon)\right) \leqslant 2\,\varepsilon \sum_{p=0}^{n-1} \eta^p + \sum_{p=0}^{n-1} \pi\,\varepsilon^2 = 2\,\varepsilon\,\frac{\eta^n - 1}{\eta \ - 1} + n\,\pi\,\varepsilon^2 \ \cdot$$

Finally the estimate $\dfrac{x^p - 1}{x \ - 1} \leqslant p\,x^p$, valid for $x > 1$, yields

$$\mu\left(\overline{\Gamma}_n(\varepsilon)\right) \leqslant 2\,\varepsilon\,n\,\eta^n + n\,\pi\,\varepsilon^2 \qquad = \varepsilon\,n\,(2\,\eta^n + \pi\,\varepsilon) \ \cdot$$

d) For every real number t, $0 \leqslant \left\langle Nt + \frac{1}{2}\right\rangle = Nt + \frac{1}{2} - \left\lfloor Nt + \frac{1}{2}\right\rfloor < 1$; this leads to

$$\left| t - \frac{\left\lfloor Nt + \frac{1}{2}\right\rfloor}{N} \right| \leqslant \frac{1}{2N} \qquad , \qquad \forall\, t \in \mathbb{R}\cdot \tag{B.36}$$

Using (2.46), Definition 2.4.2, we write

$$d_{\mathbb{T}^2}\left(\boldsymbol{x}\ ,\ \frac{\hat{\boldsymbol{x}}_N}{N}\right) \leqslant \frac{1}{\sqrt{2}N} \qquad , \qquad \forall\, \boldsymbol{x} \in \mathbb{T}^2\cdot \tag{B.37}$$

If in the triangular inequality

$$d_{\mathbb{T}^2}(\boldsymbol{x}\ ,\ \boldsymbol{y}) \leqslant d_{\mathbb{T}^2}\left(\boldsymbol{x}\ ,\ \frac{\hat{\boldsymbol{x}}_N}{N}\right) + d_{\mathbb{T}^2}\left(\frac{\hat{\boldsymbol{x}}_N}{N}\ ,\ \boldsymbol{y}\right) \qquad \forall\, \boldsymbol{y} \in \mathbb{T}^2 \qquad , \tag{B.38}$$

we take the inf over $\boldsymbol{y} \in \Gamma_n$ of (2.45), we get

$$d_{\mathbb{T}^2}\left(\frac{\hat{\boldsymbol{x}}_N}{N}\ ,\ \Gamma_n\right) \geqslant d_{\mathbb{T}^2}(\boldsymbol{x}\ ,\ \Gamma_n) - d_{\mathbb{T}^2}\left(\boldsymbol{x}\ ,\ \frac{\hat{\boldsymbol{x}}_N}{N}\right) \tag{B.39}$$

$$\geqslant d_{\mathbb{T}^2}(\boldsymbol{x}\ ,\ \Gamma_n)\,\frac{1}{\sqrt{2}N}\ , \tag{B.40}$$

that is

$$\boldsymbol{x} \in \left[\overline{\Gamma}_n\left(\varepsilon\right)\right]^{\circ} \quad \Longrightarrow \quad \frac{\hat{\boldsymbol{x}}_N}{N} \in \left[\overline{\Gamma}_n\left(\varepsilon - \frac{1}{\sqrt{2}N}\right)\right]^{\circ} \tag{B.41}$$

Therefore, from (2.49), if $\frac{\hat{\boldsymbol{x}}_N}{N}$ does not belong to $\overline{\Gamma}_n\left(\varepsilon - \frac{1}{\sqrt{2}N}\right)$. then the corresponding $\boldsymbol{x}$ in (B.41) belongs to $G_n^N\left(\varepsilon - \frac{1}{\sqrt{2}N}\right)$. Changing $\varepsilon - \frac{1}{\sqrt{2}N} \longmapsto \varepsilon$ we obtain (2.55d).

e) Writing (2.55d) in terms of complementary sets, and substituting $\varepsilon = \frac{\widetilde{N}}{2N}$, we get:

$$\left[G_n^N\left(\frac{\widetilde{N}}{2N}\right)\right]^{\circ} \subseteq \overline{\Gamma}_n\left(\frac{\widetilde{N}}{2N} + \frac{1}{\sqrt{2}N}\right) \text{ and so} \tag{B.42}$$

$$\mu\left(\left[G_n^N\left(\frac{\widetilde{N}}{2N}\right)\right]^{\circ}\right) \leqslant \mu\left(\overline{\Gamma}_n\left(\frac{\widetilde{N}}{2N} + \frac{1}{\sqrt{2}N}\right)\right) \ . \tag{B.43}$$

Now we substitute $\frac{\widetilde{N}+\sqrt{2}}{2N} = \frac{\widetilde{N}}{2N} + \frac{1}{\sqrt{2}N}$ in place of ε in (2.55c) and we get:

$$\mu\left(\left[G_n^N\left(\frac{\widetilde{N}}{2N}\right)\right]^{\circ}\right) \leqslant \frac{\widetilde{N}+\sqrt{2}}{2N}\, n\left(2\,\eta^n + \pi\,\frac{\widetilde{N}+\sqrt{2}}{2N}\right) \ . \tag{B.44}$$

Finally we use:

$$\text{inside brackets of r.h.s of (B.44)} \quad : \quad \pi\,\frac{\widetilde{N}+\sqrt{2}}{2N} < 4 < 4\eta^n \quad , \quad \forall\, N > \widetilde{N} \tag{B.45}$$

$$\text{outside brackets of r.h.s of (B.44)} \quad : \quad \widetilde{N}+\sqrt{2} < 2\widetilde{N} \quad , \quad \forall\, N > \widetilde{N} \tag{B.46}$$

and this ends the proof. ■

Outlook & Perspective

This project has been performed with the aim to inquire the footprint of chaos present in classical dynamical systems even when some quantization procedure maps these systems into quantum (or discrete) ones, with a finite number of states. The framework in which we moved is the semi–classical analysis; we developed techniques of quantization and discretization by using the well known Weyl or Anti–Wick schemes of quantization, in particular we made use of family of suitably defined Coherent States.

We used the entropy production as a parameter of chaotic behaviour: in particular two notions of quantum dynamical entropy have been used, namely the CNT and ALF entropies, both reproducing the Kolmogorov entropy if applied to classical systems.

Quantum Dynamical Systems

We have shown that both the CNT and ALF entropies reproduce the Kolmogorov metric entropy in quantum systems too, provided that we observe a strongly chaotic system on a very short logarithmic time scale. However, due to the discreteness of the spectrum of the quantizations, we know that saturation phenomena will appear. It would be interesting to study the scaling behaviour of the quantum dynamical entropies in the intermediate region between the logarithmic breaking time and the Heisenberg time. This will, however, require quite different techniques than the coherent states approach.

Discretized Dynamical Systems

We have considered discretized hyperbolic classical systems on the torus by forcing them on a squared lattice with spacing $\frac{1}{N}$. We showed how the discretization procedure is similar to quantization; in particular, following the analogous case of the classical limit $\hbar \longmapsto 0$, we have set up the theoretical framework to discuss the continuous limit $N \longmapsto \infty$. Furthermore, using the similarities between discretized and quantized classical systems, we have applied the ALF entropy to study the footprints of classical (continuous) chaos as it is expected to reveal itself, namely through the presence of characteristic time scales and corresponding breaking–times. Indeed, exactly as in quantum chaos, a discretized hyperbolic system can mimic its continuous partner only up to times which scale as $\log N$, where N^2 is the number of allowed classical phase–point.

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