Classical Limit of Quantum Dynamical Entropies

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Abstract Two non-commutative dynamical entropies are studied in connection with the classical limit. For systems with a strongly chaotic classical limit, the Kolmogorov-Sinai invariant is recovered on time scales that are logarithmic in the quantization parameter. The model of the quantized hyperbolic automorphisms of the 2-torus is examined in detail.

Keywords and phrases: Quantum dynamical entropy, coherent states, semi-classical limit, hyperbolic automorphisms of the 2-torus

1 Introduction

Classical chaos is understood as motion on compact regions with trajectories highly sensitive to initial conditions [27, 19, 11, 33]. Once quantized, the motion has discrete energy spectrum and behaves almost periodically in time. Nevertheless, nature is fundamentally quantal and, according to the correspondence principle, classical behaviour emerges in the limit $\hbar \to 0$.

Also, classical and quantum mechanics are expected to almost coincide in the semi-classical regime, that is over times scaling as $\hbar^{-\alpha}$ for some $\alpha > 0$ [33]. Actually, this is true only for regular classical limits, while for chaotic

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ones the semi-classical regime typically scales as $-\log\hbar$ [19, 11, 33]. Both time scales diverge when $\hbar \to 0$, but the shortness of the latter means that classical mechanics has to be replaced by quantum mechanics much sooner for quantum systems with chaotic classical behaviour. The logarithmic breaking time $-\log\hbar$ has been considered by some as a violation of the correspondence principle [18, 17], by others, see [11] and Chirikov in [19], as the evidence that time and classical limits do not commute.

The analytic studies of logarithmic time scales have been mainly performed by means of semi-classical tools, essentially by focusing, via coherent state techniques, on the phase space localization of specific time evolving quantum observables. In the following, we shall show how they emerge in the context of quantum dynamical entropies. As a particular example, we shall concentrate on finite dimensional quantizations of hyperbolic automorphisms of the 2-torus, which are prototypes of chaotic behaviour; indeed, their trajectories separate exponentially fast with a Lyapounov exponent $\log\lambda > 0$ [7, 31]. Standard quantization, à la Berry, of hyperbolic automorphisms [10, 14] yields Hilbert spaces of a finite dimension N. This dimension plays the role of semi-classical parameter and sets the minimal size $1/N$ of quantum phase space cells.

By the theorems of Ruelle and Pesin [21], the positive Lyapounov exponents of smooth, classical dynamical systems are related to the dynamical entropy of Kolmogorov [20] which measures the information per time step provided by the dynamics. There are several candidates for non-commutative extensions of the latter [12, 3, 30, 1, 28]: in this paper we shall use two of them [12, 3] and study their semi-classical limit. We show that, from both of them, one recovers the Kolmogorov-Sinai entropy by computing the average quantum entropy produced over a logarithmic time scale and then taking the classical limit. This confirms the numerical results in [5], where the dynamical entropy [3] is applied to the study of the quantum kicked top. In this approach, the presence of logarithmic time scales indicates the typical scaling for a joint time–classical limit suited to preserve positive entropy production in quantized classically chaotic quantum systems.

The paper is organized as follows: Section 2 contains a brief review of the algebraic approach to classical and dynamical systems, while Section 3 introduces some basic semi-classical tools. Sections 4 and 5 deal with the quantization of hyperbolic maps on finite dimensional Hilbert spaces and the relation between classical and time limits. Section 6 gives an overview of the quantum dynamical entropy of Connes, Narnhofer and Thirring [12](CNT–entropy) and of Alicki and Fannes [4, 3] (ALF-entropy, where L stands for Lindblad); finally, in Section 7 their semi-classical behaviour is studied and the emergence of a typical logarithmic time scale is showed.

2 Dynamical systems: algebraic setting

We consider reversible, discrete time, compact classical dynamical systems that can be represented by a triple $(\mathcal{X}, T, \mu)$, where:

- $\mathcal{X}$ is a compact metric space: the phase space of the system.
- T is a measurable transformation of $\mathcal{X}$ that is invertible such that T^{-1} is also measurable. The group $\{T^k \mid k \in \mathbb{Z}\}$ implements the conservative dynamics in discrete time.
- μ is a T-invariant probability measure on $\mathcal{X}$, i.e. $\mu \circ T = \mu$.

In this paper, we consider a general scheme for quantizing and dequantizing, i.e. for taking the classical limit (see [32]). Within this framework, we focus on the semi-classical limit of quantum dynamical entropies of finite dimensional quantizations of the Arnold cat map and of generic hyperbolic automorphisms of the 2-torus, cat maps for short. In order to make the quantization procedure more explicit, it proves useful to follow an algebraic approach and replace $(\mathcal{X}, T, \mu)$ with $(\mathfrak{M}_\mu, \Theta, \omega_\mu)$ where

- $\mathfrak{M}_\mu$ is the von Neumann algebra $\mathfrak{L}^\infty_\mu(\mathcal{X})$ of (equivalence classes of) essentially bounded μ-measurable functions on $\mathcal{X}$, equipped with the so-called essential supremum norm $\|\cdot\|_\infty$ [26].

- ω_μ is the state on $\mathfrak{M}_\mu$ defined by the reference measure μ

$$\omega_\mu(f) := \int_{\mathcal{X}} \mu(dx)\, f(x).$$

- $\{\Theta^k \mid k \in \mathbb{Z}\}$ is the discrete group of automorphisms of $\mathfrak{M}_\mu$ which implements the dynamics: $\Theta(f) := f \circ T^{-1}$. The invariance of the reference measure reads now $\omega_\mu \circ \Theta = \omega_\mu$.

Quantum dynamical systems are described in a completely similar way by a triple $(\mathfrak{M}, \Theta, \omega)$, the critical difference being that the algebra of observables $\mathfrak{M}$ is no longer Abelian:

- $\mathfrak{M}$ is a von Neumann algebra of operators, the observables, acting on a Hilbert space $\mathfrak{H}$.

- Θ is an automorphism of $\mathfrak{M}$.

- ω is an invariant normal state on $\mathfrak{M}$: $\omega \circ \Theta = \omega$.

Quantizing essentially corresponds to suitably mapping the commutative, classical triple $(\mathfrak{M}_\mu, \Theta, \omega_\mu)$ to a non-commutative, quantum triple $(\mathfrak{M}, \Theta, \omega)$.

3 Classical limit: coherent states

Performing the classical limit or a semi-classical analysis consists in studying how a family of algebraic triples $(\mathfrak{M}, \Theta, \omega)$ depending on a quantization $\hbar$-like parameter is mapped onto $(\mathfrak{M}_\mu, \Theta, \omega_\mu)$ when the parameter goes to zero. The most successful semi-classical tools are based on the use of coherent states.

For our purposes, we shall use a large integer N as a quantization parameter, i.e. we use $1/N$ as the $\hbar$-like parameter. In fact, we shall consider cases where $\mathfrak{M}$ is the algebra $\mathcal{M}_N$ of N-dimensional square matrices acting on $\mathbb{C}^N$, the quantum reference state is the normalized trace $\frac{1}{N}\,\mathrm{Tr}$ on $\mathcal{M}_N$, denoted by τ_N and the dynamics is given in terms of a unitary operator U_T on $\mathbb{C}^N$ in the standard way: $\Theta_N(X) := U_T^* X\, U_T$.

In full generality, coherent states will be identified as follows.

Definition 3.1 *A family* $\{|C_N(x)\rangle \mid x \in \mathcal{X}\} \in \mathfrak{H}$ *of vectors, indexed by points* $x \in \mathcal{X}$*, constitutes a set of coherent states if it satisfies the following requirements*

1. *Measurability:* $x \mapsto |C_N(x)\rangle$ *is measurable on* $\mathcal{X}$*;*
2. *Normalization:* $\|C_N(x)\|^2 = 1$, $x \in \mathcal{X}$*;*
3. *Overcompleteness:* $N \int_{\mathcal{X}} \mu(dx)\, |C_N(x)\rangle\langle C_N(x)| = \mathbb{1}$*;*
4. *Localization: given* $\varepsilon > 0$ *and* $d_0 > 0$*, there exists* $N_0(\epsilon, d_0)$ *such that for* $N \geq N_0$ *and* $d(x,y) \geq d_0$ *one has*

$$N|\langle C_N(x), C_N(y)\rangle|^2 \leq \varepsilon.$$

The overcompleteness condition may be written in dual form as

$$N \int_{\mathcal{X}} \mu(dx)\, \langle C_N(x), X\, C_N(x)\rangle = \mathrm{Tr}\, X, \quad X \in \mathcal{M}_N.$$

Indeed,

$$N \int_{\mathcal{X}} \mu(dx)\, \langle C_N(x), X\, C_N(x)\rangle = N\, \mathrm{Tr} \int_{\mathcal{X}} \mu(dx)\, |C_N(x)\rangle\langle C_N(x)|\, X = \mathrm{Tr}\, X.$$

3.1 Anti-Wick Quantization

In order to study the classical limit and, more generally, semi-classical behaviour of $(\mathcal{M}_N, \Theta_N, \tau_N)$ when $N \to \infty$, we introduce two linear maps. The first, $\gamma_{N\infty}$, (anti-Wick quantization) associates $N \times N$ matrices to functions in $\mathfrak{M}_\mu = \mathfrak{L}^\infty_\mu(\mathcal{X})$, the second one, $\gamma_{\infty N}$, maps $N \times N$ matrices to functions in $\mathfrak{L}^\infty_\mu(\mathcal{X})$.

Definition 3.2 *Given a family* $\{|C_N(x)\rangle \mid x \in \mathcal{X}\}$ *of coherent states in* $\mathbb{C}^N$*, the anti-Wick quantization scheme will be described by a (completely) positive unital map* $\gamma_{N\infty} : \mathfrak{M}_\mu \to \mathcal{M}_N$

$$\mathfrak{M}_\mu \ni f \mapsto N \int_{\mathcal{X}} \mu(dx)\, f(x)\, |C_N(x)\rangle\langle C_N(x)| =: \gamma_{N\infty}(f) \in \mathcal{M}_N \quad .$$

The corresponding dequantizing map $\gamma_{\infty N} : \mathcal{M}_N \to \mathfrak{M}_\mu$ *will correspond to the (completely) positive unital map*

$$\mathcal{M}_N \ni X \mapsto \langle C_N(x), X\, C_N(x)\rangle =: \gamma_{\infty N}(X)(x) \in \mathfrak{M}_\mu \quad .$$

Both maps are identity preserving because of the conditions imposed on the family of coherent states and are also completely positive since the domain of $\gamma_{N\infty}$ is a commutative algebra as well as the range of $\gamma_{\infty N}$. Moreover,

$$\big\|\gamma_{\infty N} \circ \gamma_{N\infty}(g)\big\|_\infty \leq \|g\|_\infty, \quad g \in \mathfrak{M}_\mu \quad , \tag{1}$$

where $\|\cdot\|_\infty$ denotes the essential norm on $\mathfrak{M}_\mu = \mathfrak{L}^\infty_\mu(\mathcal{X})$. The following two equivalent properties are less trivial:

Proposition 3.1 *For all* $f \in \mathfrak{M}_\mu$

$$\lim_{N\to\infty} \gamma_{\infty N} \circ \gamma_{N\infty}(f) = f \quad \mu\text{-}a.e.$$

Proposition 3.2 *For all* $f, g \in \mathfrak{M}_\mu$

$$\lim_{N\to\infty} \tau_N\big(\gamma_{N\infty}(f)^* \gamma_{N\infty}(g)\big) = \omega_\mu(\overline{f} g) = \int_{\mathcal{X}} \mu(dx)\, \overline{f(x)} g(x).$$

The previous two propositions can be taken as requests on any well–defined quantization–dequantization scheme for observables. In the sequel, we shall need the notion of quantum dynamical systems $(\mathcal{M}_N, \Theta_N, \tau_N)$ tending to the classical limit $(\mathcal{X}, T, \mu)$. We then not only need convergence of observables but also of the dynamics. This aspect will be considered in Section 5.

Proof of Proposition 3.1:

We first prove the assertion when f is continuous on $\mathcal{X}$ and then remove this condition. We show that the quantity

$$\begin{aligned} F_N(x) &:= \Big| f(x) - \gamma_{\infty N} \circ \gamma_{N\infty}(f)(x) \Big| \\ &= \Big| f(x) - N \int_{\mathcal{X}} \mu(dy)\, f(y)\, |\langle C_N(x), C_N(y)\rangle|^2 \Big| \\ &= N \Big| \int_{\mathcal{X}} \mu(dy)\, (f(y) - f(x))\, |\langle C_N(x), C_N(y)\rangle|^2 \Big| \end{aligned}$$

becomes arbitrarily small for N large enough, uniformly in x. Selecting a ball $B(x, d_0)$ of radius d_0, using the mean-value theorem and property (3.1.3), we derive the upper bound

$$\begin{aligned} F_N(x) \leq N &\Big| \int_{B(x,d_0)} \mu(dy)\, (f(y) - f(x))\, |\langle C_N(x), C_N(y)\rangle|^2 \Big| \\ &+ N \Big| \int_{\mathcal{X}\setminus B(x,d_0)} \mu(dy)\, (f(y) - f(x))\, |\langle C_N(x), C_N(y)\rangle^2 \Big| \qquad (2) \\ &\leq |f(c) - f(x)| + \int_{\mathcal{X}\setminus B(x,d_0)} \mu(dy)\, |f(y) - f(x)|\, N\, |\langle C_N(x), C_N(y)\rangle|^2, \qquad (3) \end{aligned}$$

where $c \in B(x, d_0)$.

Because $\mathcal{X}$ is compact, f is uniformly continuous. Therefore, we can choose d_0 in such a way that $|f(c) - f(x)| < \varepsilon$ uniformly in $x \in \mathcal{X}$. On the other hand, from the localization property (3.1.4), given $\varepsilon' > 0$, there exists an integer $N_0(\varepsilon', d_0)$ such that $N |\langle C_N(x), C_N(y)\rangle|^2 < \varepsilon'$ whenever $N > N_0(\varepsilon', d_0)$. This choice leads to the upper bound

$$\begin{aligned} F_N(x) &\leq \varepsilon + \varepsilon' \int_{\mathcal{X}\setminus B(x,d_0)} \mu(dy)\, |f(y) - f(x)| \\ &\leq \varepsilon + \varepsilon' \int_{\mathcal{X}} \mu(dy)\, |f(y) - f(x)| \leq \varepsilon + 2\varepsilon' \|f\|_\infty. \qquad (4) \end{aligned}$$

To get rid of the continuity of f, we use Lusin's theorem [26]. It states that, given $f \in \mathfrak{L}^\infty_\mu(\mathcal{X})$, with $\mathcal{X}$ compact, there exists a sequence $\{f_n\}$ of continuous functions on $\mathcal{X}$ such that $|f_n| \leq \|f\|_\infty$ and converging to f μ-almost everywhere. Thus, for $f \in \mathfrak{L}^\infty_\mu(\mathcal{X})$, we pick such a sequence and estimate

$$F_N(x) \leq |f(x) - f_n(x)| + \Big|f_n(x) - \gamma_{\infty N} \circ \gamma_{N\infty}(f_n)(x)\Big| \\ + \Big|\gamma_{\infty N} \circ \gamma_{N\infty}(f_n - f)(x)\Big|.$$

The first term can be made arbitrarily small ($\mu.a.e$) by choosing n large enough because of Lusin's theorem, while the second one goes to 0 when $N \to \infty$ since f_n is continuous. Finally, the third term becomes as well vanishingly small with $n \to \infty$ as one can deduce from

$$\begin{aligned} &\int_{\mathcal{X}} \mu(dx) \Big|\gamma_{\infty N} \circ \gamma_{N\infty}(f - f_n)(x)\Big| \\ &= \int_{\mathcal{X}} \mu(dx) \left|\int_{\mathcal{X}} \mu(dy)\,(f(y) - f_n(y))\, N\, |\langle C_N(x), C_N(y)\rangle|^2\right| \\ &\leq \int_{\mathcal{X}} \mu(dy)\, |f(y) - f_n(y)| \int_{\mathcal{X}} \mu(dx)\, N\, |\langle C_N(x), C_N(y)\rangle|^2 \\ &= \int_{\mathcal{X}} \mu(dy)\, |f(y) - f_n(y)|, \end{aligned}$$

where exchange of integration order is harmless because of the existence of the integral (1). The last integral goes to zero with n by dominated convergence and thus the result follows. ■

Proof of Proposition 3.2:

Consider

$$\begin{aligned} \Omega_N &:= \Big|\tau_N\big(\gamma_{N\infty}(f)^* \gamma_{N\infty}(g)\big) - \omega_\mu(\overline{f} g)\Big| \\ &= N \left|\int_{\mathcal{X}} \mu(dx)\, \overline{f(x)} \int_{\mathcal{X}} \mu(dy)\,(g(y) - g(x))\, |\langle C_N(x), C_N(y)\rangle|^2\right| \\ &\leq \int_{\mathcal{X}} \mu(dx)\, |f(x)| \left|\int_{\mathcal{X}} \mu(dy)\,(g(y) - g(x))\, N\, |\langle C_N(x), C_N(y)\rangle|^2\right|. \end{aligned}$$

By choosing a sequence of continuous g_n approximating $g \in \mathfrak{L}^\infty_\mu(\mathcal{X})$, and arguing as in the previous proof, we get the following upper bound:

$$\begin{aligned}\Omega_N \leq N \int_{\mathcal{X}} \mu(dx)\,|f(x)| &\left|\int_{\mathcal{X}} \mu(dy)\,(g(y) - g_n(y))\,|\langle C_N(x), C_N(y)\rangle|^2\right| \\ + N \int_{\mathcal{X}} \mu(dx)\,|f(x)| &\left|\int_{\mathcal{X}} \mu(dy)\,(g_n(y) - g_n(x))\,|\langle C_N(x), C_N(y)\rangle|^2\right| \\ + N \int_{\mathcal{X}} \mu(dx)\,|f(x)| &\left|\int_{\mathcal{X}} \mu(dy)\,(g(x) - g_n(x))\,|\langle C_N(x), C_N(y)\rangle|^2\right| .\end{aligned}$$

The integrals in the first and third lines go to zero by dominated convergence and Lusin's theorem. As regards the middle line, one can apply the argument used for the quantity $F_N(x)$ in the proof of Proposition 3.1.

■

4 Classical and quantum cat maps

In this section, we collect the basic material needed to describe both classical and quantum cat maps and we introduce a specific set of coherent states that will enable us to perform the semi-classical analysis of the dynamical entropy.

4.1 Finite dimensional quantizations

We first introduce cat maps in the spirit of the algebraic formulation introduced in the previous sections.

Definition 4.1 *Hyperbolic automorphisms of the torus, i.e. cat maps, are generically represented by triples* $(\mathfrak{M}_\mu, \Theta, \omega_\mu)$*, where*

- $\mathfrak{M}_\mu$ *is the algebra of essentially bounded functions on the two dimensional torus* $\mathbb{T} := \left\{ \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 \pmod 1 \right\}$*, equipped with the Lebesgue measure* $\mu(d\boldsymbol{x}) := d\boldsymbol{x}$*.*

- $\{\Theta^k \mid k \in \mathbb{Z}\}$ *is the family of automorphisms (discrete time evolution) given by* $\mathfrak{M}_\mu \ni f \mapsto (\Theta^k f)(\boldsymbol{x}) := f(A^{-k}\boldsymbol{x} \pmod 1)$*, where*

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *has integer entries such that* $ad - cb = 1$, $|a+d| > 2$ *and maps* $\mathbb{T}$ *onto itself.*

- ω_μ *is the expectation obtained by integration with respect to the Lebesgue measure:* $\mathfrak{M}_\mu \ni f \mapsto \omega_\mu(f) := \int_{\mathbb{T}} d\boldsymbol{x}\, f(\boldsymbol{x})$, *that is left invariant by* Θ.

The matrix A has irrational eigenvalues $1 < \lambda$, λ^{-1}, therefore distances stretch along the eigendirection $\boldsymbol{u}$ of λ, while shrink along $\boldsymbol{v}$, the eigendirection of λ^{-1}. Once the folding condition is added, the hyperbolic automorphisms of the torus become prototypes of classical chaos, with positive Lyapounov exponent $\log\lambda$.

One can quantize the associated algebraic triple $(\mathfrak{M}_\mu, \Theta, \omega_\mu)$ on either infinite [9] or finite dimensional Hilbert spaces [10, 14, 13].

In the following, we shall focus on the latter. Given an integer N, we consider an orthonormal basis $|j\rangle$ of $\mathbb{C}^N$, where the index j runs through $\mathbb{Z}_N$, namely $|j+N\rangle \equiv |j\rangle$, $j \in \mathbb{Z}$. By using this basis we define two unitary matrices U_N and V_N as follows:

$$U_N|j\rangle := \exp\big(\frac{2\pi i}{N} u\big)|j+1\rangle, \qquad \text{and} \qquad V_N|j\rangle := \exp\Big(\frac{2\pi i}{N}(v-j)\Big)|j\rangle. \quad (5)$$

$u, v \in [0,1)$ are parameters labelling the representations and

$$U_N^N = \mathrm{e}^{2i\pi u}\, \mathbb{1}_N, \quad V_N^N = \mathrm{e}^{2i\pi v}\, \mathbb{1}_N. \tag{6}$$

It turns out that

$$U_N V_N = \exp\big(\frac{2i\pi}{N}\big)\, V_N U_N. \tag{7}$$

Introducing Weyl operators labeled by $\boldsymbol{n} = (n_1, n_2) \in \mathbb{Z}^2$

$$W_N(\boldsymbol{n}) := \exp\big(\frac{i\pi}{N} n_1 n_2\big)\, V_N^{n_2} U_N^{n_1} = W_N(-\boldsymbol{n})^* \tag{8}$$

it follows that

$$W_N(N\boldsymbol{n}) = \mathrm{e}^{i\pi(Nn_1n_2+2n_1u+2n_2v)} \tag{9}$$

$$W_N(\boldsymbol{n})W_N(\boldsymbol{m}) = \exp\left(\frac{i\pi}{N}\sigma(\boldsymbol{n},\boldsymbol{m})\right) W_N(\boldsymbol{n}+\boldsymbol{m}), \tag{10}$$

where $\sigma(\boldsymbol{n},\boldsymbol{m}) := n_1m_2 - n_2m_1$.

Definition 4.2 *Quantized cat maps will be identified with algebraic triples* $(\mathcal{M}_N, \Theta_N, \tau_N)$ *where*

- $\mathcal{M}_N$ *is the full* $N \times N$ *matrix algebra linearly spanned by the Weyl operators* $W_N(\boldsymbol{n})$*.*

- $\Theta_N : \mathcal{M}_N \mapsto \mathcal{M}_N$ *is the automorphism such that*

$$W_N(\boldsymbol{p}) \mapsto \Theta_N(W_N(\boldsymbol{p})) := W_N(A\boldsymbol{p}), \quad \boldsymbol{p} \in \mathbb{Z}^2. \tag{11}$$

In the definition of above, we have omitted reference to the parameters u, v in (5): they must be chosen such that

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} + \frac{N}{2} \begin{pmatrix} ac \\ bd \end{pmatrix} \pmod 1. \tag{12}$$

Then, the folding condition (9) is compatible with the time evolution [14]. Further, the algebraic relations (10) are also preserved since the symplectic form remains invariant, i.e. $\sigma(A^t\boldsymbol{n}, A^t\boldsymbol{m}) = \sigma(\boldsymbol{n},\boldsymbol{m})$.

Useful relations can be obtained by using

$$W_N(\boldsymbol{n})\,|j\rangle = \exp\left(\frac{i\pi}{N}(-n_1n_2 + 2n_1u + 2n_2v)\right)\,\exp\left(-\frac{2i\pi}{N}jn_2\right)|j+n_1\rangle\ . \tag{13}$$

From (13) one readily derives

$$\tau_N(W_N(\boldsymbol{n})) = \mathrm{e}^{\frac{i\pi}{N}(-n_1n_2+2n_1u+2n_2v)}\,\delta^{(N)}_{\boldsymbol{n},0}\,, \tag{14}$$

$$\tau_N(W_N(A\boldsymbol{n})) = \tau_N(W_N(\boldsymbol{n})), \tag{15}$$

$$\frac{1}{N}\sum_{p_1,p_2=0}^{N-1} W_N(-\boldsymbol{p})\,W_N(\boldsymbol{n})\,W_N(\boldsymbol{p}) = \mathrm{Tr}\Big(W_N(\boldsymbol{n})\Big)\,\mathbb{1}_N, \tag{16}$$

$$\mathcal{M}_N \ni X = \sum_{p_1,p_2=0}^{N-1} \tau_N\Big(X\,W_N(-\boldsymbol{p})\Big)\,W_N(\boldsymbol{p}). \tag{17}$$

In (14), we have introduced the periodic Kronecker delta, that is $\delta^{(N)}_{\mathbf{n},0} = 1$ if and only if $\mathbf{n} = 0 \mod (N)$.

From equation (10) one derives

$$[W_N(\boldsymbol{n}), W_N(\boldsymbol{m})] = 2i\sin\big(\frac{\pi}{N}\sigma(\boldsymbol{n},\boldsymbol{m})\big)\,W_N(\boldsymbol{n}+\boldsymbol{m}),$$

which suggests that the $\hbar$-like parameter is $1/N$ and that the classical limit correspond to $N \to \infty$. In the following section, we set up a coherent state technique suited to study classical cat maps as limits of quantized cats.

4.2 Coherent states for cat maps

We shall construct a family $\{|C_N(\boldsymbol{x})\rangle \mid \boldsymbol{x} \in \mathbb{T}\}$ of coherent states on the 2-torus by means of the discrete Weyl group. We define

$$|C_N(\boldsymbol{x})\rangle := W_N([N\boldsymbol{x}])\,|C_N\rangle, \tag{18}$$

where $[N\boldsymbol{x}] = ([Nx_1],[Nx_2])$, $0 \le [Nx_i] \le N-1$ is the largest integer smaller than Nx_i and the fundamental vector $|C_N\rangle$ is chosen to be

$$|C_N\rangle = \sum_{j=0}^{N-1} C_N(j)|j\rangle, \qquad C_N(j) := \frac{1}{2^{(N-1)/2}}\sqrt{\binom{N-1}{j}}. \tag{19}$$

Measurability and normalization are immediate, over-completeness comes as follows. Let Y be the operator in the left hand side of property (3.1.3). If $\tau_N(Y\,W_N(\boldsymbol{n})) = \tau_N(W_N(\boldsymbol{n}))$ for all $\boldsymbol{n} = (n_1, n_2)$ with $0 \le n_i \le N-1$, then according to (17) applied to Y it follows that $Y = \mathbb{1}$. This is indeed the case as, using (9) and N-periodicity,

$$\begin{aligned}
\tau_N(Y\,W_N(\boldsymbol{n})) &= \int_{\mathbb{T}} d\boldsymbol{x}\, \langle C_N(\boldsymbol{x}), W_N(\boldsymbol{n})\, C_N(\boldsymbol{x})\rangle \\
&= \int_{\mathbb{T}} d\boldsymbol{x}\, \exp\Big(\frac{2\pi i}{N}\sigma(\boldsymbol{n}, [N\boldsymbol{x}])\Big)\, \langle C_N, W_N(\boldsymbol{n})\, C_N\rangle \\
&= \frac{1}{N^2}\sum_{p_1,p_2=0}^{N-1} \exp\Big(\frac{2\pi i}{N}\sigma(\boldsymbol{n}, \boldsymbol{p})\Big)\, \langle C_N, W_N(\boldsymbol{n})\, C_N\rangle \\
&= \tau_N(W_N(\boldsymbol{n})). \qquad (20)
\end{aligned}$$

In the last line we used that when $\boldsymbol{x}$ runs over $\mathbb{T}$, $[Nx_i]$, $i = 1, 2$ runs over the set of integers $0, 1, \ldots, N-1$.

The proof the localization property (3.1.4) requires several steps. First, we observe that, due to (6),

$$\begin{aligned}
E(n) &:= \Big|\langle C_N, W_N(\boldsymbol{n})\, C_N\rangle\Big| \\
&= \frac{1}{2^{N-1}}\Bigg|\sum_{\ell=0}^{N-n_1-1} \exp\Big(-\frac{2\pi i}{N}\ell n_2\Big)\sqrt{\binom{N-1}{\ell}\binom{N-1}{\ell+n_1}} \\
&\quad + \sum_{\ell=N-n_1}^{N-1} \exp\Big(-\frac{2\pi i}{N}\ell n_2\Big)\sqrt{\binom{N-1}{\ell}\binom{N-1}{\ell+n_1-N}}\Bigg| \qquad (21) \\
&\le \frac{1}{2^{N-1}}\Bigg[\sum_{\ell=0}^{N-n_1-1} \sqrt{\binom{N-1}{\ell}\binom{N-1}{\ell+n_1}} \\
&\quad + \sum_{\ell=N-n_1}^{N-1} \sqrt{\binom{N-1}{\ell}\binom{N-1}{\ell+n_1-N}}\Bigg]. \qquad (22)
\end{aligned}$$

Second, using the entropic bound of the binomial coefficients

$$\binom{N-1}{\ell} \le 2^{(N-1)\eta(\frac{\ell}{N-1})} \quad , \qquad (23)$$

where

$$\eta(t) := \begin{cases} -t\log_2 t - (1-t)\log_2(1-t) & \text{if } 0 < t \le 1 \\ 0 & \text{if } t = 0 \end{cases} , \tag{24}$$

we estimate

$$E(n) \le \frac{1}{2^{N-1}} \left[\sum_{\ell=0}^{N-1-n_1} 2^{\frac{N-1}{2}\left[\eta\left(\frac{\ell}{N-1}\right)+\eta\left(\frac{\ell+n_1}{N-1}\right)\right]} + \sum_{\ell=N-n_1}^{N-1} 2^{\frac{N-1}{2}\left[\eta\left(\frac{\ell}{N-1}\right)+\eta\left(\frac{\ell+n_1-N}{N-1}\right)\right]} \right] . \tag{25}$$

The exponents in the two sums are bounded by their maxima

$$\eta\left(\frac{\ell}{N-1}\right) + \eta\left(\frac{\ell+n_1}{N-1}\right) \le 2\eta_1(n_1), \quad (0 \le \ell \le N-n_1-1) \tag{26}$$

$$\eta\left(\frac{\ell}{N-1}\right) + \eta\left(\frac{\ell+n_1-N}{N-1}\right) \le 2\eta_2(n_1), \quad (N-n_1 \le \ell \le N-1) \tag{27}$$

where

$$\eta_1(n_1) := \eta\left(\frac{1}{2} - \frac{n_1}{2(N-1)}\right) \le 1 \tag{28}$$

$$\eta_2(n_1) := \eta\left(\frac{1}{2} + \frac{N-n_1}{2(N-1)}\right) \le \eta_2 < 1. \tag{29}$$

Notice that η_2 is automatically < 1, while $\eta_1(n_1) < 1$ if $\lim_N n_1/N \neq 0$. If so, the upper bound

$$E(n) \le N\Big(2^{-(N-1)(1-\eta_1(n_1))} + 2^{-(N-1)(1-\eta_2)}\Big) \tag{30}$$

implies $N\big|\langle C_N, W_N(\boldsymbol{n})\, C_N\rangle\big|^2 \longmapsto 0$ exponentially with $N \to \infty$.

The condition for which $\eta_1(n_1) < 1$ is fulfilled when $|x_1 - y_1| > \delta$; in fact, $\boldsymbol{n} = [N\boldsymbol{y}] - [N\boldsymbol{x}]$ and $\lim_N([Nx_1] - [Ny_1])/N = x_1 - y_1$. On the other hand, if $x_1 = y_1$ and $n_2 = [Nx_2] - [Ny_2] \neq 0$, one explicitly computes

$$N\left|\langle C_N, W_N((0,n_2))\, C_N\rangle\right|^2 = N\Big(\cos^2\big(\frac{\pi n_2}{N}\big)\Big)^{N-1}. \tag{31}$$

Again, the above expression goes exponentially fast to zero, if $\lim_N n_2/N \neq 0$ which is the case if $x_2 \neq y_2$.

5 Quantum and classical time evolutions

One of the main issues in the semi-classical analysis is to compare if and how the quantum and classical time evolutions mimic each other when a quantization parameter goes to zero.

In the case of classically chaotic quantum systems, the situation is strikingly different from the case of classically integrable quantum systems. In the former case, classical and quantum mechanics agree on the level of coherent states only over times which scale as $-\log\hbar$. As before, let T denote the evolution on the classical phase space $\mathcal{X}$ and U_T the unitary single step evolution on $\mathbb{C}^N$. We formally impose the relation between the classical and quantum evolution on the level of coherent states through:

Condition 5.1 Dynamical localization: *There exists an $\alpha > 0$ such that for all choices of $\varepsilon > 0$ and $d_0 > 0$ there exists an $N_0 \in \mathbb{N}$ with the following property: if $N > N_0$ and $k \leq \alpha \log N$, then $N|\langle U_T^k C_N(x), C_N(y)\rangle|^2 \leq \varepsilon$ whenever $d(T^k x, y) \geq d_0$.*

Remark The condition of dynamical localization is what is expected of a good choice of coherent states, namely, on a time scale logarithmic in the inverse of the semi-classical parameter, evolving coherent states should stay localized around the classical trajectories. Informally, when $N \to \infty$, the quantities

$$K_k(x,y) := \langle U_T^k C_N(x), C_N(y)\rangle \tag{32}$$

should behave as if $N|K_k(x,y)|^2 \simeq \delta(T^k x - y)$. The constraint $k \leq \alpha \log N$ is typical of hyperbolic classical behaviour and comes heuristically as follows. The maximal localization of coherent states cannot exceed the minimal coarse-graining dictated by $1/N$; if, while evolving, coherent states stayed localized forever around the classical trajectories, they would get more and more localized along the contracting direction. Since for hyperbolic systems the increase of localization is exponential with Lyapounov exponent $\lambda_{Lyap} > 0$, this sets the upper bound and indicates that $\alpha \simeq 1/\lambda_{Lyap}$.

Proposition 5.1 *Let $(\mathcal{M}_N, \Theta_N, \tau_N)$ be a general quantum dynamical system as defined in Section 3 and suppose that it satisfies Condition 5.1. Let $\|X\|_2 := \sqrt{\tau_N(X^*X)}$, $X \in \mathcal{M}_N$ denote the normalized Hilbert-Schmidt norm. In the ensuing topology*

$$\lim_{\substack{k,\ N\to\infty \\ k<\alpha\log N}} \|\Theta_N^k \circ \gamma_{N\infty}(f) - \gamma_{N\infty} \circ \Theta^k(f)\|_2 = 0. \tag{33}$$

Proof:

One computes

$$\begin{aligned}
&\|\Theta_N^k \circ \gamma_{N\infty}(f) - \gamma_{N\infty} \circ \Theta^k(f)\|_2^2 \\
&= 2N \int_{\mathcal{X}} \mu(dx) \int_{\mathcal{X}} \mu(dy)\, \overline{f(x)}\, f(y)\, |\langle C_N(x), C_N(y)\rangle|^2 \\
&\quad - 2N \mathfrak{Re}\left[\int_{\mathcal{X}} \mu(dx) \int_{\mathcal{X}} \mu(dy)\, \overline{f(y)}\, f(T^k x) |\langle U_T^k C_N(x), C_N(y)\rangle|^2\right].
\end{aligned} \tag{34}$$

The double integral in the first term goes to $\int \mu(dx)|f(x)|^2$. So, we need to show that the second integral, which we shall denote by $I_N(k)$, does the same. We will concentrate on the case of continuous f, the extension to essentially bounded f is straightforward. Explicitly, selecting a ball $B(T^k x, d_0)$, one derives

$$
\begin{aligned}
&\left| I_N(k) - \int_{\mathcal{X}} \mu(dy)\, |f(y)|^2 \right| \\
&= \left| \int_{\mathcal{X}} \mu(dx) \int \mu(dy)\, \overline{f(y)} \big(f(T^k x) - f(y) \big)\, N |\langle U_T^k C_N(x), C_N(y) \rangle|^2 \right| \\
&\leq \left| \int_{\mathcal{X}} \mu(dx) \int_{B(T^k x, d_0)} \mu(dy)\, \overline{f(y)} \big(f(T^k x) - f(y) \big) N |\langle U_T^k C_N(x), C_N(y) \rangle|^2 \right| \\
&+ \left| \int_{\mathcal{X}} \mu(dx) \int_{\mathcal{X} \setminus B(T^k x, d_0)} \mu(dy) \overline{f(y)} \big(f(T^k x) - f(y) \big) N |\langle U_T^k C_N(x), C_N(y) \rangle|^2 \right| .
\end{aligned}
$$

Applying the mean value theorem and approximating the integral of the kernel as in the proof of Proposition 3.2, we get that $\exists c \in B(T^k x, d_0)$ such that

$$
\begin{aligned}
&\left| I_N(k) - \int_{\mathcal{X}} \mu(dy)\, |f(y)|^2 \right| \\
&\leq \left| \int_{\mathcal{X}} \mu(dx)\, \overline{f(c)}\, \big(f(T^k x) - f(c) \big)\, N |\langle U_T^k C_N(x), C_N(y) \rangle|^2 \right| \\
&+ \left| \int_{\mathcal{X}} \mu(dx) \int_{\mathcal{X} \setminus B(T^k x, d_0)} \mu(dy) \overline{f(y)} \big(f(T^k x) - f(y) \big) N |\langle U_T^k C_N(x), C_N(y) \rangle|^2 \right| .
\end{aligned}
$$

By uniform continuity we can bound the first term by some arbitrary small ε, provided we choose d_0 small enough. Now, for the second integral we use our localization condition 5.1. As the constraint $k \leq \alpha \log N$ has to be enforced, we have to take a joint limit of time and size of the system with this constraint. In that case the second integral can also be bounded by an arbitrarily small ε', provided N is large enough.

■

We shall not prove the dynamical localization condition 5.1 for the quantum cat maps but instead provide a direct derivation of formula (33) based on the simple expression (11) of the dynamics when acting on Weyl operators. For this reason, we introduce the Weyl quantization:

Definition 5.1 *Let f be a function in $\mathfrak{L}^\infty_\mu(\mathbb{T})$, $\mathbb{T}$ denoting the two–dimensional torus, whose Fourier series $\hat{f}$ has only finitely many non-zero terms. We shall* *denote by* $\mathrm{Supp}(\hat{f})$ *the support of* $\hat{f}$ *in* $\mathbb{Z}^2$*. Then, in the Weyl quantization scheme, one associates to* f *the* $N \times N$ *matrix*

$$W_N(f) := \sum_{\boldsymbol{k} \in \mathrm{Supp}(\hat{f})} \hat{f}(\boldsymbol{k})\, W_N(\boldsymbol{k}).$$

Our aim is to prove:

Proposition 5.2 *Let* $(\mathcal{M}_N, \Theta_N, \tau_N)$ *be a sequence of quantum cat maps tending with* $N \to \infty$ *to a classical cat map with Lyapounov exponent* $\log \lambda$*; then*

$$\lim_{\substack{k,\ N \to \infty \\ k < \log N/(2\log\lambda)}} \|\Theta_N^k \circ \gamma_{N\infty}(f) - \gamma_{N\infty} \circ \Theta^k(f)\|_2 = 0 \quad ,$$

where $\|\cdot\|_2$ *is the Hilbert-Schmidt norm of Proposition 5.1.*

First we prove an auxiliary result.

Lemma 5.1 *If* $\boldsymbol{n} = (n_1, n_2) \in \mathbb{Z}^2$ *is such that* $0 \leq n_i \leq N-1$ *and* $\lim_N \frac{n_i}{\sqrt{N-1}} = 0$*, then the expectation of Weyl operators* $W_N(\boldsymbol{n})$ *with respect to the state* $|C_N\rangle$ *given in* (19) *is such that*

$$\lim_{N\to\infty} \langle C_N, W_N(\boldsymbol{n})\, C_N \rangle = 1.$$

Proof:
The idea of the proof is to use the fact that, for large N, the binomial coefficients $\binom{N-1}{j}$ contribute to the binomial sum only when j stays within a neighbourhood of $(N-1)/2$ of width $\simeq \sqrt{N}$, in which case they can be approximated by a normalized Gaussian function. We also notice that, by expanding the exponents in the bounds (30) and (31), the exponential decay fails only if $n_{1,2}$ grow with N slower than $\sqrt{N}$, which is surely the case for fixed finite n, whereby it also follows that we can disregard the second term in the sum comprising the contributions (21). We then write the j's in the binomial coefficients as

$$j = \left[\frac{N-1}{2}\right] + k = \frac{N-1}{2} + k - \alpha, \quad \alpha \in \{0, \tfrac{1}{2}\},$$

and consider only $k = \mathrm{O}(\sqrt{N})$. Stirling's formula

$$L! = L^{L+1/2}\,\mathrm{e}^{-L}\,\sqrt{2\pi}\,\Big(1 + \mathrm{O}(L^{-1})\Big),$$

allows us to rewrite the first term in the r.h.s. of (21) as

$$\exp\left(-\frac{1}{2}\frac{n_1^2}{N-1}\right)\sum_{k=-\left[\frac{N-1}{2}\right]}^{N-1-\left[\frac{N-1}{2}\right]+n_1}\frac{2\,\mathrm{e}^{\frac{2\pi i}{N}n_2\left(k+\left[\frac{N-1}{2}\right]\right)}}{\sqrt{2\pi(N-1)}}$$
$$\times\exp\left(-\frac{2(k-\alpha+\frac{n_1}{2})^2}{N-1}\right)\Big(1+\mathrm{O}(N^{-1})+\mathrm{O}((k+n_1)^3\,N^{-2})\Big). \qquad (35)$$

For any fixed, finite $\boldsymbol{n}$, both the sum and the factor in front tend to 1, the sum becoming the integral of a normalized Gaussian.

■

Proof of Proposition 5.2: Given $f \in \mathfrak{L}^\infty_\mu(\mathcal{X})$ and $\varepsilon > 0$, we choose N_0 such that the Fourier approximation f_ε of f with $\#(\mathrm{Supp}(\hat{f})) = N_0$ is such that $\|f - f_\varepsilon\| \le \varepsilon$, where $\|\cdot\|$ denotes the usual Hilbert space norm. Next, we estimate

$$\begin{aligned} I_N(f) &:= \big\|\Theta_N^k \circ \gamma_{N\infty}(f) - \gamma_{\infty N} \circ \Theta^k(f)\big\|_2 \\ &\le \big\|\Theta_N^k \circ \gamma_{N\infty}(f - f_\varepsilon)\big\|_2 + \big\|\gamma_{N\infty} \circ \Theta^k(f - f_\varepsilon)\big\|_2 \\ &\quad + \big\|\Theta_N^k \circ \gamma_{N\infty}(f_\varepsilon) - \gamma_{N\infty} \circ \Theta^k(f_\varepsilon)\big\|_2 \\ &\le 2\|f - f_\varepsilon\| + I_N(f_\varepsilon). \end{aligned}$$

This follows from Θ_N-invariance of the norm $\|\cdot\|_2$, from T-invariance of the measure μ and from the fact that the positivity inequality for unital completely positive maps such as $\gamma_{N\infty}$ gives:

$$\begin{aligned} \big\|\gamma_{N\infty}(g)\big\|_2^2 &= \tau_N\big(\gamma_{N\infty}(g)^*\gamma_{N\infty}(g)\big) \le \tau_N\Big(\gamma_{N\infty}(|g|^2)\Big) \\ &= \int_{\mathbb{T}} d\boldsymbol{x}\,|g|^2(\boldsymbol{x}) = \|g\|^2 \quad . \end{aligned}$$

We now use that f_ε is a function with finitely supported Fourier transform and, inserting the Weyl quantization of f_ε, we estimate

$$I_N(f_\varepsilon) \leq \big\|\gamma_{N\infty}(f_\varepsilon) - W_N(f_\varepsilon)\big\|_2 + \big\|\gamma_{N\infty} \circ \Theta^k(f_\varepsilon) - \Theta_N^k(W_N(f_\varepsilon))\big\|_2. \quad (36)$$

Then, we concentrate on the square of the second term, which we denote by $G_{N,k}(f_\varepsilon)$ and explicitly reads

$$\begin{aligned} G_{N,k}(f_\varepsilon) &= \tau_N\big(\gamma_{N\infty} \circ \Theta^k(f_\varepsilon^* \gamma_{N\infty} \circ \Theta^k(f_\varepsilon)\big) + \tau_N\big(W_N(f_\varepsilon)^* W_N(f_\varepsilon)\big) \\ &\quad - 2\Re\mathfrak{e}\Big(\tau_N\big(\gamma_{N\infty} \circ \Theta^k(f_\varepsilon)^* \Theta_N^k(W_N(f_\varepsilon))\big)\Big). \end{aligned} \quad (37)$$

The first term tends to $\|f_\varepsilon\|^2$ as $N \to \infty$, because of Proposition 3.2 and the same is true of the second term; indeed,

$$\tau_N\big(W_N(f_\varepsilon)^* W_N(f_\varepsilon)\big) = \sum_{\boldsymbol{k},\boldsymbol{q} \in \mathrm{Supp}(\hat{f}_\varepsilon)} \overline{\hat{f}_\varepsilon(\boldsymbol{k})}\, \hat{f}_\varepsilon(\boldsymbol{q})\, \mathrm{e}^{\frac{i\pi}{N}\sigma(\boldsymbol{q},\boldsymbol{k})}\, \tau_N\big(W_N(\boldsymbol{q} - \boldsymbol{k})\big).$$

Now, since $\mathrm{Supp}(\hat{f}_\varepsilon)$ is finite, the vector $\boldsymbol{k} - \boldsymbol{q}$ is uniformly bounded with respect to N. Therefore, with N large enough, (14) forces $\boldsymbol{k} = \boldsymbol{q}$, whence the claim. It remains to show that the same for the third term in (37) which amounts to twice the real part of

$$\begin{aligned} &\int_{\mathbb{T}} d\boldsymbol{x}\, \overline{f_\varepsilon(A^{-k}\boldsymbol{x})} \langle C_N(\boldsymbol{x}), \Theta_N^k(W_N(f_\varepsilon)) C_N(\boldsymbol{x})\rangle \\ &\quad = \sum_{\boldsymbol{p} \in \mathcal{S}(f_\varepsilon)} \overline{\hat{f}_\varepsilon(\boldsymbol{p})} \langle C_N, W_N(A^k\boldsymbol{p}) C_N\rangle \int_{\mathbb{T}} d\boldsymbol{x} \overline{f_\varepsilon(A^{-k}\boldsymbol{x})} \exp\Big(\frac{2\pi i}{N}\sigma(A^k\boldsymbol{p}, [N\boldsymbol{x}])\Big). \end{aligned}$$

According to Lemma 5.1, the matrix element $\langle C_N, W_N(A^k\boldsymbol{p}) C_N\rangle$ tends to 1 as $N \to \infty$ whenever the vectorial components $(A^k\boldsymbol{p})_j$, $j = 1, 2$, satisfy

$$\lim_N \frac{(A^k\boldsymbol{p})_j^2}{N} = C_{\boldsymbol{u}}(\boldsymbol{p})(\boldsymbol{u})_j \lim_N \frac{\lambda^{2k}}{N} = 0,$$

where we expanded $\boldsymbol{p} = C_{\boldsymbol{u}}(\boldsymbol{p})\boldsymbol{u} + C_{\boldsymbol{v}}(\boldsymbol{p})\boldsymbol{v}$ along the stretching and squeezing eigendirections of A (see Definition 4.1). This fact sets the logarithmic time scale $k < \frac{1}{2}\frac{\log N}{\log \lambda}$. Notice that, when $k = 0$, $G_{N,k}(f_\varepsilon)$ equals the first term in (36) and this concludes the proof. ∎

Remark The previous result essentially points to the fact that the time evolution and the classical limit do commute over time scales that are logarithmic in the semi–classical parameter N. The upper bound of this time, which goes like const. $\times \frac{\log N}{\log \lambda}$, is typical of quantum chaos and is known as logarithmic *breaking–time*. Such a scaling has been found numerically in [8] also for discrete classical cat maps, converging in a suitable classical limit to continuous cat maps.

6 Dynamical entropies

Intuitively, one expects the instability proper to the presence of a positive Lyapounov exponent to correspond to some degree of unpredictability of the dynamics: classically, the metric entropy of Kolmogorov provides the link [16].

In the usual setting, one considers partitions $\mathcal{C} = \{C_0, C_1, \dots, C_{q-1}\}$ of the phase space $\mathcal{X}$ into finitely many measurable disjoint subsets C_j (atoms). Under the dynamics T, $\mathcal{C}$ evolves into another finite partition $T(\mathcal{C}) := \{T^{-1}(C_0), T^{-1}(C_1), \dots, T^{-1}(C_{q-1})\}$. Moreover, by intersecting atoms of partitions at different times one gets disjoint atoms

$$C_{\boldsymbol{i}} := \bigcap_{j=0}^{k-1} T^j(C_{i_j}) \quad \text{for } \boldsymbol{i} = (i_0, i_1, \dots, i_{k-1}),$$

which constitute the refined partition

$$\mathcal{C}^{(k)} := \bigvee_{j=0}^{k-1} T^j(\mathcal{C}) \ .$$

Given the invariant measure μ on $\mathcal{X}$, the probability for the system to belong to the atoms C_{i_0}, C_{i_1}, ..., $C_{i_{k-1}}$ at the successive times $0 \le j \le k-1$ is $\mu(C_{\boldsymbol{i}})$.

In terms of symbolic dynamics, one gets a stationary stochastic process. This amounts to a right-shift along a classical spin half-chain with respect to a translation-invariant state. At each site of the half-chain, one has the state space $\{0,1,\ldots,q-1\}$. The atom $C_{\boldsymbol{i}}$ of the refined partition $\mathcal{C}^{(k)}$ is identified with the local configurations $\boldsymbol{i} \in \{0,1,\ldots,q-1\}^k$ and has a weight

$$\mu^{\mathcal{C}}_{(k)}(\boldsymbol{i}) := \mu(C_{\boldsymbol{i}}).$$

The local states $\mu^{\mathcal{C}}_{(k)}$ are compatible and define a global state on the set of extended configurations $\{0,1,\ldots,q-1\}^{\mathbb{N}}$. Such a state is invariant under the right-shift and has a well-defined mean entropy

$$h^{\mathrm{KS}}_\mu(T,\mathcal{C}) := \lim_{k\to\infty} \frac{1}{k} S\big(\mu^{\mathcal{C}}_{(k)}\big), \tag{38}$$

where, for a discrete measure λ, $S(\lambda) := -\sum_j \lambda_j \log \lambda_j$. The entropy density (38) is also interpretable as average entropy production. It consistently measures how predictable the dynamics is on the coarse grained scale provided by the finite partition $\mathcal{C}$. Then, removal of the dependence on finite partitions leads to

Definition 6.1 *The* KS*-entropy of a classical dynamical system* $(\mathcal{X},T,\mu)$ *is*

$$h^{\mathrm{KS}}_\mu(T) := \sup_{\mathcal{C}} h^{\mathrm{KS}}_\mu(T,\mathcal{C}).$$

For the automorphisms of the 2-torus, we have the well-known result [20]:

Proposition 6.1 *Let* $(\mathfrak{M}_\mu,\Theta,\omega_\mu)$ *be as in Definition 4.1, then* $h^{\mathrm{KS}}_\mu(T) = \log\lambda$.

The idea behind the notion of dynamical entropy is that information can be obtained by repeatedly observing a system in the course of its time evolution. Due to the uncertainty principle, or, in other words, to non-commutativity, if observations are intended to gather information about the intrinsic dynamical properties of quantum systems, then non-commutative extensions of the KS-entropy ought first to decide whether quantum disturbances produced by observations have to be taken into account or not.

Concretely, let us consider a quantum system described by a density matrix ρ acting on a Hilbert space $\mathfrak{H}$. Via the wave packet reduction postulate, generic measurement processes may reasonably well be described by finite sets $\mathcal{Y} = \{y_0, y_1, \ldots, y_{q-1}\}$ of bounded operators $y_j \in \mathfrak{B}(\mathfrak{H})$ such that $\sum_j y_j^* y_j = \mathbb{1}$. These sets are called partitions of unity and describe the change in the state of the system caused by the corresponding measurement process:

$$\rho \mapsto \Gamma^*_{\mathcal{Y}}(\rho) := \sum_j y_j \, \rho \, y_j^* . \tag{39}$$

It looks rather natural to rely on partitions of unity to describe the process of collecting information through repeated observations of an evolving quantum system [3]. Yet, most of these measurements interfere with the quantum evolution, possibly acting as a source of unwanted extrinsic randomness. Nevertheless, the effect is typically quantal and rarely avoidable. Quite interestingly, as we shall see later, pursuing these ideas leads to quantum stochastic processes with a quantum dynamical entropy of their own, the ALF-entropy, that is also useful in a classical context.

An alternative approach [12] leads to the CNT-entropy. This approach lacks the operational appeal of the ALF-construction, but is intimately connected with the intrinsic relaxation properties of quantum systems [12, 22] and possibly useful in the rapidly growing field of quantum communication. The CNT-entropy is based on decomposing quantum states rather than on reducing them as in (39). Explicitly, if the state ρ is not a one dimensional projection, any partition of unity $\mathcal{Y}$ yields a decomposition

$$\rho = \sum_j \mathrm{Tr}\big(\rho \, y_j^* y_j\big) \, \frac{\sqrt{\rho}\, y_j^* y_j \sqrt{\rho}}{\mathrm{Tr}\big(\rho \, y_j^* y_j\big)} . \tag{40}$$

When $\Gamma^*_{\mathcal{Y}}(\rho) = \rho$, reductions also provide decompositions, but not in general.

6.1 CNT-entropy

The CNT-entropy is based on decomposing quantum states into convex linear combinations of other states. The information content attached to the quantum dynamics is not based on modifications of the quantum state or on perturbations of the time evolution. Let $(\mathfrak{M}, \Theta, \omega)$ represent a quantum dynamical system in the algebraic setting and assume ω to be decomposable. The construction runs as follows.

- Classical partitions are replaced by finite dimensional C*-algebras $\mathfrak{N}$ with identity embedded into $\mathfrak{M}$ by completely positive, unity preserving (cpu) maps $\gamma : \mathfrak{N} \mapsto \mathfrak{M}$. Given γ, consider the cpu maps $\gamma_\ell := \Theta^\ell \circ \gamma$ that result from successive iterations of the dynamical automorphism Θ, and associate to each of them an index set I_ℓ. These index sets I_ℓ will be coupled to the cpu maps γ_ℓ through the variational problem (43).

- If $0 \leq \ell < k$ then consider multi-indices $\boldsymbol{i} = (i_0, i_1, \ldots, i_{k-1}) \in I^{(k)} := I_0 \times \cdots \times I_{k-1}$ as labels of states $\omega_{\boldsymbol{i}}$ on $\mathfrak{M}$ and of weights $0 < \mu_{\boldsymbol{i}} < 1$ such that $\sum_{\boldsymbol{i}} \mu_{\boldsymbol{i}} = 1$ and $\omega = \sum_{\boldsymbol{i}} \mu_{\boldsymbol{i}} \omega_{\boldsymbol{i}}$. These states are given by elements $0 \leq x'_{\boldsymbol{i}} \in \mathfrak{M}'$, the commutant of $\mathfrak{M}$, such that $\sum_{\boldsymbol{i}} x'_{\boldsymbol{i}} = \mathbb{1}_N$. Explicitly

$$y \in \mathfrak{M} \longmapsto \omega_{\boldsymbol{i}}(y) := \frac{\omega(x'_{\boldsymbol{i}}\, y)}{\omega(x'_{\boldsymbol{i}})}\ , \quad \mu_{\boldsymbol{i}} := \omega(x'_{\boldsymbol{i}})\ \cdot \tag{41}$$

  The decomposition has be done with elements x' in the commutant in order to ensure the positivity of the expectations $\omega_{\boldsymbol{i}}$.

- From $\omega = \sum_{\boldsymbol{i}} \mu_{\boldsymbol{i}} \omega_{\boldsymbol{i}}$, one obtains subdecompositions $\omega = \sum_{i_\ell \in I_\ell} \mu^\ell_{i_\ell} \omega^\ell_{i_\ell}$, where

$$\omega^\ell_{i_\ell} := \sum_{\substack{\boldsymbol{i} \\ i_\ell \text{ fixed}}} \frac{\mu_{\boldsymbol{i}}}{\mu^\ell_{i_\ell}}\, \omega_{\boldsymbol{i}} \qquad \text{and} \qquad \mu^\ell_{i_\ell} := \sum_{\substack{\boldsymbol{i} \\ i_\ell \text{ fixed}}} \mu_{\boldsymbol{i}}. \tag{42}$$

- Since $\mathfrak{N}$ is finite dimensional, the states $\omega \circ \Theta^\ell \circ \gamma = \omega \circ \gamma$ and $\omega^\ell_{i_\ell} \circ \Theta^\ell \circ \gamma$, have finite von Neumann entropies $S(\omega \circ \gamma)$ and $S(\omega^\ell_{i_\ell} \circ \Theta^\ell \circ \gamma)$. With $\eta(x) := -x\ \log x$ if $0 < x \leq 1$ and $\eta(0) = 0$, one defines the

k subalgebra functional

$$H_\omega(\gamma_0,\gamma_1,\ldots,\gamma_{k-1}) := \sup_{\omega=\sum_{\boldsymbol{i}}\mu_{\boldsymbol{i}}\,\omega_{\boldsymbol{i}}} \left\{ \sum_{\boldsymbol{i}} \eta(\mu_{\boldsymbol{i}}) - \sum_{\ell=0}^{k-1}\sum_{i_\ell\in I_\ell}\eta(\mu^\ell_{i_\ell}) \right.$$
$$\left. + \sum_{\ell=0}^{k-1}\Big(S(\omega\circ\gamma_\ell) - \sum_{i_\ell\in I_\ell}\mu^\ell_{i_\ell}\, S(\omega^\ell_{i_\ell}\circ\gamma_\ell)\Big)\right\}. \tag{43}$$

We list a number of properties of k-subalgebra functionals, see [12], that will be used in the sequel:

- positivity: $0 \leq H_\omega(\gamma_0,\gamma_1,\ldots,\gamma_{k-1})$
- subadditivity:

$$\begin{aligned} H_\omega(\gamma_0,\gamma_1,\ldots,\gamma_{k-1}) &\leq H_\omega(\gamma_0,\gamma_1,\ldots,\gamma_{\ell-1}) \\ &+ H_\omega(\gamma_\ell,\gamma_{\ell+1},\ldots,\gamma_{k-1}) \end{aligned}$$

- time invariance: $H_\omega(\gamma_0,\gamma_1,\ldots,\gamma_{k-1}) = H_\omega(\gamma_\ell,\gamma_{\ell+1},\ldots,\gamma_{\ell+k-1})$
- boundedness: $H_\omega(\gamma_0,\gamma_1,\ldots,\gamma_{k-1}) \leq kH_\omega(\gamma) \leq kS(\omega\circ\gamma)$
- The k-subalgebra functionals are invariant under interchange and repetitions of arguments:

$$H_\omega(\gamma_0,\gamma_1,\ldots,\gamma_{k-1}) = H_\omega(\gamma_{k-1},\ldots,\gamma_0,\gamma_0). \tag{44}$$

- monotonicity: If $i_\ell : \mathfrak{N}_\ell \mapsto \mathfrak{N}$, $0\leq \ell \leq k-1$, are cpu maps from finite dimensional algebras $\mathfrak{N}_l$ into $\mathfrak{N}$, then the maps $\tilde\gamma_\ell := \gamma\circ i_\ell$ are cpu and

$$H_\omega(\tilde\gamma_0,\Theta\circ\tilde\gamma_1,\ldots,\Theta^{k-1}\circ\tilde\gamma_{k-1}) \leq H_\omega(\gamma_0,\gamma_1,\ldots,\gamma_{k-1}). \tag{45}$$

- continuity: Let us consider for $\ell = 0,1,\ldots,k-1$ a set of cpu maps $\tilde\gamma_\ell : \mathfrak{N} \mapsto \mathfrak{M}$ such that $\|\gamma_\ell - \tilde\gamma_\ell\|_\omega \leq \epsilon$ for all ℓ, where

$$\|\gamma_\ell-\tilde\gamma_\ell\|_\omega := \sup_{x\in\mathfrak{N},\ \|x\|\leq 1} \sqrt{\omega\Big((\gamma_\ell(x)-\tilde\gamma_\ell(x))^*(\gamma_\ell(x)-\tilde\gamma_\ell(x))\Big)}\ . \tag{46}$$

Then [12], there exists $\delta(\epsilon) > 0$ depending on the dimension of the finite dimensional algebra $\mathcal{N}$ and vanishing when $\epsilon \to 0$, such that

$$\Big|H_\omega(\gamma_0, \gamma_1 \dots, \gamma_{k-1}) - H_\omega(\tilde\gamma_0, \tilde\gamma_1 \dots, \tilde\gamma_{k-1})\Big| \leq k\,\delta(\epsilon). \qquad (47)$$

On the basis of these properties, one proves the existence of the limit

$$h_\omega^{\mathrm{CNT}}(\theta, \gamma) := \lim_k \frac{1}{k} H_\omega(\gamma_0, \gamma_1, \dots, \gamma_{k-1}) \qquad (48)$$

and defines [12]:

Definition 6.2 *The* CNT*-entropy of a quantum dynamical system* $(\mathfrak{M}, \Theta, \omega)$ *is*

$$h_\omega^{\mathrm{CNT}}(\Theta) := \sup_\gamma h_\omega^{\mathrm{CNT}}(\Theta, \gamma)\ .$$

6.2 ALF-entropy

The idea underlying the ALF-entropy is that the evolution of a quantum dynamical system can be modelled by repeated measurements at successive equally spaced times, the measurements corresponding to partitions of unity as defined in Section 6 which we shall refer to as *p.u.*, for the sake of shortness.

Such a construction associates a quantum dynamical system with a symbolic dynamics corresponding to the right-shift along a quantum spin half-chain [29].

Generic *p.u* $\mathcal{Y} = \{y_0, y_1, \dots, y_{\ell-1}\}$ need not preserve the state, but disturbances are kept under control by suitably selecting the y_j. The construction of the ALF-entropy for a quantum dynamical system $(\mathfrak{M}, \Theta, \omega)$ can be resumed as follows:

- One selects a sub-algebra $\mathfrak{M}_0 \subseteq \mathfrak{M}$ which is invariant under Θ and a *p.u.* $\mathcal{Y} = \{y_0, y_1, \dots, y_{\ell-1}\}$ of finite size ℓ with $y_j \in \mathfrak{M}_0$. After j time steps $\mathcal{Y}$ will have evolved into another *p.u.* from $\mathfrak{M}_0$: $\Theta^j(\mathcal{Y}) := \{\Theta^j(y_0), \Theta^j(y_1), \dots, \Theta^j(y_{\ell-1})\} \subset \mathfrak{M}_0$.

- Every *p.u.* $\mathcal{Y}$ of size ℓ gives rise to an ℓ-dimensional density matrix

$$\rho[\mathcal{Y}]_{i,j} := \omega(y_j^* y_i), \tag{49}$$

with von Neumann entropy $H_\omega[\mathcal{Y}] := S(\rho[\mathcal{Y}])$.

- Given two *p.u.* $\mathcal{Y} = \{y_0, y_1, \ldots, y_{\ell-1}\}$ and $\mathcal{Z} = \{z_0, z_1, \ldots, z_{k-1}\}$, of sizes ℓ and k, their ordered refinement is the size ℓk *p.u.*

$$\mathcal{Y} \circ \mathcal{Z} := \{y_0 z_0, y_0 z_1, \ldots, y_0 z_{k-1}, \ldots, y_{\ell-1} z_{k-1}\}. \tag{50}$$

- Given a size ℓ *p.u.* $\mathcal{Y}$ and the ordered time refinements

$$\mathcal{Y}^{(k)} := \Theta^{k-1}(\mathcal{Y}) \circ \Theta^{k-2}(\mathcal{Y}) \circ \cdots \circ \mathcal{Y}, \tag{51}$$

the density matrices $\rho_{\mathcal{Y}}^{(k)} := \rho[\mathcal{Y}^{(k)}]$ define states on the k-fold tensor product $\mathcal{M}_\ell^{\otimes k}$ of ℓ-dimensional matrix algebras $\mathcal{M}_\ell$.

- Given a *p.u.* $\mathcal{Y}$ of size ℓ, let $\Phi_{\mathcal{Y}} : \mathcal{M}_\ell \otimes \mathfrak{M} \mapsto \mathfrak{M}$ and $e_M : \mathfrak{M} \mapsto \mathfrak{M}$, with $M \in \mathcal{M}_\ell$, be linear maps defined by

$$\Phi_{\mathcal{Y}}(M \otimes x) := \sum_{i,j} y_i^* x\, y_j\, M_{ij} \qquad \text{and} \qquad e_M(x) := \sum_{i,j} y_i^* \Theta(x)\, y_j\, M_{ij}. \tag{52}$$

$\Phi_{\mathcal{Y}}$ is a cpu map, while $e_{\mathbb{1}}(\mathbb{1}) = \mathbb{1}$. One readily computes

$$\omega\Big(e_{M_0} \circ e_{M_1} \cdots \circ e_{M_{k-1}}(\mathbb{1})\Big) = \mathrm{Tr}\Big(\rho_{\mathcal{Y}}^{(n)}\, M_0 \otimes M_1 \cdots \otimes M_{k-1}\Big).$$

The states $\rho_{\mathcal{Y}}^{(k)}$ are compatible and define therefore a global state $\omega_{\mathcal{Y}}$ on the quantum spin half-chain $\mathcal{M}_\ell^{\mathbb{N}}$, which is the uniform closure of $\bigcup_{n\in\mathbb{N}} \mathcal{M}_\ell^{\otimes n}$. Along the same line as in Section 6, one associates with the quantum dynamical system $(\mathfrak{M}, \Theta, \omega)$ the right shift σ along the quantum spin half-chain. However, non-commutativity shows up in that $\omega_{\mathcal{Y}}$ is shift-invariant only if $\omega\Big(\sum_{j=0}^{\ell} y_j^* x y_j\Big) = \omega(x)$ for all $x \in \mathcal{M}_\ell^{\mathbb{N}}$. Note that this is the case when *p.u.* give rise to decompositions of ω as in CNT-construction, (compare (39) and

(40)). This leads to

Definition 6.3 *The* ALF*-entropy of a quantum dynamical system* $(\mathfrak{M},\Theta,\omega)$ *is*

$$h^{\mathrm{ALF}}_{(\omega,\mathfrak{M}_0)}(\Theta) := \sup_{\mathcal{Y}\subset\mathfrak{M}_0} h^{\mathrm{ALF}}_{\omega}(\Theta,\mathcal{Y}) \quad \textit{with} \quad h^{\mathrm{ALF}}_{\omega}(\Theta,\mathcal{Y}) := \limsup_k \frac{1}{k} H_\omega[\mathcal{Y}^{(k)}]. \tag{53}$$

6.3 Quantum Dynamical Entropies Compared

In this section we outline some of the main features of both quantum dynamical entropies. The first thing to notice is that the CNT- and the ALF-entropy coincide with the KS-entropy when $\mathfrak{M} = \mathfrak{M}_\mu$ is the Abelian von Neumann algebra $\mathfrak{L}^\infty_\mu(\mathcal{X})$ and $(\mathfrak{M},\Theta,\omega)$ represents a classical dynamical system. The next observation is that when, as for the quantized hyperbolic automorphisms of the torus considered in this paper, $\mathfrak{M}$ is a finite-dimensional algebra, both the CNT- and the ALF-entropy are zero, see [12, 3]. Consequently, if we decide to take the strict positivity of quantum dynamical entropies as a signature of quantum chaos, quantized hyperbolic automorphisms of the torus cannot be called chaotic.

The complete proofs of the above facts can be found in [12] for the CNT and [2, 3] for the ALF-entropy. Here, we just sketch them, emphasizing those parts that are important to the study of their classical limit.

Proposition 6.2 *Let* $(\mathfrak{M}_\mu,\Theta,\omega_\mu)$ *represent a classical dynamical system. Then, with the notations of the previous sections*

$$h^{\mathrm{CNT}}_{\omega_\mu}(\Theta) = h^{\mathrm{KS}}_{\mu}(T) = h^{\mathrm{ALF}}_{(\omega_\mu,\mathfrak{M}_\mu)}(\Theta).$$

Proof:

CNT-Entropy. In this case, $h^{\mathrm{CNT}}_{\omega_\mu}(\Theta)$ is computable by using natural embeddings of finite dimensional subalgebras of $\mathfrak{M}_\mu$ rather than generic cpu maps γ. Partitions $\mathcal{C} = \{C_0, C_1, \ldots, C_{n-1}\}$ of $\mathcal{X}$ can be identified with the finite dimensional subalgebras $\mathcal{N}_\mathcal{C} \in \mathfrak{M}_\mu$ generated by the characteristic functions

χ_{C_j} of the atoms of the partition, with $\omega_\mu(\chi_C) = \mu(C)$. Also, the refinements $\mathcal{C}^{(k)}$ of the evolving partitions $T^{-j}(\mathcal{C})$ correspond to the subalgebras $\mathcal{N}_\mathcal{C}^{(k)}$ generated by $\chi_{C_{\boldsymbol{i}}} = \prod_{j=0}^{k-1} \chi_{T^{-j}(C_{i_j})}$.
Thus, if $\imath_{\mathcal{N}_\mathcal{C}}$ embeds $\mathcal{N}_\mathcal{C}$ into $\mathfrak{M}_\mu$, then $\omega_\mu \circ \imath_{\mathcal{N}_\mathcal{C}}$ corresponds to the state $\omega_\mu \upharpoonright \mathcal{N}_\mathcal{C}$, which is obtained by restriction of ω_μ to $\mathcal{N}_\mathcal{C}$ and is completely determined by the expectation values $\omega_\mu(\chi_{C_j})$, $1 \leq j \leq n-1$.
Further, identifying the cpu maps $\gamma_\ell = \Theta^\ell \circ \imath_{\mathcal{N}_\mathcal{C}}$ with the corresponding subalgebras $\Theta^\ell(\mathcal{N})$, $h^{\mathrm{CNT}}_{\omega_\mu}(\Theta) = h^{\mathrm{KS}}_\mu(T)$ follows from

$$H_\omega(\mathcal{N}_\mathcal{C}, \Theta(\mathcal{N}_\mathcal{C}), \ldots, \Theta^{k-1}(\mathcal{N}_\mathcal{C})) = S_\mu(\mathcal{C}^{(k)}), \quad \forall\, \mathcal{C}, \tag{54}$$

see (38). In order to prove (54), we decompose the reference state as

$$\omega_\mu = \sum_{\boldsymbol{i}} \mu_{\boldsymbol{i}}\, \omega_{\boldsymbol{i}} \qquad \text{with} \qquad \omega_{\boldsymbol{i}}(f) := \frac{1}{\mu_{\boldsymbol{i}}} \int_\mathcal{X} \mu(dx)\, \chi_{C_{\boldsymbol{i}}}(x)\, f(x)$$

where $\mu_{\boldsymbol{i}} = \mu(C_{\boldsymbol{i}})$, see (42). Then, $\sum_{\boldsymbol{i}} \eta(\mu_{\boldsymbol{i}}) = S_\mu(\mathcal{C}^{(k)})$.
On the other hand,

$$\omega^\ell_{i_\ell}(f) = \frac{1}{\mu^\ell_{i_\ell}} \int_\mathcal{X} \mu(dx)\, \chi_{T^{-\ell}(C_{i_\ell})}(x)\, f(x) \qquad \text{and} \qquad \mu^\ell_{i_\ell} = \mu(C_{i_\ell}).$$

It follows that $\omega_\mu \circ \imath_{\mathcal{N}_\mathcal{C}} = \omega_\mu \upharpoonright \mathcal{N}_\mathcal{C}$ is the discrete measure $\{\mu^\ell_0, \mu^\ell_1, \ldots, \mu^\ell_{n-1}\}$ for all $\ell = 0, 1, \ldots, k-1$ and, finally, that $S(\omega^\ell_{i_\ell} \circ \gamma_\ell) = 0$ as $\omega^\ell_{i_\ell} \circ j_\ell = \omega^\ell_{i_\ell} \upharpoonright \Theta^\ell(\mathcal{N}_\mathcal{C})$ is a discrete measure with values 0 and 1.

ALF-Entropy. One expects that (53), computed over all possible *p.u.* from $\mathfrak{M}_\mu$ should equal $h^{\mathrm{KS}}_\mu(T)$. Notice, however, that, even if the dynamical system is classical, still (53) has to be computed within the non-commutative setting of density matrices as in (49). In [2], it is shown that $h^{\mathrm{ALF}}_{(\omega, \mathfrak{M}_\mu)}(\Theta) = h^{\mathrm{KS}}_\mu(T)$. ■

In the particular case of the hyperbolic automorphisms of the torus, we may restrict our attention to *p.u.* whose elements belong to the $*$-algebra $\mathcal{D}_\mu$ of complex functions f on $\mathbb{T}$ such that the support of $\hat{f}$ is bounded:

$$h^{\mathrm{KS}}_\mu(T) = h^{\mathrm{ALF}}_{(\omega_\mu, \mathfrak{M}_\mu)}(\Theta) = h^{\mathrm{ALF}}_{(\omega_\mu, \mathcal{D}_\mu)}(\Theta).$$

Remarkably, the computation of the classical KS-entropy via the quantum mechanical ALF-entropy yields a proof of Proposition 6.1 that is much simpler than the standard ones [7, 31].

Proposition 6.3 *Let $(\mathfrak{M}, \Theta, \omega)$ be a quantum dynamical system with $\mathfrak{M}$, a finite dimensional C^*-algebra, then,*

$$h_\omega^{\rm CNT}(\Theta) = 0 \qquad \textit{and} \qquad h^{\rm ALF}_{(\omega,\mathfrak{M})}(\Theta) = 0.$$

Proof:

CNT-Entropy: as in the commutative case, $h_\omega^{\rm CNT}(\Theta)$ is computable by means of cpu maps γ that are the natural embeddings $\imath_{\mathcal{N}}$ of subalgebras $\mathcal{N} \subseteq \mathfrak{M}$ into $\mathfrak{M}$. Since each $\Theta^\ell(\mathcal{N})$ is obviously contained in the algebra $\mathcal{N}^{(k)} \subseteq \mathfrak{M}$ generated by the subalgebras $\Theta^j(\mathcal{N})$, $j = 0, 1, \ldots, k-1$, from the properties of the k-subalgebra functionals H and identifying again the natural embeddings $\gamma_\ell := \Theta^\ell \circ \imath_{\mathcal{N}}$ with the subalgebras $\Theta^\ell(\mathcal{N}) \subseteq \mathfrak{M}$, we derive

$$\begin{aligned} H_\omega(\mathcal{N}, \Theta(\mathcal{N}), \ldots, \Theta^{k-1}(\mathcal{N})) &\leq H_\omega(\mathcal{N}^{(k)}, \mathcal{N}^{(k)}, \ldots, \mathcal{N}^{(k)}) \\ &\leq H_\omega(\mathcal{N}^{(k)}) \leq qS\big(\omega \upharpoonright \mathcal{N}^{(k)}\big) \leq \log d, \end{aligned}$$

where $\mathfrak{M} \subseteq \mathcal{M}_d$. In fact, $\omega \upharpoonright \mathcal{N}$ amounts to a density matrix with eigenvalues λ_ℓ and von Neumann entropy $S(\omega \upharpoonright \mathcal{N}) = -\sum_{\ell=1}^d \lambda_\ell \log \lambda_\ell \leq \log d$. Therefore, for all $\mathcal{N} \subseteq \mathfrak{M}$, $h_\omega^{\rm CNT}(\Theta, \mathcal{N}) = 0$.

ALF-Entropy: Let the state ω on $\mathcal{M}_d$ be given by $\omega(x) = \mathrm{Tr}(\rho\, x)$, where ρ is a density matrix in $\mathcal{M}_d$. Given a partition of unity $\mathcal{Y}$ of size ℓ, the cpu map $\Phi_{\mathcal{Y}}$ in (52) can be used to define a state $\Phi^*_{\mathcal{Y}}(\rho)$ on $\mathcal{M}_\ell \otimes \mathfrak{M}$ which is dual to ω:

$$\Phi^*_{\mathcal{Y}}(\rho)(M \otimes x) = \mathrm{Tr}\,\Big(\rho\, \Phi_{\mathcal{Y}}(M \otimes x)\Big), \quad M \in \mathcal{M}_\ell,\ x \in \mathfrak{M}.$$

Since $\sum_{j=0}^\ell y_j^* y_j = \mathbb{1}$, it follows that $\Phi^*_{\mathcal{Y}}(\rho^k) = \big(\Phi^*_{\mathcal{Y}}(\rho)\big)^k$. Therefore, ρ and $\Phi^*_{\mathcal{Y}}(\rho)$ have the same spectrum, apart possibly from the eigenvalue zero, and thus the same von Neumann entropy. Moreover, $\Phi^*_{\mathcal{Y}}(\rho) \upharpoonright \mathcal{M}_\ell = \rho[\mathcal{Y}]$ and $\Phi^*_{\mathcal{Y}}(\rho) \upharpoonright \mathfrak{M} = \Gamma^*_{\mathcal{Y}}(\rho)$ as in (39). Applying the triangle inequality for the entropy [25]

$$S\big(\Phi^*_{\mathcal{Y}}(\rho)\big) \geq \Big|S\big(\Phi^*_{\mathcal{Y}}(\rho) \restriction \mathcal{M}_\ell\big) - S\big(\Phi^*_{\mathcal{Y}}(\rho) \restriction \mathfrak{M}\big)\Big|,$$

one obtains $S(\rho[\mathcal{Y}]) \leq 2\log d$. Finally, as evolving *p.u.* $\Theta^j(\mathcal{Y})$ and their ordered refinements (50), (51) remain in $\mathfrak{M}$, one gets

$$\limsup_k \frac{1}{k} H_\omega[\mathcal{Y}^{(k)}] = 0, \quad \mathcal{Y} \subset \mathfrak{M}.$$

■

From the considerations of above, it is clear that the main field of application of the CNT- and ALF-entropies are infinite quantum systems, where the differences between the two come to the fore [6]. The former has been proved to be useful to connect randomness with clustering properties and asymptotic commutativity. A rather strong form of clustering and asymptotic Abeliannes is necessary to have a non-vanishing CNT-entropy [22, 23, 24]. In particular, the infinite dimensional quantization of the automorphisms of the torus has vanishing CNT-entropy for most of irrational values of the deformation parameter ϕ, whereas, independently of the value of ϕ, the ALF-entropy is always equal to the positive Lyapounov exponent. These results reflect the different perspectives upon which the two constructions are based.

7 Classical limit of quantum dynamical entropies

Proposition 6.3 confirms the intuition that finite dimensional, discrete time, quantum dynamical systems, however complicated the distribution of their quasi-energies might be, cannot produce enough information over large times to generate a non-vanishing entropy per unit time. This is due to the fact that, despite the presence of almost random features over finite intervals, the time evolution cannot bear random signatures if watched long enough, because almost periodicity would always prevail asymptotically.

In this section we take the the CNT and the ALF-entropy as good indicators of the degree of randomness of a quantum dynamical system. Then, we show that underlying classical chaos plus Hilbert space finiteness make a characteristic logarithmic time scale emerge over which these systems can be called chaotic.

7.1 CNT-entropy

Theorem 7.1 *Let $(\mathcal{X}, T, \mu)$ be a classical dynamical system which is the classical limit of a sequence of finite dimensional quantum dynamical systems $(\mathcal{M}_N, \Theta_N, \tau_N)$. We also assume that the dynamical localization condition 5.1 holds. If*

1. $\mathcal{C} = \{C_0, C_1, \ldots, C_{q-1}\}$ *is a finite measurable partition of* $\mathcal{X}$,
2. $\mathcal{N}_{\mathcal{C}} \subset \mathfrak{M}_\mu$ *is the finite dimensional subalgebra generated by the characteristic functions* χ_{C_j} *of the atoms of* $\mathcal{C}$,
3. $\imath_{\mathcal{N}_{\mathcal{C}}}$ *is the natural embedding of* $\mathcal{N}_{\mathcal{C}}$ *into* $\mathfrak{M}_\mu = \mathfrak{L}_\mu(\mathcal{X})$, $\gamma_{N\infty}$ *the anti-Wick quantization map and*

$$\gamma^\ell_{\mathcal{C}} := \Theta^\ell_N \circ \gamma_{N\infty} \circ \imath_{\mathcal{N}_{\mathcal{C}}}, \quad \ell = 0, 1, \ldots, k-1,$$

then there exists an α *such that*

$$\lim_{\substack{k,\ N \to \infty \\ k \le \alpha \log N}} \frac{1}{k} \left| H(\gamma^0_{\mathcal{C}}, \gamma^1_{\mathcal{C}}, \ldots, \gamma^{k-1}_{\mathcal{C}}) - S_\mu\big(\mathcal{C}^{(k)}\big) \right| = 0.$$

Proof:

We split the proof in two parts:

1. We relate the quantal evolution $\gamma^\ell_{\mathcal{C}} = \Theta^\ell_N \circ \gamma_{N\infty} \circ \imath_{\mathcal{N}_{\mathcal{C}}}$ to the classical evolution $\tilde{\gamma}^\ell_{\mathcal{C}} := \gamma_{N\infty} \circ \Theta^\ell \circ \imath_{\mathcal{N}_{\mathcal{C}}}$ using the continuity property of the entropy functional.
2. We find an upper and a lower bound to the entropy functional that converge to the KS-entropy in the long time limit.

We define for convenience the algebra $\mathcal{N}_\mathcal{C}^\ell := \Theta^\ell(\mathcal{N}_\mathcal{C})$ and the algebra $\mathcal{N}_\mathcal{C}^{(k)}$ corresponding to the refinements $\mathcal{C}^{(k)} = \bigvee_{\ell=0}^{k-1} T^{-\ell}(\mathcal{C})$ which consist of atoms $\mathcal{C}_{\boldsymbol{i}} := \bigcap_{\ell=0}^{k-1} T^{-\ell}(C_{i_\ell})$ labeled by the multi-indices $\boldsymbol{i} = (i_0, i_1, \ldots, i_{k-1})$. Thus the algebra $\mathcal{N}_\mathcal{C}^{(k)}$ is generated by the characteristic functions $\chi_{C_{\boldsymbol{i}}}$.

Step 1

The maps $\gamma_\mathcal{C}^\ell$ and $\tilde{\gamma}_\mathcal{C}^\ell$ connect the quantum and classical time evolution. Indeed, using Proposition 5.1

$$k \leq \alpha \log N \Rightarrow \|\Theta_N^k \circ \gamma_{N\infty} \circ \imath_{\mathcal{N}_\mathcal{C}}(f) - \gamma_{N\infty} \circ \Theta^k \circ \imath_{\mathcal{N}_\mathcal{C}}(f)\|_2 \leq \varepsilon,$$

or

$$k \leq \alpha \log N \Rightarrow \|\gamma_\mathcal{C}^k - \tilde{\gamma}_\mathcal{C}^k\|_2 \leq \varepsilon$$

This in turn implies, due to strong continuity,

$$\left|H(\gamma_\mathcal{C}^0, \gamma_\mathcal{C}^1, \ldots, \gamma_\mathcal{C}^{k-1}) - H(\tilde{\gamma}_\mathcal{C}^0, \tilde{\gamma}_\mathcal{C}^1, \ldots, \tilde{\gamma}_\mathcal{C}^{k-1})\right| \leq k\delta(\varepsilon)$$

with $\delta(\varepsilon) > 0$ depending on the dimension of the space $\mathcal{N}_\mathcal{C}$ and vanishing when $\varepsilon \to 0$. From now on we can concentrate on the classical evolution and benefit from its properties.

Step 2, upper bound

We now show that

$$H(\tilde{\gamma}_\mathcal{C}^0, \tilde{\gamma}_\mathcal{C}^1, \ldots, \tilde{\gamma}_\mathcal{C}^{k-1}) \leq S_\mu(\mathcal{C}^{(k)}).$$

Notice that we can embed $\mathcal{N}_\mathcal{C}^\ell$ into $\mathfrak{M}_\mu$ by first embedding it into $\mathcal{N}_\mathcal{C}^{(k)}$ with $\imath_{\mathcal{N}_\mathcal{C}^\ell \mathcal{N}_\mathcal{C}^{(k)}}$ and then embedding $\mathcal{N}_\mathcal{C}^{(k)}$ into $\mathfrak{M}_\mu$ with $\imath_{\mathcal{N}_\mathcal{C}^{(k)}}$:

$$\imath_{\mathcal{N}_\mathcal{C}^\ell} = \imath_{\mathcal{N}_\mathcal{C}^{(k)}} \circ \imath_{\mathcal{N}_\mathcal{C}^\ell \mathcal{N}_\mathcal{C}^{(k)}}.$$

We now estimate:

$$\begin{aligned}
&H(\tilde\gamma^0_{\mathcal{C}}, \tilde\gamma^1_{\mathcal{C}}, \ldots, \tilde\gamma^{k-1}_{\mathcal{C}}) \\
&\quad = H(\gamma_{N\infty} \circ \imath_{\mathcal{N}^{(k)}_{\mathcal{C}}} \circ \imath_{\mathcal{N}^0_{\mathcal{C}}\mathcal{N}^{(k)}_{\mathcal{C}}}, \ldots, \gamma_{N\infty} \circ \imath_{\mathcal{N}^{(k)}_{\mathcal{C}}} \circ \imath_{\mathcal{N}^{k-1}_{\mathcal{C}}\mathcal{N}^{(k)}_{\mathcal{C}}}) \\
&\quad \le H(\gamma_{N\infty} \circ \imath_{\mathcal{N}^{(k)}_{\mathcal{C}}}, \ldots, \gamma_{N\infty} \circ \imath_{\mathcal{N}^{(k)}_{\mathcal{C}}}) \le H(\gamma_{N\infty} \circ \imath_{\mathcal{N}^{(k)}_{\mathcal{C}}}) \\
&\quad \le S\left(\tau_N \circ \gamma_{N\infty} \circ \imath_{\mathcal{N}^{(k)}_{\mathcal{C}}}\right).
\end{aligned} \tag{55}$$

The first inequality follows from monotonicity of the entropy functional, the second from invariance under repetitions and the third from boundedness in terms of von Neumann entropies. The state $\tau_N \circ \gamma_{N\infty} \circ \imath_{\mathcal{N}^{(k)}_{\mathcal{C}}}$ takes the values

$$\begin{aligned}
\tau_N\big(\gamma_{N\infty}(\chi_{C_{\boldsymbol{i}}})\big) &= \tau_N\Big(N \int_{\mathcal{X}} \mu(dx)\, \chi_{C_{\boldsymbol{i}}}(x)\, |C_N(x)\rangle\langle C_N(x)|\Big) \\
&= \int_{\mathcal{X}} \mu(dx)\, \chi_{C_{\boldsymbol{i}}}(x)\, \langle C_N(x), C_N(x)\rangle = \omega_\mu(\chi_{C_{\boldsymbol{i}}}) = \mu(C_{\boldsymbol{i}}).
\end{aligned}$$

This gives, together with $S(\mu(C_{\boldsymbol{i}})) = S_\mu(\mathcal{C}^{(k)})$, the desired upper bound.

Step 2, lower bound

We show that $\forall \varepsilon > 0$ there exists an N such that

$$H(\tilde\gamma^0_{\mathcal{C}}, \tilde\gamma^1_{\mathcal{C}}, \ldots, \tilde\gamma^{k-1}_{\mathcal{C}}) \ge S_\mu(\mathcal{C}^{(k)}) - k\varepsilon.$$

As $H(\tilde\gamma^0_{\mathcal{C}}, \tilde\gamma^1_{\mathcal{C}}, \ldots, \tilde\gamma^{k-1}_{\mathcal{C}})$ is defined as a supremum over decompositions of the state τ_N, we can construct a lower bound by picking a good decomposition. Consider the decomposition $\tau_N = \sum_{\boldsymbol{i}} \mu_{\boldsymbol{i}}\, \omega_{\boldsymbol{i}}$ with

$$\begin{aligned}
&\omega_{\boldsymbol{i}} : \mathcal{M}_N \ni x \mapsto \omega_{\boldsymbol{i}}(x) := \frac{\tau_N\big(\gamma_{N\infty}(\chi_{C_{\boldsymbol{i}}})\big)(x)}{\tau_N\big(\gamma_{N\infty}(\chi_{C_{\boldsymbol{i}}})\big)} \\
&\mu_{\boldsymbol{i}} := \tau_N\big(\gamma_{N\infty}(\chi_{C_{\boldsymbol{i}}})\big)
\end{aligned}$$

and the subdecompositions $\tau_N = \sum_{j_\ell} \mu^\ell_{j_\ell}\, \omega^\ell_{j_\ell}$, $\ell = 0, 1, \ldots, k-1$, with

$$\omega^\ell_{j_\ell} : \mathcal{M}_N \ni x \mapsto \omega^\ell_{j_\ell}(x) := \frac{\tau_N(\gamma_{N\infty}(\chi_{T^{-\ell}(C_{j_\ell})}))(x)}{\tau_N\big(\gamma_{N\infty}(\chi_{C_{j_l}})\big)}$$
$$\mu^\ell_{j_\ell} := \tau_N(\gamma_{N\infty}(\chi_{C_{j_\ell}})).$$

In comparison with (41), it is not necessary to go to the commutant for one can use the cyclicity property of the trace. We then have:

$$H(\tilde{\gamma}^0_{\mathcal{C}}, \tilde{\gamma}^1_{\mathcal{C}}, \ldots, \tilde{\gamma}^{k-1}_{\mathcal{C}}) \geq S_\mu(\mathcal{C}^{(k)}) - \sum_{i=0}^{k-1} \sum_{i_\ell \in I_\ell} \mu^\ell_{i_\ell}\, S(\omega^\ell_{i_\ell} \circ \tilde{\gamma}^\ell_{\mathcal{C}}).$$

The inequality stems from the fact that $H(\tilde{\gamma}^0_{\mathcal{C}}, \tilde{\gamma}^1_{\mathcal{C}}, \ldots, \tilde{\gamma}^{k-1}_{\mathcal{C}})$ is a supremum, whereas the middle terms in the original definition of the entropy functional drop out because they are equal in magnitude but opposite in sign. For $s = 0, 1, \ldots, k-1$, $\omega^\ell_{i_\ell} \circ \tilde{\gamma}^\ell_{\mathcal{C}}$ takes on the values

$$\omega^\ell_{i_\ell}(\gamma_{N\infty}(\chi_{T^{-\ell}(C_s)})) = \mu^{-1}_{i_\ell}\tau_N\Big(\gamma_{N\infty}(\chi_{T^{-\ell}(C_{j_\ell})})\, \gamma_{N\infty}(\chi_{T^{-\ell}(C_s)})\Big).$$

Due to Proposition 3.2, these converge to $\omega_\mu(\chi_{T^{-\ell}(C_{j_\ell})}\, \chi_{T^{-\ell}(C_s)}) = \delta_{i_\ell, s}$. This means that in the limit the von Neumann entropy will be zero. Or stated more carefully, $\forall \varepsilon' : \exists N'$ such that

$$\sum_{i=0}^{k-1} \sum_{i_\ell \in I_\ell} \mu^\ell_{i_\ell}\, S(\omega^\ell_{i_\ell} \circ \tilde{\gamma}^\ell_{\mathcal{C}}) \leq k\varepsilon'.$$

We thus obtain a lower bound.

Combining our results and choosing $\tilde{N} := \max(N, N')$, we conclude

$$S_\mu(\mathcal{C}^{(k)}) - k\varepsilon' - k\delta(\varepsilon) \leq H(\gamma^0_{\mathcal{C}}, \gamma^1_{\mathcal{C}}, \ldots, \gamma^{k-1}_{\mathcal{C}}) \leq S_\mu(\mathcal{C}^{(k)}) + k\delta(\varepsilon).$$

■

7.2 ALF-entropy

Theorem 7.2 *Let* $(\mathcal{X}, T, \mu)$ *be a classical dynamical system which is the classical limit of a sequence of finite dimensional quantum dynamical systems* $(\mathcal{M}_N, \Theta_N, \tau_N)$. *We also assume that the dynamical localization condition 5.1 holds. If*

1. $\mathcal{C} = \{C_0, C_1, \ldots, C_{q-1}\}$ *is a finite measurable partition of* $\mathcal{X}$,
2. $\mathcal{Y}_N = \{y_0, y_1, \ldots, y_q\}$ *is a bistochastic partition of unity, which is the quantization of the previous partition, namely* $y_i = \gamma_{N\infty}(\chi_{C_i})$ *for* $i = 0, 1, \ldots, q-1$ *and* $y_q := \sqrt{\mathbb{1} - \sum_{i=0}^{q-1} y_i^* y_i}$,

then there exists an α *such that*

$$\lim_{\substack{k,N\to\infty \\ k\leq\alpha\log N}} \frac{1}{k} \left| H[\mathcal{Y}_N^{(k)}] - S_\mu(\mathcal{C}^{(k)}) \right| = 0.$$

Proof:
First notice that $\mathcal{Y}_N = \{y_0, y_1, \ldots, y_q\}$ is indeed a bistochastic partition. We have

$$\begin{aligned} y_i^* &= \gamma_{N\infty}(\chi_{C_i})^* = \gamma_{N\infty}(\overline{\chi_{C_i}}) = \gamma_{N\infty}(\chi_{C_i}) = y_i \\ 0 &\leq \gamma_{N\infty}(\chi_{C_i})^2 = y_i^2 \leq \gamma_{N\infty}(\chi_{C_i}^2) = \gamma_{N\infty}(\chi_{C_i}) \end{aligned}$$

Summing the last line over i from 1 to $q-1$, we see that $\sum_{i=1}^{q-1} y_i^2 \leq \mathbb{1}$, This means that $\{y_0, y_1, \ldots, y_{q-1}\}$ is not a partition of unity, but we can use this property to define an extra element y_q which completes it to a bistochastic partition of unity, $\mathcal{Y}_N = \{y_0, y_1, \ldots, y_q\}$:

$$y_q := \sqrt{\mathbb{1} - \sum_{i=0}^{q-1} y_i^* y_i}$$

The bistochasticity is a useful property because it implies translation invariance of the state on the quantum spin chain, state which arises during the construction of the ALF-entropy.

The density matrix $\rho[\mathcal{Y}^{(k)}]$ of the refined partition reads

$$\begin{aligned}\rho\left[\mathcal{Y}^{(k)}\right] &= \sum_{\boldsymbol{i},\boldsymbol{j}} \rho\left[\mathcal{Y}^{(k)}\right]_{\boldsymbol{i},\boldsymbol{j}} |e_{\boldsymbol{i}}\rangle\langle e_{\boldsymbol{j}}| \\ &= \sum_{\boldsymbol{i},\boldsymbol{j}} \tau_N\Big(y^*_{j_1}\cdots\Theta_N^{k-1}(y^*_{j_k})\Theta_N^{k-1}(y_{i_k})\cdots y_{i_1}\Big)|e_{\boldsymbol{i}}\rangle\langle e_{\boldsymbol{j}}|\end{aligned}$$

Now we will expand this formula using the operators y_i defined above, the quantities $K_a(x,y)$ defined in (32) and controlling the element y_q as follows:

$$\begin{aligned}\|y_q\|_2^2 &= \left\|\sqrt{\mathbb{1} - \sum_{i=1}^{q-1} y_i^* y_i}\right\|_2^2 = \tau_N\Big(\mathbb{1} - \sum_{i=1}^{q-1} y_i^* y_i\Big) \\ &= \int dydz \sum_{i\neq j} \chi_i(y)\,\chi_j(z)\,N|K_0(y,z)|^2. \end{aligned} \tag{56}$$

Thus, in the limit of large N, $N|K_0(y,z)|^2$ is just $\delta(y-z)$ (see (32)) so that (56) tends to $\int dz \sum_{i\neq j}\chi_i(z)\,\chi_j(z) = 0$ and we can consistently neglect those entries of $\rho[\mathcal{Y}^{(k)}]$ containing y_q.

By means of the properties of coherent states, we write out explicitly the elements of the density matrix

$$\begin{aligned}\rho\left[\mathcal{Y}^{(k)}\right]_{\boldsymbol{i},\boldsymbol{j}} &= N^{2k-1}\int d\boldsymbol{y}d\boldsymbol{z}\prod_{\ell=1}^{k}\chi_{C_{j_\ell}}(y_\ell)\chi_{C_{i_\ell}}(z_\ell)\,\times \\ &\times K_0(z_1,y_1)\left(\prod_{p=1}^{k-1}K_1(y_p,y_{p+1})\right)K_0(y_k,z_k)\left(\prod_{q=1}^{q-1}K_{-1}(z_{k-q+1},z_{k-q})\right)\quad.\end{aligned} \tag{57}$$

We now use that for N large enough,

$$\left|N\int dy\,\chi_C(y)\,K_m(x,y)\,K_n(y,z) - \chi_{T^{-m}C}(x)\,K_{m+n}(x,z)\right| \leq \varepsilon_m(N)\,, \tag{58}$$

where $\varepsilon_m(N)\to 0$ with $N\to\infty$ uniformly in $x,y\in\mathcal{X}$. This is a consequence of the dynamical localization condition 5.1 and can be rigorously proven in the same way as Proposition 3.1. However, the rough idea is the following:

from the property 3.1.3 of coherent states, one derives

$$N\int dy\,\chi_C(y)\,K_m(x,y)\,K_n(y,z) = K_{m+n}(x,z) \\ +N\int dy\,\Big(\chi_C(y)-1\Big)\,K_m(x,y)\,K_n(y,z)\ .$$

For large N, the condition 5.1 makes the integral in (58) negligible small unless $x \in T^{-m}(C)$, in which case it is the second integral in the formula of above which can be neglected.

By applying (58) to the couples of products in (57) one after the other, we finally arrive at the upper bound

$$\left|\rho\left[\mathcal{Y}^{(k)}\right]_{\boldsymbol{i},\boldsymbol{j}} - \delta_{\boldsymbol{i},\boldsymbol{j}}\,\mu(C_{\boldsymbol{i}})\right| \le \Big(2\sum_{m=1}^{k}\varepsilon_m(N)+\varepsilon_0(N)\Big) =: \epsilon(N),$$

where $C_{\boldsymbol{i}} := \bigcap_{\ell=1}^{k} T^{-\ell+1}C_{i_\ell}$ is an element of the partition $\mathcal{C}^{(k)}$.

We now set $\sigma\left[\mathcal{C}^{(k)}\right] := \sum_{\boldsymbol{i}}\mu(C_{\boldsymbol{i}})|e_{\boldsymbol{i}}\rangle\langle e_{\boldsymbol{i}}|$ and use the following estimate: let A be an arbitrary matrix of dimension d and let $\{e_1,e_2,\ldots,e_d\}$ and $\{f_1,f_2,\ldots,f_d\}$ be two orthonormal bases of $\mathbb{C}^d$, then $\|A\|_1 := \mathrm{Tr}\,|A| \le \sum_{i,j}|\langle e_i, A\,f_j\rangle|$. This yields

$$\Delta(k) := \|\rho\left[\mathcal{Y}^{(k)}\right]-\sigma\left[\mathcal{C}^{(k)}\right]\|_1 = \mathrm{Tr}\Big|\rho\left[\mathcal{Y}^{(k)}\right]-\sigma\left[\mathcal{C}^{(k)}\right]\Big| \le q^{2k}\epsilon(N).$$

Finally, by the continuity of the von Neumann entropy [15], we get

$$\left|S(\rho\left[\mathcal{X}^{(k)}\right]) - S(\sigma\left[\mathcal{C}^{(k)}\right])\right| \le \Delta(k)\log q^k\ +\ \eta(\Delta(k))\ .$$

Since, from $k \le \alpha\log N$, $q^{2k} \le N^{2\alpha\log q}$, if we want the bound $q^{2k}\epsilon(N)$ to converge to zero with $N\to\infty$, the parameter α has to be chosen accordingly. Then, the result follows because the von Neumann entropy of σ reduces to the Shannon entropy of the refinements of the classical partition.

■

8 Conclusions

In this paper, we have shown that both the CNT and ALF entropies reproduce the Kolmogorov-Sinai invariant if we observe a strongly chaotic system at a very short time scale. However, due to the discreteness of the spectrum of the quantizations, we know that saturation phenomena will appear. It would be interesting to study the scaling behaviour of the quantum dynamical entropies in the intermediate region between the random breaking time and the Heisenberg time. This will, however, require quite different techniques than the coherent states approach.

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