Quantum states and maps modeled by random matrices

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1 Summary

- Irreversible quantum dynamics
  - geometry and statistics of the set of quantum states;
  - completely positive trace preserving maps = quantum operations;
  - probability measure in the set of quantum operations;
  - completely positive trace nonincreasing maps (CP–TNI Maps);
  - spectral properties of superoperators associated with stochastic maps;
  - quantum analogue of the classical Frobenius-Perron theorem on spectra of stochastic matrices.
- Library of ForTran codes for producing:
  - all random matrices representing states and maps of previous list;
  - tools for dealing with states and maps like partial tracing procedures or reshuffling operations;
  - random matrices out of classical compact Lie groups O(N), U(N) and USp(2N);
  - random matrices out of the classical universality classes of Random Matrix Theory.

2 Methods

The theory of random matrices was also applied to construct and investigate ensembles of random mixed states. In particular, the Ginibre ensemble was used to design an algorithm to generate random states with respect to the Hilbert-Schmidt measure, while the technique of orthogonal polynomials allowed us to find out expectation values and mean correlation functions.

3 Quantum states

Define the set $\mathcal{M}_N$ of all density operators ρ which act in an N–dimensional Hilbert space $\mathcal{H}_N$ and satisfy:

- Hermiticity ($\rho^\dagger = \rho$);
- normalization ($\mathrm{Tr}\rho = 1$);
- positivity ($\rho \geq 0$).

For any $\rho, \sigma \in \mathcal{M}^{(N)} \subset \mathbf{R}^{N^2-1}$ one defines the following distances:

a) Hilbert–Schmidt distance, $D_{\mathrm{HS}}(\rho,\sigma) = \sqrt{\mathrm{Tr}(\rho-\sigma)^2}$;

b) Bures distance, $D_{\mathrm{B}}(\rho,\sigma) = \sqrt{2-2\sqrt{F}}$, where **fidelity** between both states reads $F(\rho,\sigma) = (\mathrm{Tr}|\rho^{1/2}\sigma\rho^{1/2}|)^2$.

Example: Mixed states of a two level system, $N = 2$

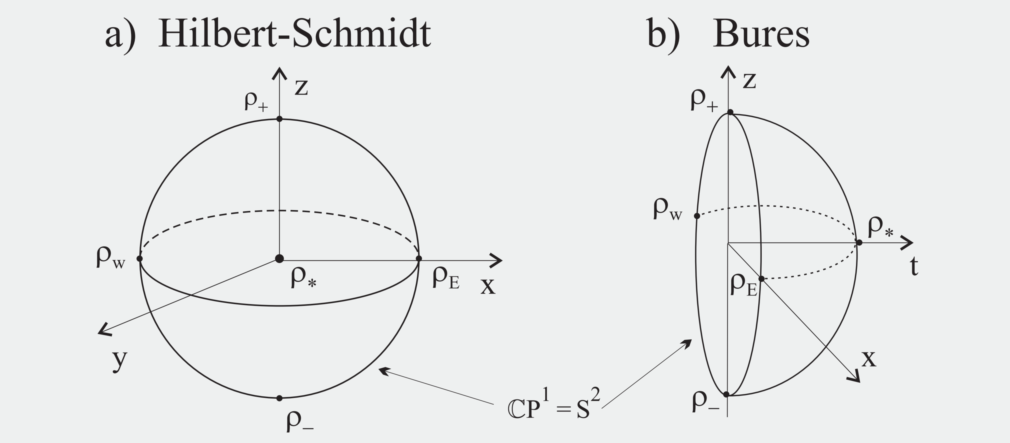

Fig. 1: *Space of mixed states of a qubit:* **a)** *HS–metric induces a flat geometry of a Bloch ball $B^3 \subset \mathbf{R}^3$,* **b)** *the Bures metric induces a curved space - the Uhlmann hemisphere, $\frac{1}{2}S^3 \subset \mathbf{R}^4$. one dimension suppressed.*

Results obtained:

a) characterization of properties of the set $\mathcal{M}_N$ [5].

- the volume and hyperarea of the boundary of this N^2-1 dimensional convex set was computed with respect to the Hilbert–Schmidt measure [2], and Bures measure [1],
- the spectra of random density matrices were analyzed and average purity, $\langle \mathrm{Tr}\rho^2 \rangle$ and average entropy $\langle \mathrm{Tr}\rho \ln \rho \rangle$ of random mixed states was calculated in [3],
- the mean fidelity $\langle F(\rho,\sigma) \rangle$ between two random mixed states ρ and σ was found as a function of the system size [4]

b) properties of the set of pure states for a composite $N \times K$ system.

- We characterized the moments and distribution of G–concurrence [6]

$$G := N D^{\frac{1}{N}} = N \left[\det \rho\right]^{\frac{1}{N}} = N \left[\det\left(\mathrm{Tr}_B |\psi\rangle\langle\psi|\right)\right]^{\frac{1}{N}}$$

  over random pure bipartite states (both real or complex, see figure 2);
- in the limit of high dimensional bipartite $J\ell_1 \times J\ell_2$ systems, the probability distribution for the G–concurrence approaches a Dirac delta

$$P_q^{(\beta)}(G) := \lim_{J\to\infty} P^{(\beta)}_{J\ell_1, J\ell_2}(G) = \delta(G - X_q)$$

$$(\ell_1, \ell_2 \in \mathbf{Z}, \quad 1 < q := \ell_1/\ell_2 \in \mathbf{Q}) \quad ;$$

- the (non trivial) accumulation point is given by $X_q := \frac{1}{\mathrm{e}}\left(\frac{q}{q-1}\right)^{q-1}$.

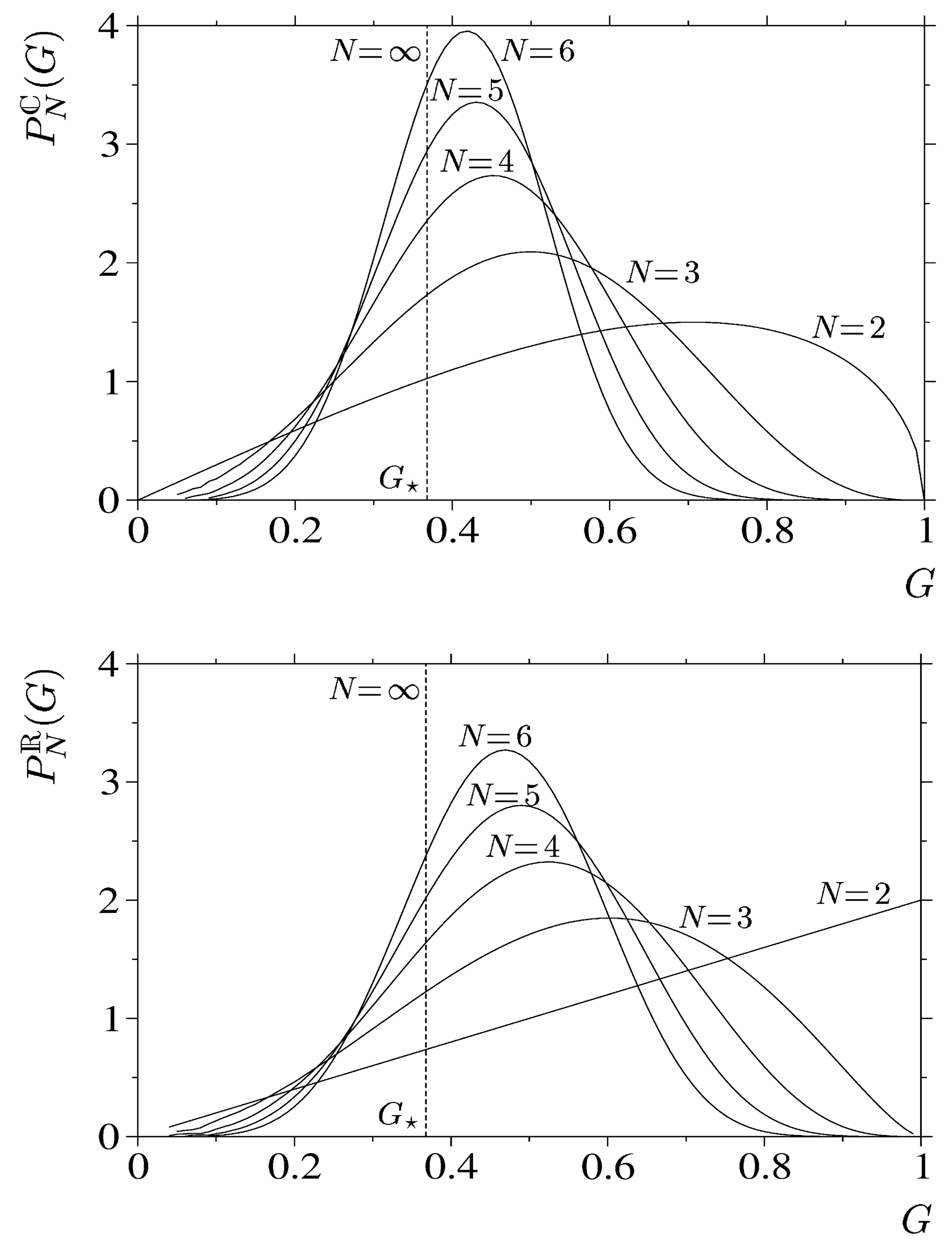

Fig. 2: *G–concurrence's distributions $P^{(\beta)}_{N,K=N}(G)$ are compared for different N in the case of* **(a)** *complex and* **(b)** *real random pure states. Dashed vertical line centered in $G_\star = 1/\mathrm{e}$ denotes the position of the accumulation point X_1*

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[3] H.-J. Sommers and K. Życzkowski
Statistical properties of random density matrices, J. Phys. A **37** (2004) 8457-8466

[4] K. Życzkowski and H.-J. Sommers
Average fidelity between random quantum states, Phys. Rev. A **71** (2005) 032313 (1-11)

[5] I. Bengtsson and K. Życzkowski
Geometry of Quantum States, Cambridge University Press (Cambridge, UK, 2006)

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Distribution of G–concurrence of random pure states, Phys. Rev. A **74** (2006) 062322 (1-8)

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4 Quantum Operations

Quantum operation are defined as completely positive (CP), trace preserving maps acting in the N^2-1 dimensional space of density matrices $\mathcal{M}^{(N)}$. Any quantum operation $\mathbf{\Phi}$ may be represented as

$$\rho' = \mathbf{\Phi}\rho = \sum_{l=1}^{L} A_l \rho A_l^\dagger$$

with the Kraus operators A_l fulfilling $\sum_{l=1}^{L} A_l^\dagger A_l = \mathbf{1}$. Due to that the trace is preserved, $\mathrm{Tr}\ \mathbf{\Phi}\rho = \mathrm{Tr}\ \rho = 1$.

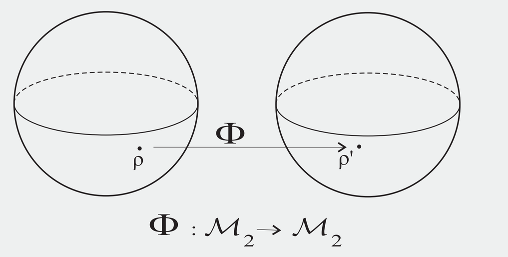

Fig.3: Quantum operation: completely positive, trace preserving map, $\Phi : \mathcal{M}_N \rightarrow \mathcal{M}_N$

Future work on quantum operations:

- Investigation of physically motivated measures in the set of quantum operations, and the measures in the set of mixed states they induce if applied to random pure states, $\rho = \Phi(|\psi\rangle\langle\psi|)$.
- Studying spectra of superoperators $\Phi : \rho' = \Phi(\rho)$ in attempt to extend for the quantum case the classical Frobenius–Perron theorem, which characterizes the spectrum of a stochastic transition matrix $S : \ p' = Sp$.
- Defining a procedure to obtain random bistochastic operation, Φ_B, such that $\Phi_B(\mathbf{1}) = (\mathbf{1})$.
- Comparison of random quantum operations with the maps describing chaotic (non–unitary) dynamics in open quantum systems.

5 Trace–nonincreasing maps

Any completely positive trace–nonincreasing maps (CP–TNI maps) acting in the N^2-1 dimensional space of density matrices $\mathcal{M}^{(N)}$ may be represented as [7]

$$\rho' = \mathbf{\Phi}\rho = \sum_{l=1}^{L} A_l \rho A_l^\dagger$$

with the Kraus operators A_l fulfilling $\mathbf{0} \leq \sum_{l=1}^{L} A_l^\dagger A_l \leq \mathbf{1}$.
Due to that we have $\mathrm{Tr}\ \mathbf{\Phi}\rho \leq \mathrm{Tr}\ \rho = 1$.

CP–TNI maps correspond to situations in which some possible ways of the evolution of the quantum system remain unknown, so the sum of the probabilities of all the events analyzed is smaller then unity.

6 Library of FORTRAN Codes

We design a Library of FORTRAN Codes [8], using algorithms taken from [1], [5], [6] and [9], with the purpose of supplying our analysis with numerical simulations.

Utilities: Useful subroutines non concerned with the other points here listed, including random number generators, auxiliary data files, etc.

Random Pure States: Four subroutines for producing both REAL or CPLX Random Pure States, distributed according to the Haar measure of O(N), respectively U(N). Such a pure states can be produced as vectors $|\psi\rangle$ or as rank 1 projectors $\rho = |\psi\rangle\langle\psi|$.

Random Mixed States: Two subroutines for producing both REAL or CPLX Random Mixed States ρ, distributed according to the so-called "induced" distribution family $P_{N,K}(\rho)$. As a particular case, the "Hilbert-Schmidt" distribution $P_{\mathrm{HS}}(\rho)$ can also be obtained.

Traces: All this subroutines are designed with the aim of performing "total" or "partial" traces of operators M mapping a finite dimensional bipartite Hilbert space

$$\mathcal{H}_{\mathrm{tot}} = \mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}} = \mathbb{C}^N \otimes \mathbb{C}^K$$

into itself. Some distinctions are in order:

- M can either be REAL or CPLX;
- partial traces can either be performed with respect to the system "A' (that is $\mathcal{H}_{\mathrm{A}}$) or respect to the system "B" (that is $\mathcal{H}_{\mathrm{B}}$);
- in the input/output of the subroutine, M can either be expressed in the **"product basis"** of $\mathbb{C}^N \otimes \mathbb{C}^K$, namely $|m,\mu\rangle = |m\rangle_{\mathrm{A}} \otimes |\mu\rangle_{\mathrm{B}}$, or in the **"overall basis"** of $\mathbb{C}^{NK}$, namely $|i\rangle_{\mathrm{tot}}$.

According to the last of the points listed above, when M is given in the **overall basis**, it will be considered as a 2–dimensional $NK \times NK$ matrix whose matrix elements $M(i,j)$ are given by

$$M(i,j) = {}_{\mathrm{tot}}\langle i|M|j\rangle_{\mathrm{tot}} \ .$$

Conversely, when M is given in the **product basis**, it will be considered as a 4–dimensional $N \times K \times N \times K$ matrix whose matrix elements $M(m,\mu,n,\nu)$ are given by

$$M(m,\mu,n,\nu) = \langle m,\mu|M|n,\nu\rangle \ .$$

Random Matrix Tools: Subroutines for producing random matrices out of classical compact Lie groups (O(N), U(N) and USp(2N)), Ginibre's Ensembles (GinOE(N), GinUE(N) and GinSE(N)), Gaussian's Ensembles (GOE(N), GUE(N) and GSE(N)) and Dyson's Circular Ensembles (COE(N), CUE(N) and CSE(N)).

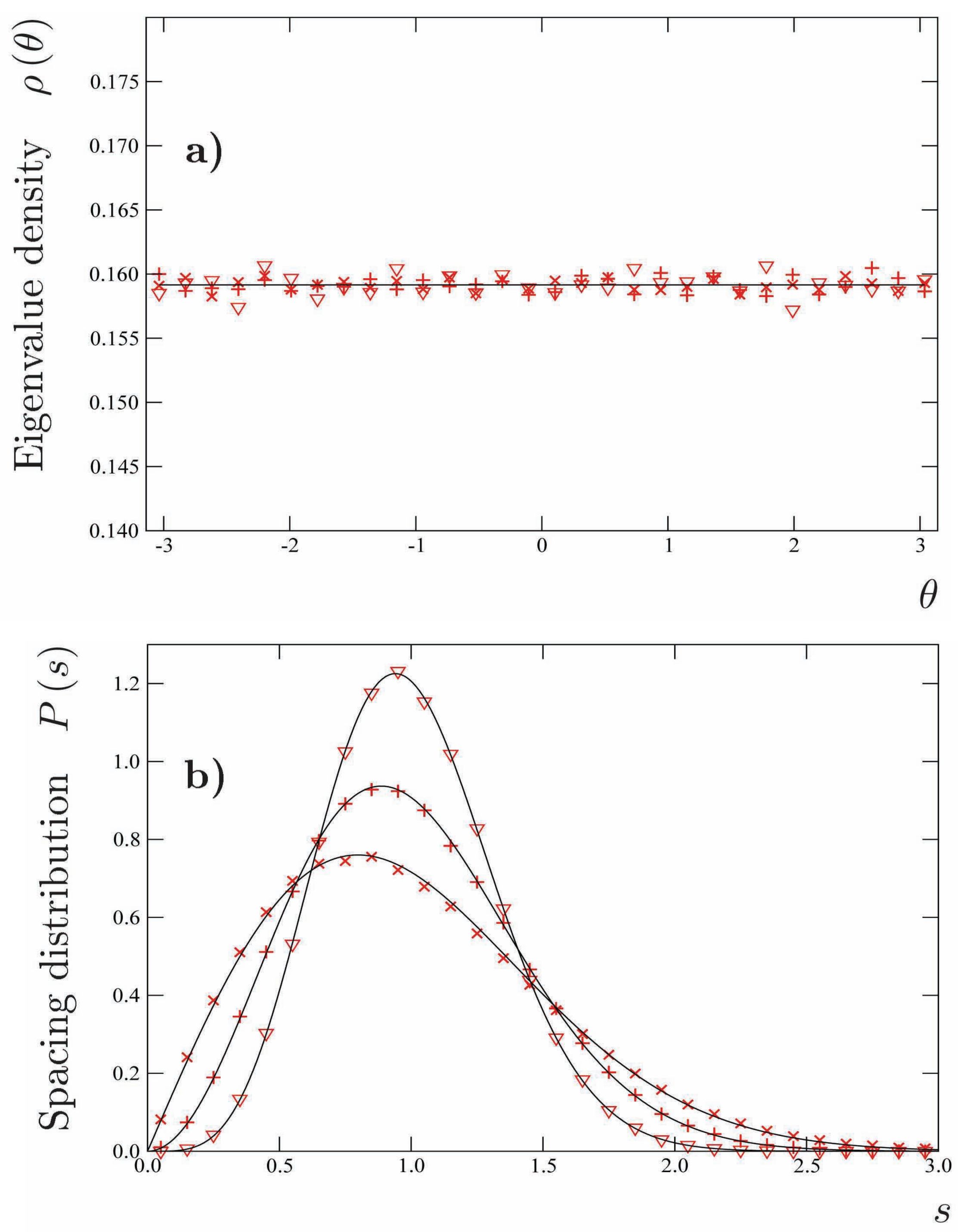

Fig. 4: **a test on our codes on the Dyson's Circular Ensembles.** *Empirical histograms of the density of the eigenvalues and of the spacing distributions for the COE, CUE and CSE. The data are computed from the eigenvalues of ten thousand 50×50 matrices. In panel* **(a)** *the density of the phases of the unimodular eigenvalues are collected in a 30 bins histogram for all the 3 universality classes, whereas in panel* **(b)** *a similar analysis is performed on the level spacing of the phases. Wigner Surmises are also plotted in solid black lines for comparison.*

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