Average G–concurrence of random pure states

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Abstract

Average entanglement of random pure states of an $N \times N$ composite system is analyzed. We compute the average value of the determinant D of the reduced state, which forms an entanglement monotone. Calculating higher moments of the determinant we characterize the probability distribution $P(D)$. Similar results are obtained for the rescaled N^{th} root of the determinant, called G–concurrence. We show that in the limit $N \to \infty$ this quantity becomes concentrated at a single point $G_\star = 1/\mathrm{e}$. The position of the concentration point changes if one consider an arbitrary $N \times K$ bipartite system, in the joint limit $N, K \to \infty$, K/N fixed.

Introduction

The measures of quantum entanglement [1] for a pure state of a bipartite system, $|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, rely on Schmidt coefficients [2] $\{\Lambda_i\}_i$ equivalent to the spectrum $\vec{\Lambda}$ of the reduced system, $\rho = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$. An important set of monotones may be constructed out of symmetric polynomials of the Schmidt coefficients of order $k = 2, \ldots, N$ [3]. Taking the N-th root of the polynomials does not spoil the monotonicity [4]. The last polynomial is equivalent to the determinant of the reduced matrix $D = \det \rho$; its rescaled N–th root is proportional to the geometric mean of all Schmidt coefficients

$$G := N D^{\frac{1}{N}} = N\,[\det \rho]^{\frac{1}{N}} = N\,[\det(\mathrm{Tr}_B |\psi\rangle\langle\psi|)]^{\frac{1}{N}} \tag{1}$$

and was called G–concurrence in [4].

The aim of this work is to compute mean values and to describe probability distributions for the determinant D and its root G of random pure states of a bipartite system, generated with respect to the natural, unitary invariant measure on the space of pure states, also called Fubini–Study (FS) measure.

Random pure states and induced measures

Consider a pure state of a bipartite $N \times K$ system represented in a product basis

$$|\psi\rangle = \sum_{i=1}^{N} \sum_{j=1}^{K} A_{ij} |i\rangle \otimes |j\rangle .$$

The Schmidt coefficients Λ_i coincide with the eigenvalues of a positive matrix $\rho_N = AA^\dagger$, equal to the density matrix obtained by a partial trace on the K–dimensional space. The matrix A needs not to be Hermitian, the only constraint is the trace condition, $\mathrm{Tr}\, AA^\dagger = 1$.

The natural measure (FS) on the space of pure states induces the following ρ_N–eigenvalues's distributions [5]

$$P^{(\beta)}_{N,K}(\Lambda_1, \ldots, \Lambda_N) = C^{(\beta)}_{N,K}\, \delta(1 - \sum_i \Lambda_i) \times \prod_i \Lambda_i^{(\beta(K-N)+\beta-2)/2} \theta(\Lambda_i) \prod_{i<j} |\Lambda_i - \Lambda_j|^\beta , \tag{2}$$

in which the cases of real or complex $|\psi\rangle$ are distinguished by the *repulsion exponent* β being equal 1, respectively 2 and $C^{(\beta)}_{N,K}$ is just a normalization constant.

Induced distribution (2) coincide with the Hilbert–Schmidt distribution [6] of $N \times N$ density matrices ρ_N, provide that

$$K = \begin{cases} N & \text{for complex } |\psi\rangle \quad (\beta = 2 \text{ or abbr. } \mathbb{C}) \\ N+1 & \text{for real } |\psi\rangle \quad (\beta = 1 \text{ or abbr. } \mathbb{R}) \end{cases}$$

Average moments of G–concurrence

The moments of the G–concurrence (1) on the HS–probability distribution $P^{(\beta)}_N(G)$ are given by

$$\begin{cases} \langle G^M_{\mathbb{C}} \rangle_N &= N^M \dfrac{\Gamma(N^2)}{\Gamma(M+N^2)} \prod_{j=1}^{N} \dfrac{\Gamma\left(\frac{M}{N}+j\right)}{\Gamma(j)} \\ \langle G^M_{\mathbb{R}} \rangle_N &= N^M \dfrac{\Gamma\left(\frac{N^2+N}{2}\right)}{\Gamma\left(M+\frac{N^2+N}{2}\right)} \prod_{j=1}^{N} \dfrac{\Gamma\left(\frac{M}{N}+\frac{j+1}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right)} \end{cases} ; \tag{3}$$

similar expression have been found for $\langle D^M_{(\beta)} \rangle_N$. The firsts two moments are depicted in Figure 1.

HS–Probability distribution $P^{(\beta)}_N(G)$

When $N = 2$, the HS–Probability distribution $P^{(\beta)}_2(G)$ is simply given by

$$\begin{cases} P^{\mathbb{C}}_2(G) &= 3\,G\,\sqrt{1-G^2} \\ P^{\mathbb{R}}_2(G) &= 2\,G \end{cases} , \qquad G \in [0, 1] . \tag{4}$$

For $N > 2$ we construct the HS–distribution from all moments $\langle G^M_{(\beta)} \rangle_N$ given by equation (3). $P^{(\beta)}_N(G)$ is simply given by an inverse Laplace transforming procedure, consisting in the following integral along the imaginary M–axis (see Figure 2):

$$P^{(\beta)}_N(G) = \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}M}{2\pi i}\, G^{-(1+M)}\, \langle G^M_{(\beta)} \rangle_N \tag{5}$$

The same relation holds between $P^{(\beta)}_N(D)$ and $\langle D^M_{(\beta)} \rangle_N$.

Asymptotically, for $D \to (1/N)^N$, we find (see Figure 3)

$$\begin{cases} P^{\mathbb{C}}_N(D) \simeq A^{\mathbb{C}}_N \cdot \dfrac{(-\log D - N \log N)^{(N^2-3)/2}}{D\,\left[(N^2-3)/2\right]!} \\ P^{\mathbb{R}}_N(D) \simeq A^{\mathbb{R}}_N \cdot \dfrac{(-\log D - N \log N)^{(N^2+N-6)/4}}{D\,\left[(N^2+N-6)/4\right]!} \end{cases} \tag{6}$$

whereas for $D \to 0$ we have (see Figure 4)

$$\begin{cases} P^{\mathbb{C}}_N(D) \simeq Z^{\mathbb{C}}_N + X^{\mathbb{C}}_N \cdot D \log D + \widetilde{X}^{\mathbb{C}}_N \cdot D + \mathcal{O}\left(D^2 (\log D)^2\right) \\ P^{\mathbb{R}}_N(D) \simeq Z^{\mathbb{R}}_N + Y^{\mathbb{R}}_N \cdot D^{\frac{1}{2}} + X^{\mathbb{R}}_N \cdot D \log D + \widetilde{X}^{\mathbb{R}}_N \cdot D + \\ \qquad + W^{\mathbb{R}}_N \cdot D^{\frac{3}{2}} \log D + \widetilde{W}^{\mathbb{R}}_N \cdot D^{\frac{3}{2}} + \mathcal{O}\left(D^2 (\log D)^2\right) \end{cases} \tag{7}$$

Asymptotic behavior for $P^{(\beta)}_N(G)$ at large N

When the system becomes eventually large, we found the general result

$$G(M) := \lim_{N \to \infty} \left\langle G^M_{(\beta)} \right\rangle_N = \mathrm{e}^{-M} \tag{8}$$

that holds for both real and complex density matrices HS–distributed.

Moreover (8) display that $\langle G \rangle_N$ is $1/\mathrm{e} = 0.367\,879\,441\ldots$ and variance is 0; such behavior can be recognized in Figure 1.

For what concerns the limiting distribution, we have

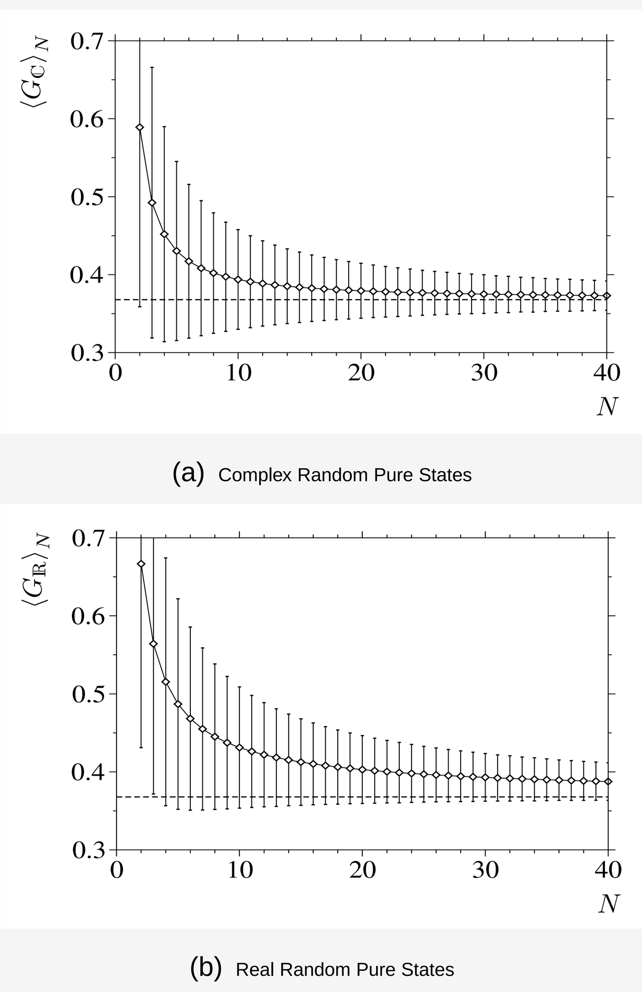
$$P^{(\beta)}(G) := \lim_{N \to \infty} P^{(\beta)}_N(G) = \delta(G - \mathrm{e}^{-1}) \tag{9}$$

again for both $\beta = 1, 2$.

A concentration of reduced density matrices around the maximally mixed state has been quantified [7] for bipartite $N \times K$ systems in the case $K \gg N$. Concerning the G–concurrence, a similar concentration effect occurs even if $K = N$, provided that $N \to \infty$.

Conclusion

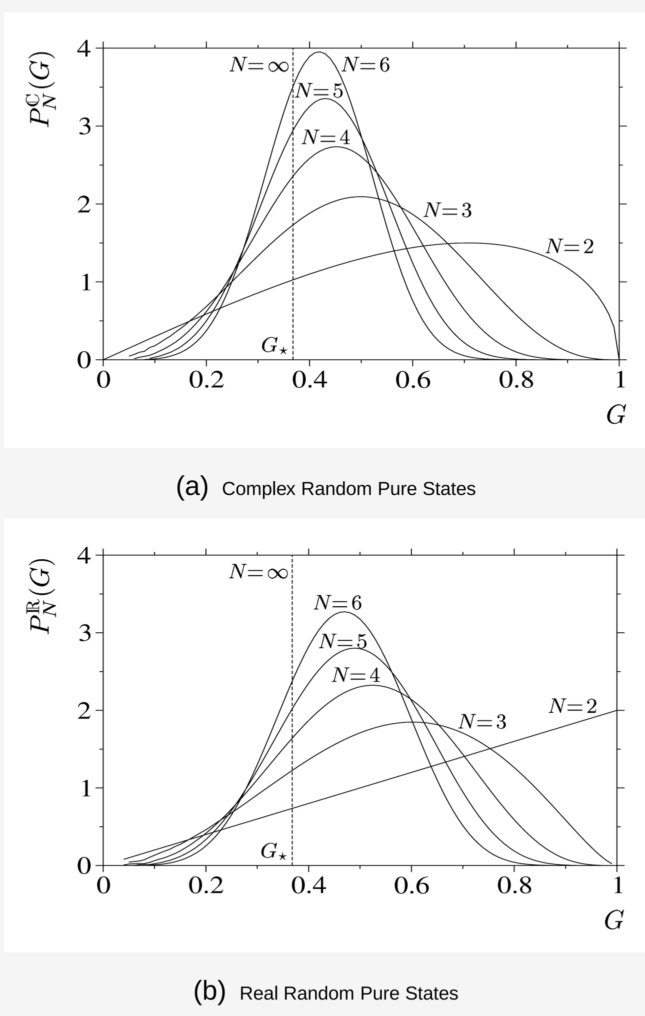
The generalized G–concurrence is likely to be the first measure of pure state entanglement for which one could find not only the mean value over the set of random pure states, but also compute explicitly all moments and describe its probability distribution, deriving an analytic expression in the large N limit. Our work may also be considered as a contribution to the random matrix theory: we have found the distribution of the determinants of random Wishart matrices $AA^\dagger$, normalized by fixing their trace.



(a) Complex Random Pure States

(b) Real Random Pure States

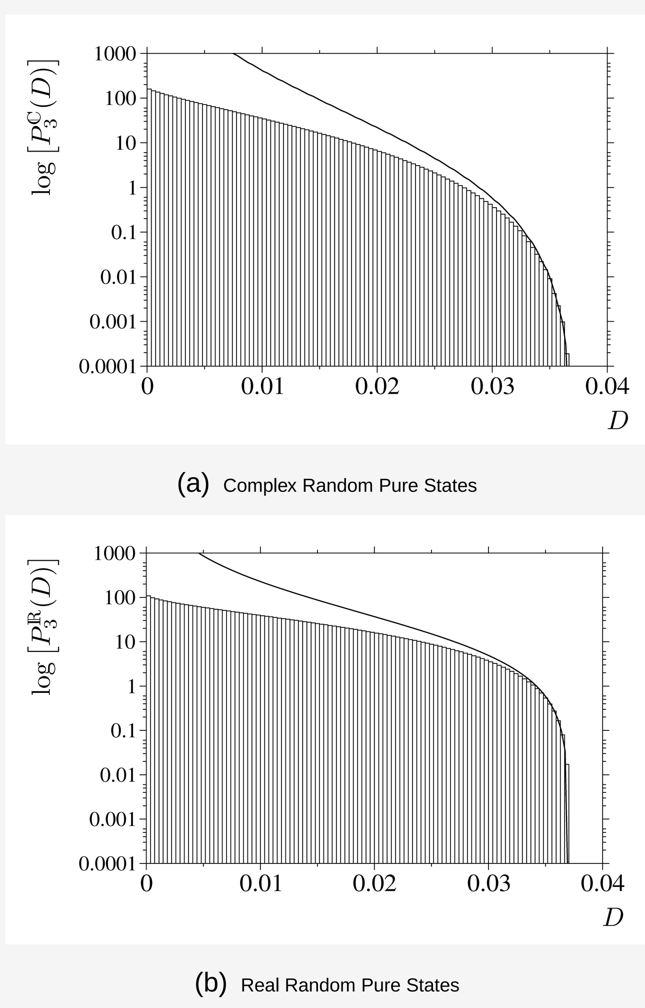
Figure 1: Average of G–concurrence for **(a)** complex and **(b)** real random pure states of a $N \times (N+2-\beta)$ system distributed accordingly to the FS measure. The average is computed by means of equation (3); error bars represent the variance of $P^{(\beta)}_N(G)$. Dashed line represent the asymptote $G_\star = 1/\mathrm{e}$ of eq. (8).



(a) Complex Random Pure States

(b) Real Random Pure States

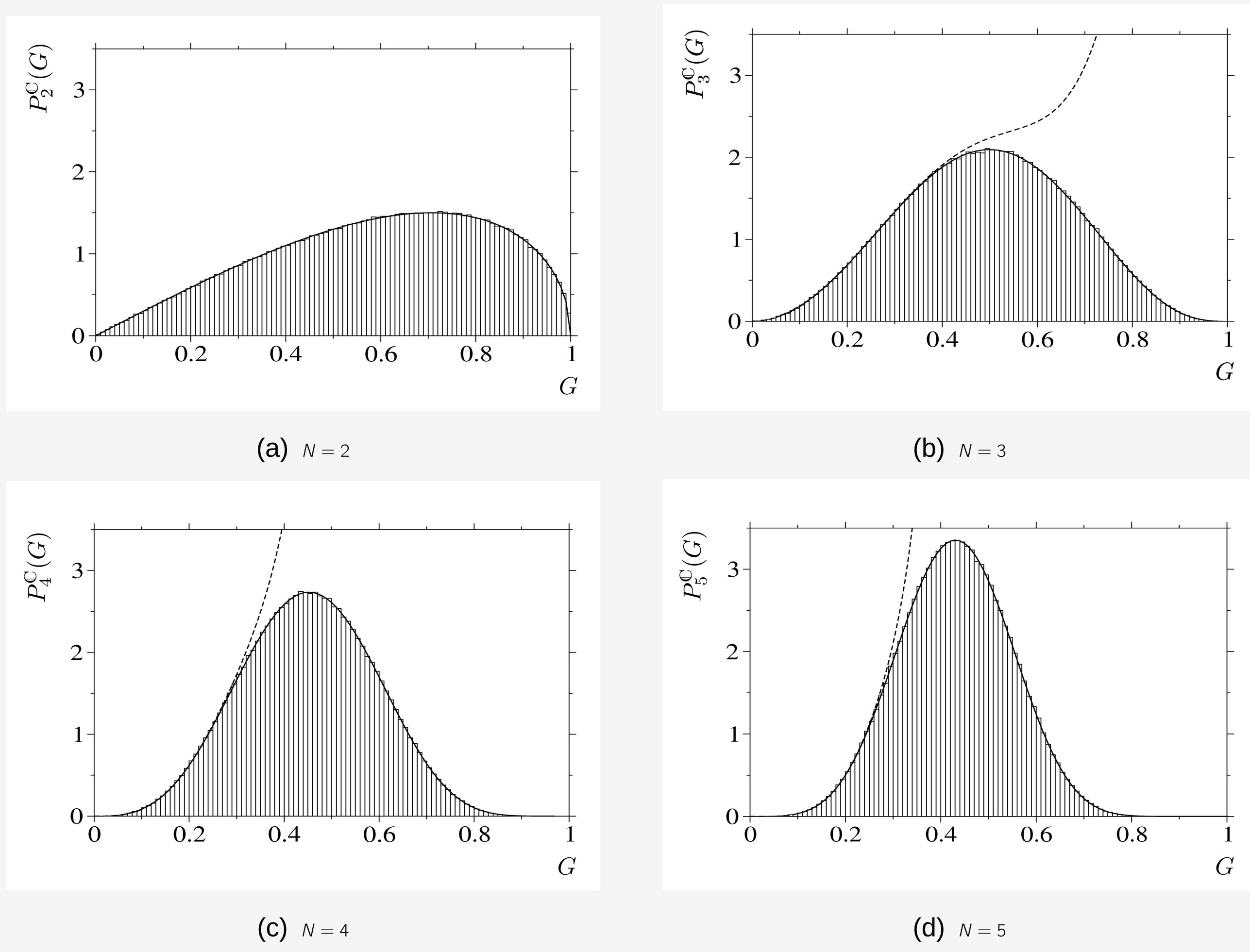
Figure 2: G–concurrence's distributions $P^{(\beta)}_N(G)$ are compared for different N. The distributions are obtained by performing numerically the inverse Laplace transform of equation (5). Dashed vertical line centered in $G_\star = 1/\mathrm{e}$ denotes the position of the Dirac–delta corresponding to $P^{(\beta)}(G)$, as in eq. (9).



(a) Complex Random Pure States

(b) Real Random Pure States

Figure 3: 100 bins histogram of 10^8 determinants of 3×3 density matrices distributed accordingly to the HS measure is compared with the right asymptote given by equation (6) (plotted in solid line).



(a) $N = 2$

(b) $N = 3$

(c) $N = 4$

(d) $N = 5$

Figure 4: In panel **(a)**, formula (4) is compared with a 100 bins histogram of 10^6 G–concurrence of 2×2 complex density matrices distributed accordingly to the HS measure. The other panels shows histograms (for different N) together with the distribution of G–concurrence obtained by inverse Laplace transforming as in equation (5) (plotted in solid lines). The left asymptote given by eq. (7) is also plotted in dashed line for comparison.

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