Average \( G \)-concurrency of random pure states

Valerio Cappellini\(^1\), Hans-Jürgen Sommers and Karol Życzkowski

“Mark Kac” Complex Systems Research Centre, Jagiellonian University of Cracow
\(^1\)valerio@cft.edu.pl


Abstract

Average entanglement of random pure states of an \( N \times N \) composite system is analyzed. We compute the average value of the determinant \( D \) of the reduced state, which forms an entanglement monotone. Calculating higher moments of the determinant we characterize the probability distribution \( P(D) \). Similar results are obtained for the rescaled \( N^\beta \) root of the determinant, called \( G \)-concurrency. We show that in the limit \( N \to \infty \) this quantity becomes concentrated at a single point \( G = 1/e \). The position of the concentration point changes if one consider an arbitrary \( N \times K \) bipartite system, in the joint limit, \( N, K \to \infty \), \( K/N \) fixed.

Introduction

The measures of quantum entanglement [1] for a pure state of a bipartite system, \( |\psi\rangle \in \mathbb{C}^N \otimes \mathbb{C}^N \), rely on Schmidt coefficients \( 2 \) for the Schmidt decompositions [2] \((\Lambda_i, |\psi_i\rangle) \) to the spectrum \( \Lambda \) of the reduced system, \( \rho = \text{Tr}_R(|\psi\rangle \langle \psi|) \). An important set of monotones may be constructed out of symmetric polynomials of the Schmidt coefficients of order \( k = 2, \ldots, N \) [3]. Taking the \( N \)-th root of the polynomials does not spoil the monotonicity [4]. The last polynomial is equivalent to the determinant of the reduced matrix \( D = \det \rho \); its rescaled \( N \)-th root is proportional to the geometric mean of all Schmidt coefficients

\[
G = N^{\frac{1}{2N}} \det |\psi\rangle \langle \psi| = N^{\frac{1}{2N}} \det (\text{Tr}_R(|\psi\rangle \langle \psi|))
\]

and was called \( G \)-concrence in [4].

The aim of this work is to compute mean values and to describe probability distributions for the determinant \( D \) and its root \( G \) for random pure states of a bipartite system, generated with respect to the natural, unitary invariant measure on the space of pure states, also called Fubini–Study (FS) measure.

Random pure states and induced measures

Consider a pure state of a bipartite \( N \times K \) system represented in a product basis

\[
|\psi\rangle = \sum_{i=1}^N \sum_{j=1}^K \beta_i |i\rangle \otimes |j\rangle
\]

The Schmidt coefficients \( \Lambda_i \) coincide with the eigenvalues of a positive matrix \( \rho_{ij} = \Lambda_i \) equal to the density matrix obtained by a partial trace on the \( K \)-dimensional space. The matrix \( \Lambda \) needs not to be Hermitian, the only constraint is the trace condition, \( \text{Tr} \Lambda^2 = 1 \).

The natural measure (FS) on the space of pure states induces the following \( \rho_{ij} \)-eigenvalues distributions [5]

\[
\rho_{ij} \equiv \{ \rho_{ij} \}_{N,K} = C_{ij}^N \sum_{(\Lambda_i, |\psi_i\rangle)} |\Lambda|^2 \prod_{i<j} (\Lambda_i - \Lambda_j)^2 / (N-1)!
\]

where \( C_{ij}^N \) is the binomial coefficient. These distributions are obtained by performing numerically the inverse Laplace transform of equation (5). Dashed vertical line centered in \( G = 1/e \) denotes the position of the Dresdella corresponding to \( \rho_{ij} \), as in eq. (8).

![Figure 1: Average of \( G \)-concrence for (a) complex and (b) real random pure states of a \( N \times (N+1-\beta) \) system distributed according to the FS measure. The average is computed by means of equation (3); error bars represent the variance of \( \rho_{ij} \).](image)

![Figure 2: \( G \)-concrence’s distributions \( \rho_{ij} \) are compared for different \( \beta \). The distributions are obtained by performing numerically the inverse Laplace transform of equation (5). Dashed vertical line centered in \( G = 1/e \) denotes the position of the Dresdella corresponding to \( \rho_{ij} \), as in eq. (8).](image)

![Figure 3: 10^6 bins histogram of 10^6 determinants of 3 x 3 density matrices distributed according to the FS measure is compared with the right asymptote given by equation (8) (glotted in solid line).](image)

![Figure 4: In panel (a), formula (4) is compared with a 10^6 bins histogram of 10^6 \( G \)-concrence of 2 x 2 complex density matrices distributed according to the FS measure. The other panels shows histograms, (for different \( \beta \)) together with the distribution of \( G \)-concrence obtained by inverse Laplace transforming as in equation (5) (glotted in solid line). The left asymptote given by eq. (7) is also plotted in dashed line for comparison.](image)

Asymptotic behavior for \( \rho_{ij} \) at large \( N \)

When the system becomes eventually large, we found the general result

\[
G(M) \sim \text{km} N^{-\beta} (\rho_{ij})_N^{1/2} N^{-e^{-\beta}}
\]

for both real and complex density matrices HS-distributed. Moreover, \( \rho_{ij} \) display that \( (\rho_{ij})_N \) is \( 1/e \approx 0.3678741 \ldots \) and variance is 0, such behavior can be recognized in Figure 1.

For what concerns the limiting distribution, we have

\[
\rho_{ij}(G) = \lim_{N \to \infty} \rho_{ij}^{\beta}(G) = \delta(G-e^{-\beta})
\]

again for both \( \beta > 1 \). A concentration of reduced density matrices around the maximally mixed state has been quantified [7] for bipartite N x K systems in the case \( K > N \). Concerning the \( G \)-concrence, a similar concentration effect occurs even if \( K > N \), provided that \( N \to \infty \).

Conclusion

The generalized \( G \)-concrence is likely to be the first measure of pure state entanglement for which one could find not only the mean value over the set of random pure states, but also compute explicitly all moments and describe its probability distribution, deriving an analytic expression in the large \( N \) limit. Our work may also be considered as a contribution to the random matrix theory: we have found the distribution of the determinants of random Wishart matrices \( A \), normalized by their trace.

References