LOGARITHMIC BREAKING TIMES IN DISCRETIZED CLASSICAL DYNAMICAL SYSTEMS

Valerio Cappellini [*]
Dipartimento di Fisica Teorica, Università di Trieste
Strada Costiera 11, 34014 Trieste, Italy

Fabio Benatti [†] Federico Zertuche [‡]

Classical Chaos: exponential amplification of small errors

$$|\delta x| \underset{n}{\longrightarrow} |\delta x(n)| = \lambda^n \, |\delta x| = e^{n \log \lambda} |\delta x|$$

$\log \lambda$ is Lyapounov Exponent

$|\delta x(n)|$ increase no longer in compact systems...

$$\lambda := \lim_{n \to +\infty} \lim_{\delta x \to 0} \frac{1}{n} \log \frac{|\delta x(n)|}{|\delta x|}$$

... and $|\delta x|$ can't decrease in discrete systems!

For a discrete system $|\delta x| > a$ (a = lattice spacing)

$$\forall a : \lambda(a) = 0$$

*E-mail: *valerio.cappellini@ts.infn.it*

†Dipartimento di Fisica Teorica, Università di Trieste, Strada Costiera 11, 34014 Trieste, Italy and Istituto Nazionale di Fisica Nucleare, Sezione di Trieste. E-mail: *fabio.benatti@ts.infn.it*

‡Instituto de Matemáticas, UNAM, Unidad Cuernavaca, A.P. 273-3, Admon. 3, 62251 Cuernavaca, Morelos, México. E-mail: *zertuche@matcuer.unam.mx*

- Cfr. Quantum Chaos:

$$\forall \hbar : \lambda(\hbar) = 0 \quad \text{but} \quad \lim_{n\to\infty} \lim_{\hbar\to 0} \lambda(\hbar) > 0$$

An indicator of chaos in classical continuous systems

- Entropy $[S(n)]$: information on the evolving system up to time n.
- Loosely speaking, the Kolmogorov-Sinai metric entropy is $h_\mu(T) := \lim_{n\to\infty} \frac{S(n)}{n}$ (entropy per unit time).

Theorem 1 (Pesin)

$$\text{Ergodicity} \Longrightarrow h_\mu(T) = \sum \text{positive Lyapounov exponent}$$

Theorem 2 (Brudno)

$$\text{Ergodicity} \Longrightarrow h_\mu(T) = \text{Algorithmic Complexity}$$

Classical Sistems

↓

Algorithmic Complexity ←(Thm. Brudno)— Kolmogorov–Sinai metric Entropy —(Thm. Pesin)→ Lyapunouv Exponent

Which indicator of Chaos can we use
for discrete classical systems?

A Quantum concept is of help

The Alicki Lindblad Fannes Dynamical Entropy

For Quantum Dynamical Systems $(\mathcal{M}, \omega, \Theta)$

$$\begin{cases} \mathcal{M} : \text{a (not generally commutative) *algebra} \\ \Theta : \text{a *automorphism on } \mathcal{M} \text{ that implements the dynamic} \\ \omega : \text{an expectation (state) on } \mathcal{M} \text{ that is } \Theta\text{-invariant} \end{cases}$$

Let us Introduce:

- $\mathcal{Y} := \{y_\ell\}_{\ell=1}^D$; $\sum_{\ell=1}^D y_\ell^* y_\ell = \mathbb{1}_{\mathcal{M}_0}$ (Partition of unit)
  $y_\ell \in \mathcal{M}_0 \subseteq \mathcal{M}$; $\mathcal{M}_0$ (subalgebra) s.t. $\Theta(\mathcal{M}_0) = \mathcal{M}_0$
- the time-evolving partition of unit: $\Theta^k(\mathcal{Y}) := \left\{\Theta^k(y_i)\right\}_{i=1}^D$
- the refined partition:
  $\mathcal{Y}_\Theta^{[0,n-1]} = \left\{\Theta^{n-1}\left(y_{i_{n-1}}\right)\Theta^{n-2}\left(y_{i_{n-2}}\right)\cdots\Theta(y_{i_1})\,y_{i_0}\right\}$
- the $D^n \times D^n$ density matrices $\rho\left[\mathcal{Y}_\Theta^{[0,n-1]}\right]$ with elements
  $\left[\rho\left[\mathcal{Y}_\Theta^{[0,n-1]}\right]\right]_{\boldsymbol{i},\boldsymbol{j}} := \omega\left(y_{j_0}^*\Theta\left(y_{j_1}^*\right)\cdots\Theta^{n-1}\left(y_{j_{n-1}}^* y_{i_{n-1}}\right)\cdots\Theta\left(y_{i_1}\right)y_{i_0}\right)$.
- the Von Neumann Entropy $H_{\omega,\mathcal{M}_0}\left[\mathcal{Y}_\Theta^{[0,n-1]}\right]$

The Alicki Lindblad Fannes Entropy $h^{ALF}_{\omega,\mathcal{M}_0}(\Theta)$ of $(\mathcal{M},\omega,\Theta)$

$$h^{ALF}_{\omega,\mathcal{M}_0}(\Theta) := \sup_{\mathcal{Y}\subset\mathcal{M}_0} \limsup_{n} \frac{1}{n} H_{\omega,\mathcal{M}_0}\left[\mathcal{Y}_{\Theta}^{[0,n-1]}\right]$$

The ALF-entropy can also be used for Classical Dynamical System $(\mathcal{X},\mu,T)$. Let's take as a system $(\mathbb{T}^2,\mathrm{d}\boldsymbol{x},T_\alpha)$ that is the torus $\mathbb{R}^2/\mathbb{Z}^2$ equipped with the Lebesgue measure $\mathrm{d}\boldsymbol{x}$, on which the dynamic is implemented by

$$\boldsymbol{x}_n \mapsto \boldsymbol{x}_{n+1} := T_\alpha\,\boldsymbol{x}_n$$
$$T_\alpha := \begin{pmatrix} 1 & 1 \\ \alpha & 1+\alpha \end{pmatrix} \quad , \quad \alpha\in\mathbb{Z}$$

T_α is a toral automorphism and a generalization of the so called Arnold Cat Map. Depending on α we have two kind of dynamics:

- $\alpha\in(-\infty,-4)\cup(0,+\infty)$ Chaotic System
- $\alpha\in[-4,0]$ Regular System

Algebraically the system $(\mathbb{T}^2,\mathrm{d}\boldsymbol{x},T_\alpha)$ can be described by $(\mathcal{A}_{\mathcal{X}},\omega,\Theta_\alpha)$ where

$$\begin{cases} \mathcal{A}_{\mathcal{X}} : \text{The }{}^*\text{algebra of bounded function } f \text{ on } \mathbb{T}^2 \\ \omega : \text{the state defined by } \omega(f) := \int_{\mathbb{T}^2} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ \Theta_\alpha : \text{ the }{}^*\text{automorphism } \Theta_\alpha(f(\boldsymbol{x})) =: f(T_\alpha\,\boldsymbol{x}) \end{cases}$$

In order to get a partition of unit on $\mathcal{A}_{\mathcal{X}}$ we shall use a partition on $\mathbb{T}^2$:

$$\mathcal{E} := \{E_\ell\}_{\ell=1,2,\cdots,D} : \text{ Partition } \left(\bigcup_{\ell=1}^{D} E_\ell = \mathbb{T}^2 \; ; \; E_\ell \cap E_k = \emptyset \;\; \forall \ell \neq k \right)$$

$$\mathcal{Y} := \{\mathcal{X}_{E_\ell}\}_{\ell=1,2,\cdots,D} : \text{ Partition of unit } \left(\sum_{\ell=1}^{D} \mathcal{X}^*_{E_\ell} \mathcal{X}_{E_\ell} = \mathbb{1}_{\mathcal{A}_{\mathcal{X}}} \right)$$

Here $\mathcal{X}_{E_\ell} \in \mathcal{A}_0 \subseteq \mathcal{A}_{\mathcal{X}}$ where $\mathcal{A}_0$ is the subalgebra of simple function.

With our algebraic description $(\mathcal{A}_{\mathcal{X}}, \omega, \Theta_\alpha)$, using the partition $\mathcal{Y}$, we can obtain the ALF-entropy $h^{ALF}_{\omega,\mathcal{A}_0}(\Theta_\alpha)$.
As a result:

$$h^{ALF}_{\omega,\mathcal{A}_{\mathcal{X}}}(\Theta_\alpha) = h_\mu(T_\alpha)$$

Then a quantum entropy is useful in a classical system, in which is equivalent to our “chaos indicator” $h_\mu(T_\alpha)$.

What about a discrete system?

- Finite number of states $\Longrightarrow$ Periodicity
- $\Longrightarrow$ No Chaoticity at all
- With a sufficient number of states, a Discrete System can represent a Continuous System (also Chaotic), but only in a definite time range ...
- When do they lose chaoticity ? Why ?
- What does chaos means for a Discrete system ?

We use $h^{ALF}_{\omega,\mathcal{M}_0}(\Theta_\alpha)$ in order to study the evolution of chaoticity in time, for discrete classical systems

From $(\mathbb{T}^2, \mathrm{d}\boldsymbol{x}, T_\alpha)$ we get a discrete dynamical system by replacing the torus with a periodic grid with lattice spacing $a = \frac{1}{N}$. Algebraically, we define a *morphism $\mathcal{J}_{N,\infty} : (\mathcal{A}_{\mathcal{X}}, \omega, \Theta_\alpha) \mapsto (D_{N^2}, \omega_{N^2}, \Theta_\alpha)$ where

$$\begin{cases} D_{N^2} : \text{The abelian C}^* \text{ algebra of diagonal } N^2 \times N^2 \text{ matrices} \\ \omega_{N^2} : \text{the } \textit{tracial state } (\Theta_\alpha\text{-invariant}) \ \omega_{N^2}(D) := \frac{1}{N^2}\operatorname{Tr}(D) \\ \Theta_\alpha : \text{ the *automorphism } \Theta_\alpha(f(\boldsymbol{n})) := f(T_\alpha \boldsymbol{n}) \\ \boldsymbol{n} : \text{ the coordinates on the lattice } \left(\boldsymbol{n} = \left(\frac{\ell_1}{N}, \frac{\ell_2}{N}\right); 0 \leqslant \ell_i < N\right) \end{cases}$$

$\mathcal{J}_{N,\infty}(f)$ is the diagonal matrix that has as (N^2) diagonal elements the N^2 values that the function f assumes on the finite grid on the torus

Discretization $\left(\frac{1}{N}\right) \iff$ Weyl Quantization $(\hbar)$

Partition: In order to get a partition of unit on D_{N^2} we use a subset Λ (D-dimensional) of the toral coordinates:

$$\Lambda = \{\boldsymbol{n}_1, \boldsymbol{n}_2, \cdots, \boldsymbol{n}_D\} \ : \text{ collection of coordinates}$$

$$\mathcal{Y}_\alpha := \{y_\ell\}_{\ell=1}^{D} \ : \text{ Partition of unit } \ y_\ell := \frac{1}{\sqrt{D}} \mathcal{J}_{N,\infty}\left(e^{\,2\pi i\, \boldsymbol{n}_\ell \boldsymbol{x}}\right)$$

With $\boldsymbol{i}$ we indicate the string $\boldsymbol{i} = \{i_0, i_1, \cdots, i_{n-1}\}$ each of $i_\ell \in \{1, 2, \cdots, D\}$. — **Def.:** $\{\boldsymbol{i}\} =: \Omega_D^{(n)}$.

In order to compute the evolving density matrix, we define a function

$$\Xi^{(n),N}_{\Lambda,\alpha} : \Omega^{(n)}_{D} \mapsto (\mathbb{Z}/N\mathbb{Z})^2 = \left\{ \begin{array}{c} \text{points of} \\ \text{the lattice} \end{array} \right\}$$

$$\Xi^{(n),N}_{\Lambda,\alpha}\left(\{i_0, i_1, \cdots, i_{n-1}\}\right) := \sum_{\ell=0}^{n-1} T^{\ell}_{\alpha}\, \boldsymbol{n}_{i_\ell} \pmod{N}$$

The Density matrix is given by

$$\left[\rho\left[\mathcal{Y}^{[0,n-1]}_{\alpha}\right]\right]_{\boldsymbol{i},\boldsymbol{j}} = \frac{1}{D^n}\, \delta^{(N)}_{\Xi^{(n),N}_{\Lambda,\alpha}(\boldsymbol{i})\,,\,\Xi^{(n),N}_{\Lambda,\alpha}(\boldsymbol{j})}$$

where $\delta^{(N)}$ is the periodic Kronecker delta $\pmod{N}$
Let us define two more functions:

$$m^{(n),N}_{\Lambda,\alpha}(\boldsymbol{n}) := \#\left\{ \boldsymbol{i} \in \Omega^{(n)}_{D} \,\middle|\, \Xi^{(n),N}_{\Lambda,\alpha}(\boldsymbol{i}) \equiv \boldsymbol{n} \pmod{N} \right\}$$

$$\nu^{(n),N}_{\Lambda,\alpha}(\boldsymbol{n}) := m^{(n),N}_{\Lambda,\alpha}(\boldsymbol{n}) \Big/ D^n$$

The set of $\nu^{(n),N}_{\Lambda,\alpha}$ different from zero coincide with the set of non null eigenvalues of $\rho\left[\mathcal{Y}^{[0,n-1]}_{\alpha}\right]$.

With this set we can compute the Von Neumann Entropy $H_{\omega,D_{N^2}}\left[\mathcal{Y}^{[0,n-1]}_{\alpha}\right]$.

Note that $$\#\left\{\nu^{(n),N}_{\Lambda,\alpha} \neq 0\right\} \leqslant N^2.$$

Then $H_{\omega,D_{N^2}}\left[\mathcal{Y}^{[0,n-1]}_{\alpha}\right]$ cannot grow indefinitely...
$\Longrightarrow h^{ALF}_{\omega,D_{N^2}}(\Theta_\alpha) \xrightarrow[n\to\infty]{} 0$ (No chaos)

- For an Hyperbolic System, $\Xi^{(n),N}_{\Lambda,\alpha}$ tends to the maximum spreading over $(\mathbb{Z}/N\mathbb{Z})^2$, until we get a saturation. (see Figs. 1, 3)
- $H_{\omega,D_{N^2}}$ grows linearly whence $h^{ALF}_{\omega,D_{N^2}}(\Theta_\alpha)$ is constant. (see Figs. 4, 5, 6)

Thus the System shows chaoticity

- Beyond this time scale, frequencies $\nu^{(n),N}_{\Lambda,\alpha}$ tends to equipartite, and $H_{\omega,D_{N^2}}$ tends to a constant; $h^{ALF}_{\omega,D_{N^2}}(\Theta_\alpha)$ starts to decrease. (see Figs. 4, 5, 6)

Thus the System remembers to be Discrete

- The Breaking Time, that is the time $\bar{n}$ at which we have the change of behaviour, scales with N as (see Fig. 4):

$$\bar{n} \simeq 2\log N = \log\left(\#\left\{\text{states of the system}\right\}\right)$$

- Breaking Time is order of the time at which minimal errors permitted by the discrete structure of the phase space become of the order of the (compact) phase space bound.
- Elliptic System doesn't exhibit a satisfactory spreading of Ran $\left(\Xi^{(n),N}_{\Lambda,\alpha}\right)$ and $H_{\omega,D_{N^2}}$ is just monotonically increasing and bounded (no linear increase).
  We cannot split the time evolution in two different ranges.
  (see Figs. 1, 3, 4, 5, 6)

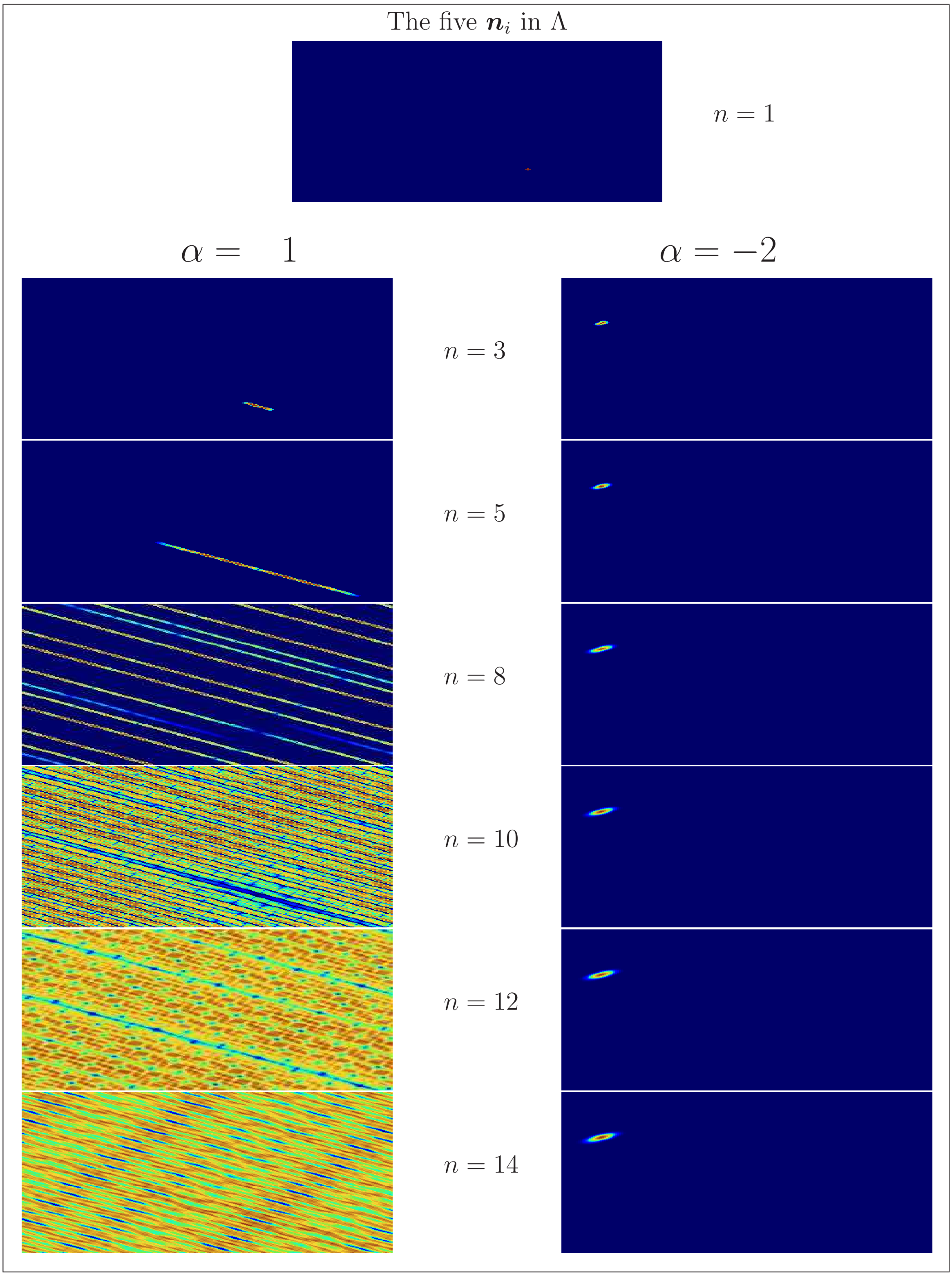


Figure 1: A density plot that shows the distribution of $\nu_{\Lambda,\alpha}^{(n),N}$ values in the hyperbolic ($\alpha = 1$) and elliptic ($\alpha = -2$) case, for five very near $\boldsymbol{n}_i$ in Λ ($N = 200$). Blue, in the plot, correspond to null value of $\nu_{\Lambda,\alpha}^{(n),N}$.

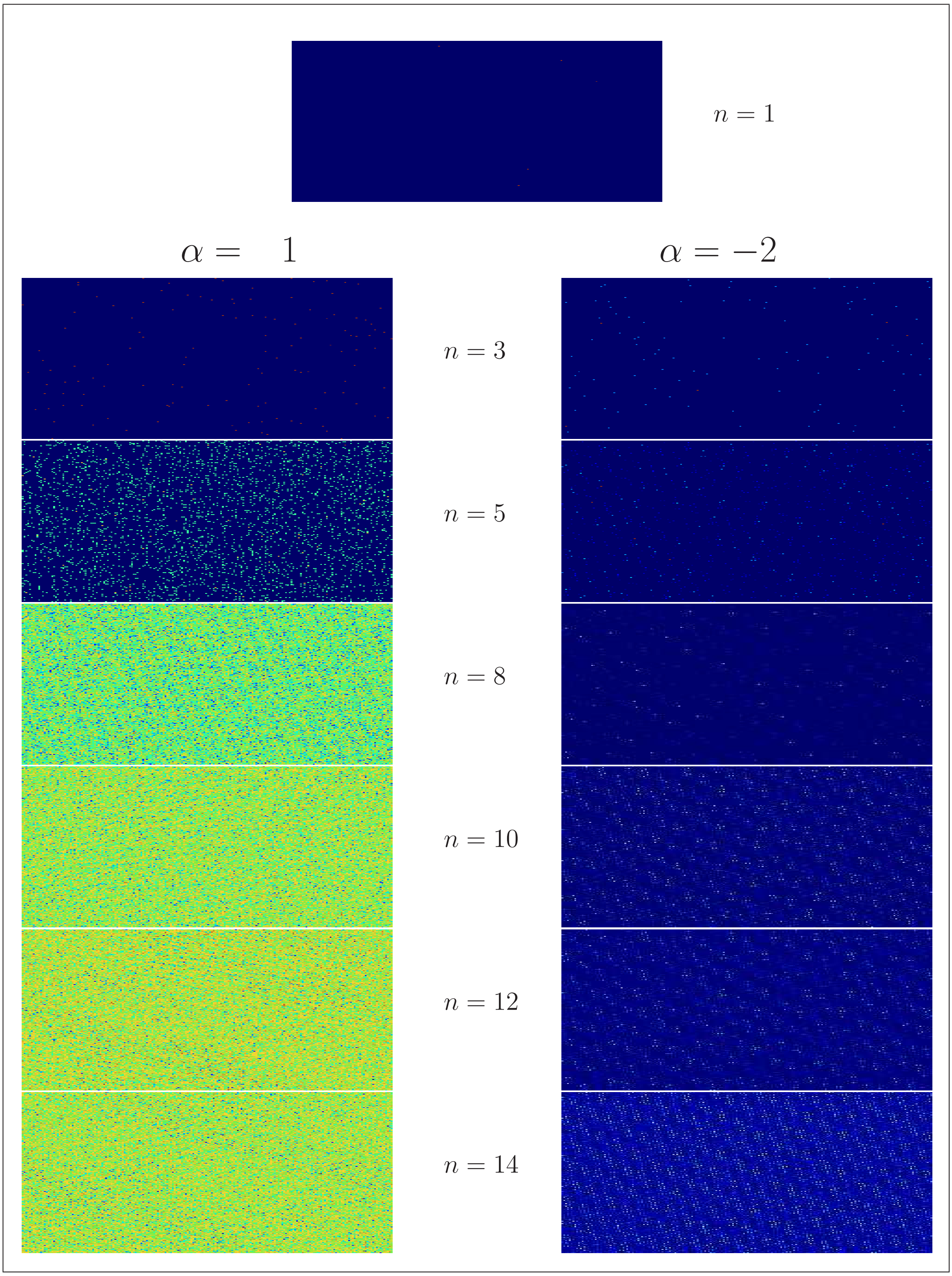


Figure 2: A density plot that shows the distribution of $\nu^{(n),N}_{\Lambda,\alpha}$ values in the hyperbolic ($\alpha = 1$) and elliptic ($\alpha = -2$) case, for five randomly distributed $\boldsymbol{n}_i$ in Λ ($N = 200$). Blue, in the plot, correspond to null value of $\nu^{(n),N}_{\Lambda,\alpha}$.

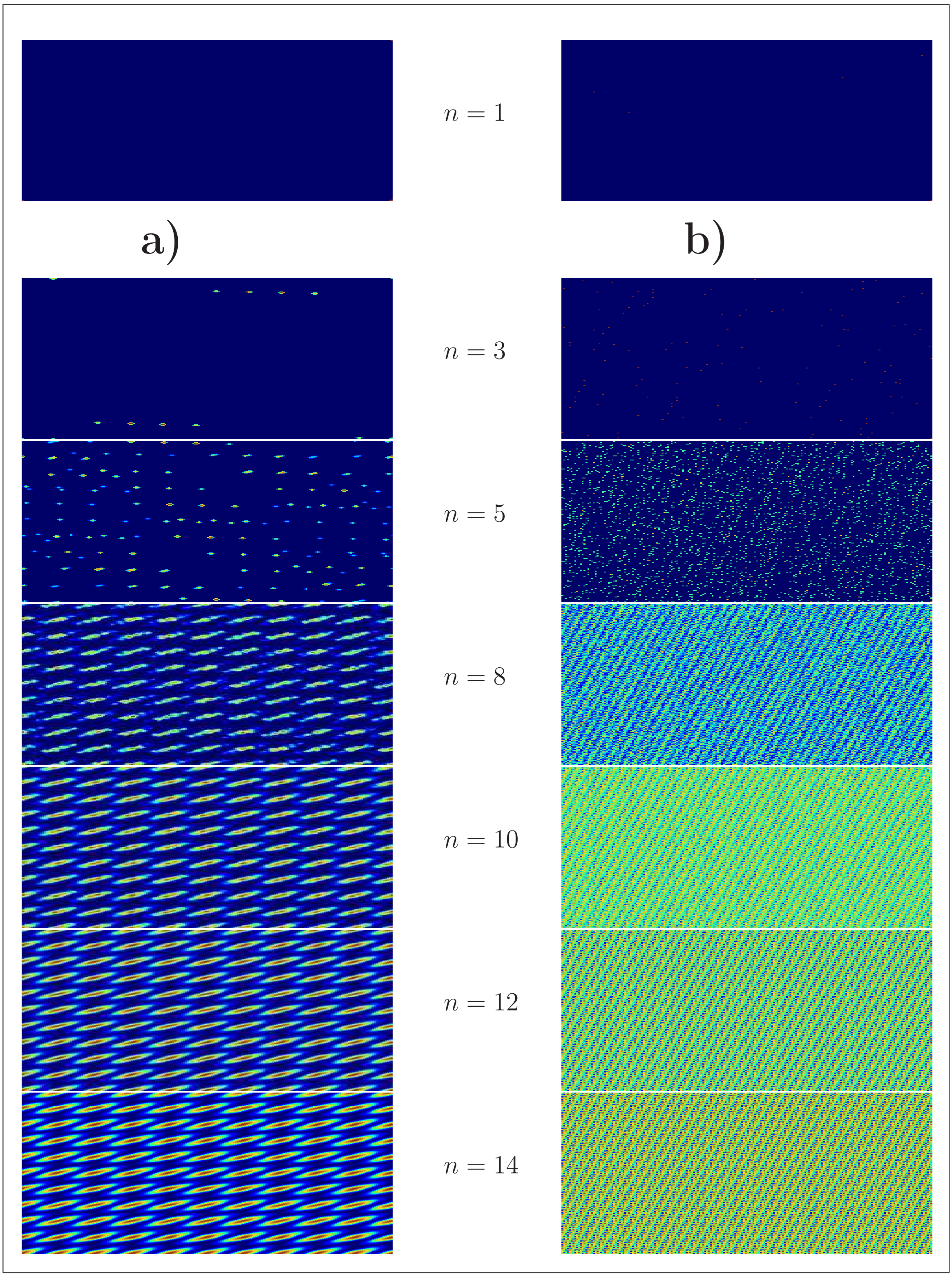
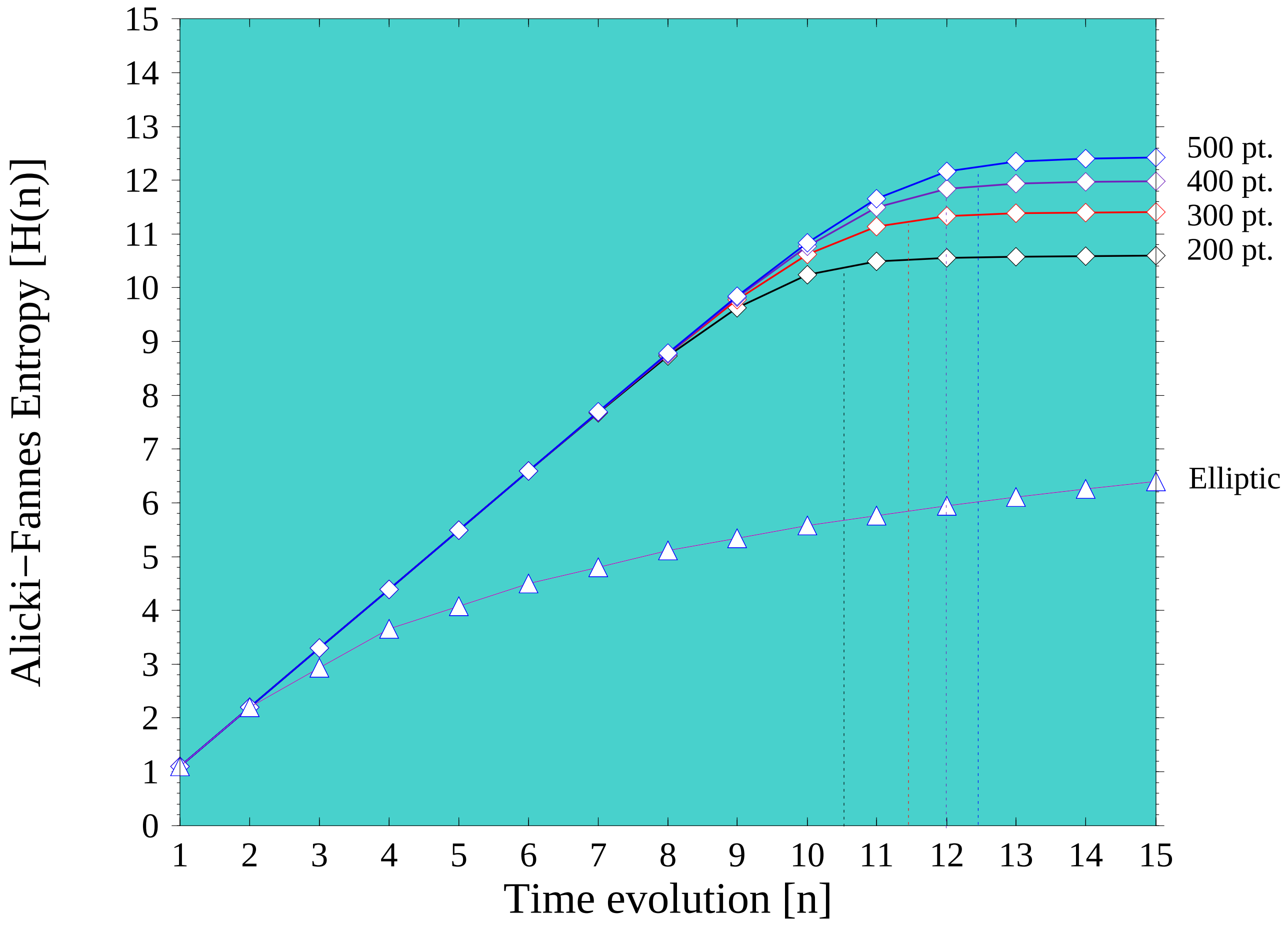


Figure 3: A density plot that shows the distribution of $\nu_{\Lambda,\alpha}^{(n),N}$ values in an hyperbolic case ($\alpha = 17$), with ($N = 200$) for:
a) five very close $\boldsymbol{n}_i$ in Λ; **b)** five randomly distributed $\boldsymbol{n}_i$ in Λ.



N	Breaking Time $= \log N^2$
$N = 200$	10.60
$N = 300$	11.41
$N = 400$	11.98
$N = 500$	12.43

$$N = \frac{1}{a} \quad ; \quad a = \text{lattice spacing}$$

Figure 4: $H_{\omega, D_{N^2}}$ for four different values of N on an hyperbolic system ($\alpha = 1$) and on an elliptic one ($\alpha = -2$); $D = 3$

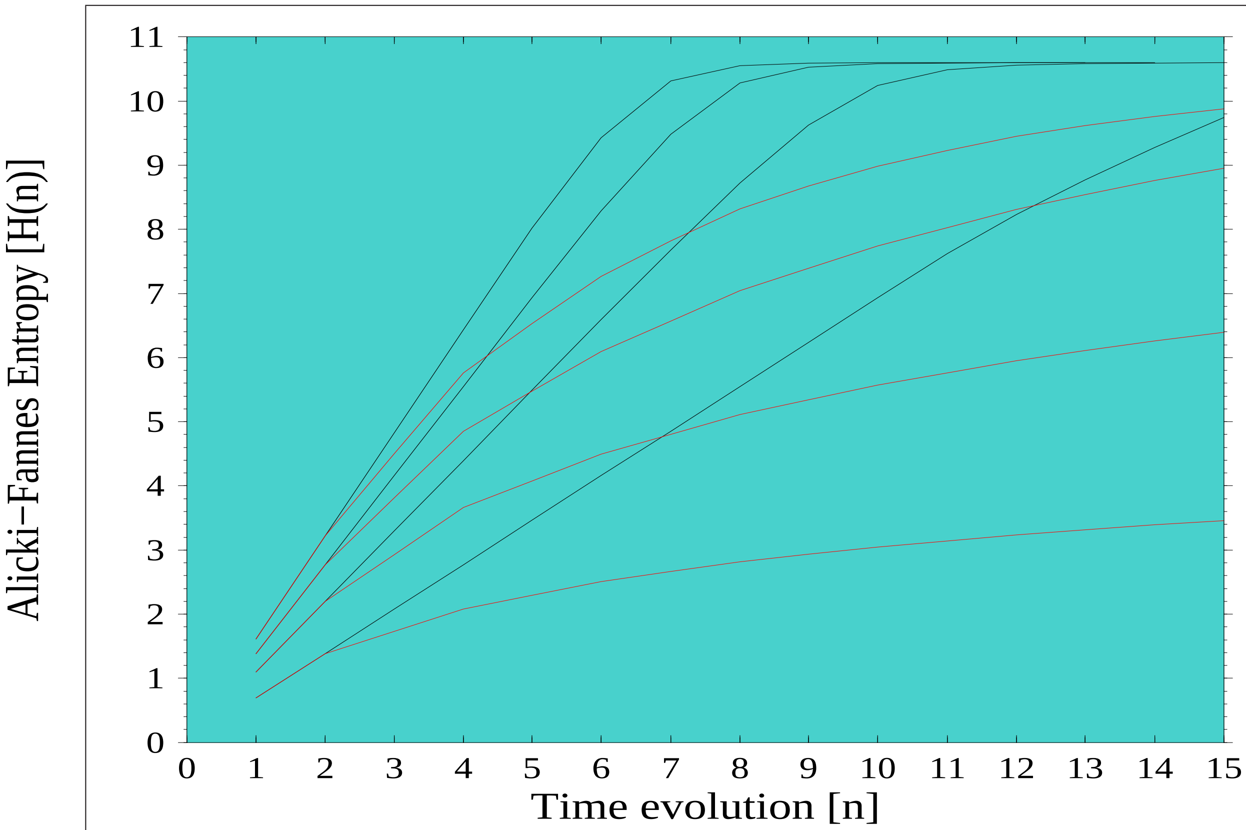


Figure 5: $H_{\omega,D_{N^2}}$ for four different values of D $(2,3,4,5)$ on hyperbolic systems ($\alpha = 1$, black lines) and elliptic systems ($\alpha = -2$, red lines); $N = 200$

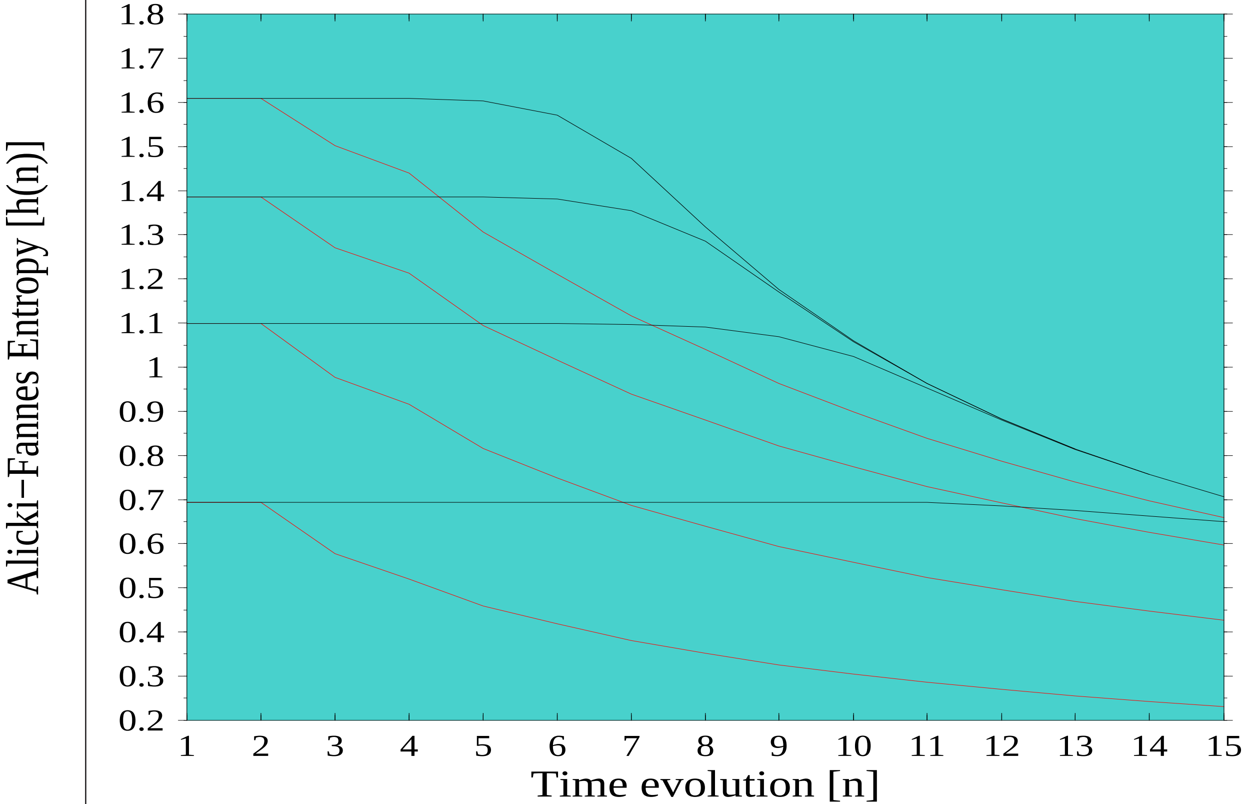


Figure 6: $h^{ALF}_{\omega,D_{N^2}}(\Theta_\alpha)$ for four different values of D $(2,3,4,5)$ on hyperbolic systems ($\alpha = 1$, black lines) and elliptic systems ($\alpha = -2$, red lines); $N = 200$