OUTLINE(1)

- CHAOS and its characterization (Lyapounov Exponents)
- Classical Dynamical Systems (CDS)
- An equivalent indicator of chaos for CDS:
  - the Kolmogorov–Sinai (metric) Entropy [Pesin Thm.]

- Algebraic description of CDS
- Systems with finite number of states:
  - Quantum Dynamical Systems (QDS)
  - Discrete Systems (DDS)

ABSENCE OF CHAOS FOR THESE SYSTEMS

CHAOS over times windows (entropy production analysis)

- Quantization/Discretization (Weyl and Anti–Wick)
- Classical/Continuous Limit (of phase–space Quant./Discret.)
- Coherent States/Lattice States on the torus

OUTLINE(2)

- Quantum Dynamical Entropies (QDE)
  - ALF Entropy
  - CS Entropy
- Relation between the QDE and the KS entropy
- QDE show absence of chaos for systems with finite number of states

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- Classical (Continuous) limit and Temporal Evolution

““““VIOLATION OF THE CORRESPONDENCE PRINCIPLE””””

BREAKING TIME

- Analytical and Numerical estimation of the Breaking Time by means of QDE production analysis
- CONCLUSION

The problem of defining quantum chaos

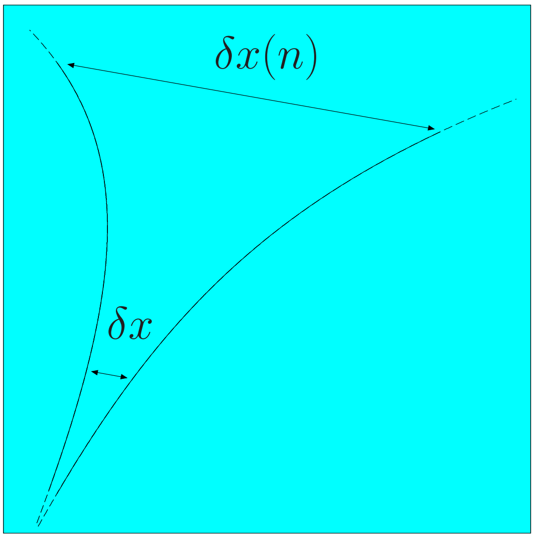
Classical Chaos: exponential amplification of small errors

$$|\delta x| \underset{n}{\longrightarrow} |\delta x(n)| = \lambda^n \, |\delta x| = e^{n \log \lambda} |\delta x|$$

$\log \lambda$ is $\boxed{\text{Lyapounov Exponent}}$

$|\delta x(n)|$ increase no longer in compact systems...

$$\log \lambda := \lim_{n \to +\infty} \lim_{\delta x \to 0} \frac{1}{n} \log \frac{|\delta x(n)|}{|\delta x|}$$



...and $|\delta x|$ cannot be let go to zero in quantum or discrete systems!
Indeed:

- for a discrete system $|\delta x| > a$ ($a = \dfrac{1}{N}$ = lattice spacing);
- for a quantum system $|\delta x| > \dfrac{\hbar}{|\delta p|}$ ($|\delta p|$ bounded on compact).

$$\forall \hbar : \log \lambda(\hbar) = 0 \quad \text{but} \quad \lim_{n\to\infty} \lim_{\hbar\to 0} \log \lambda(\hbar) > 0$$

$$\forall N : \log \lambda(N) = 0 \quad \text{but} \quad \lim_{n\to\infty} \lim_{N\to\infty} \log \lambda(N) > 0$$

An equivalent indicator of chaos in classical continuous systems

- Entropy $[S(n)]$: information on the evolving system up to time n.
- Loosely speaking, the Kolmogorov-Sinai metric entropy is $h_\mu(T) := \lim_{n\to\infty} \frac{S(n)}{n}$ (entropy per unit time).

Theorem 1 (Pesin)

$$\boxed{\text{Ergodicity} \Longrightarrow h_\mu(T) = \sum \text{positive Lyapounov exponent}}$$

CLASSICAL SYSTEMS : $\{\lambda_j^+\}$ $\xrightarrow[\text{Theorem}]{\text{Pesin}}$ $h_\mu(T)$

Correspondence Principle $\uparrow$ $\hbar \to 0$

QUANTUM SYSTEMS : TEMPORAL CHAOS (?) ?

DISCRETE SYSTEMS : TEMPORAL CHAOS (?) ?

Classical Dynamical Systems $(\mathcal{X}, \mu, T)$

$$\begin{cases} \mathcal{X} : \text{Measurable space } (\text{Torus } \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2) \\ \mu : \text{normalized measure } (\mu(\mathcal{X}) = 1) \ (\text{Lebesgue } \mu(\mathrm{d}\boldsymbol{x}) = \mathrm{d}x_1 \mathrm{d}x_2) \\ T : \text{an invertible measurable map } T : \mathcal{X} \mapsto \mathcal{X} \ (\text{described later}) \\ \quad \mu\text{–invariant } (\mu \circ T = \mu) \end{cases}$$

Partition over $\mathcal{X}$

$$\mathcal{E} := \{E_\ell\}_{\ell=1,2,\cdots,D} \qquad \text{such that} \qquad \begin{cases} E_\ell \subset \mathcal{X} \\ \bigcup_{\ell=1}^D E_\ell = \mathcal{X} \\ E_\ell \cap E_k = \emptyset \quad , \quad \forall\, \ell \neq k \end{cases}$$

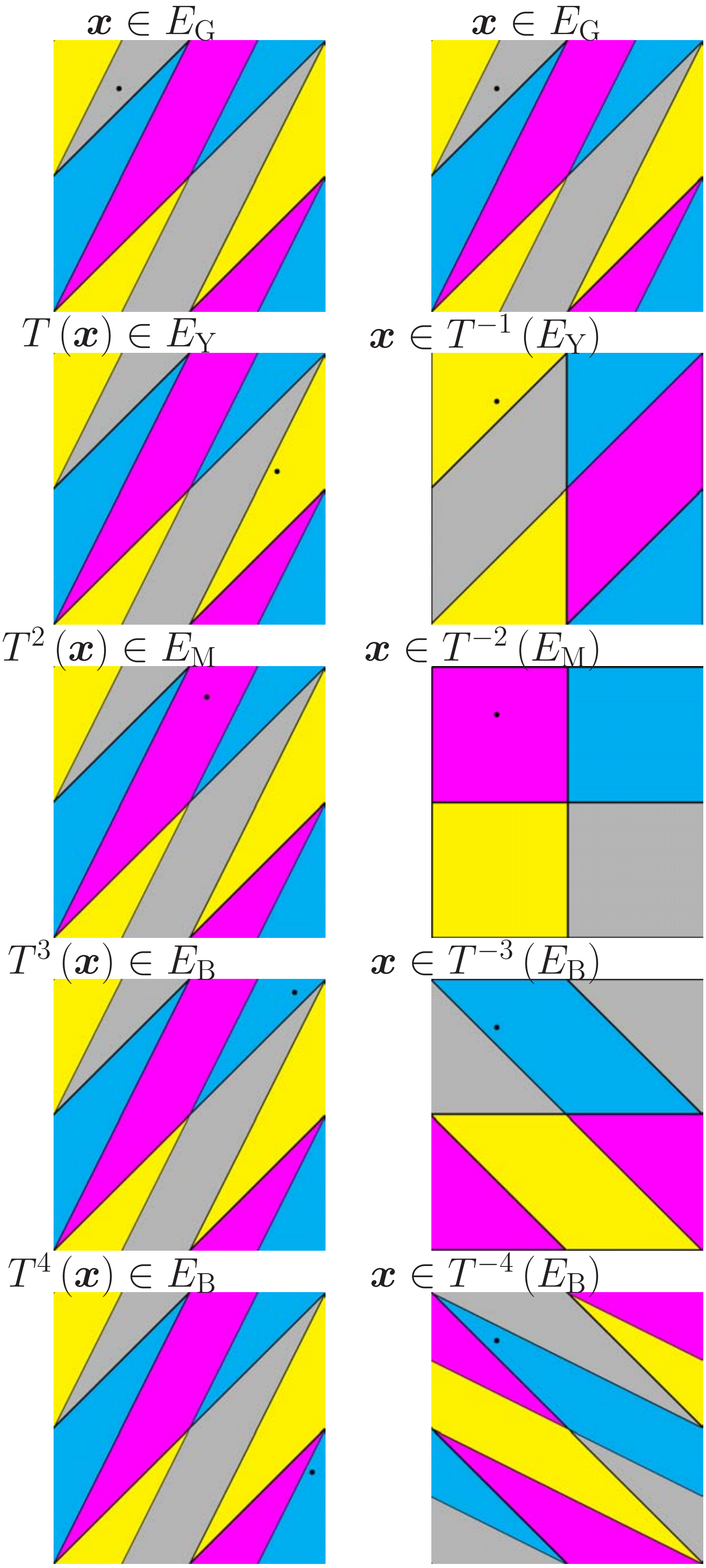
Evolved partition at time j

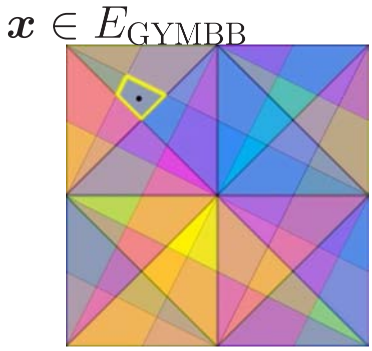
$$T^j(\mathcal{E}) := \left\{T^{-j}(E_\ell)\right\}_{\ell=1,2,\cdots,D}$$

Definition

Ω_D^n is the set of strings $\boldsymbol{i} := \{i_0, i_1, \cdots, i_{n-1}\}$,
i_j belonging to the alphabet $\{1, 2, \cdots, D\}$

$$\mathcal{E} = \{E_{\text{Blue}}, E_{\text{Gray}}, E_{\text{Magenta}}, E_{\text{Yellow}}\}$$





All trajectories starting in $E_{\text{GYMBB}} \coloneqq E_{\text{G}} \cap \ldots \cap T^{-4}(E_{\text{B}})$ are encoded by the same string {GYMBB} (up to time 4)

Their probability is: $\mu_{\text{GYMBB}} \coloneqq \mu(E_{\text{GYMBB}})$

Refined partition (up to time n)

$$\mathcal{E}_{[0,n-1]} := \{E_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\Omega_D^n} \quad , \quad E_{\boldsymbol{i}} := \bigcap_{j=0}^{n-1} T^{-j}\left(E_{i_j}\right)$$

strings $\boldsymbol{i} \in \Omega_D^n$ encode trajectories $\left\{T^k \boldsymbol{x}\right\}$, $\boldsymbol{x} \in E_{\boldsymbol{i}}$

Richness in trajectories of $E_{\boldsymbol{i}} \in \mathcal{E}_{[0,n-1]}$ is measured by the volume $\mu_{\boldsymbol{i}} := \mu\left(E_{\boldsymbol{i}}\right)$.

Kolmogorov metric entropy

With the probabilities $\mu_{\boldsymbol{i}}$ we can compute the SHANNON ENTROPY

$$S_\mu(\mathcal{E}_{[0,n-1]}) := -\sum_{\boldsymbol{i}\in\Omega_D^n} \mu_{\boldsymbol{i}} \log \mu_{\boldsymbol{i}}$$

the entropy production per time step

$$h_\mu(T,\mathcal{E}) := \lim_{n\to\infty} \frac{1}{n} S_\mu(\mathcal{E}_{[0,n-1]})$$

and the KS entropy

$$h_\mu(T) := \sup_{\mathcal{E}} h_\mu(T,\mathcal{E})$$

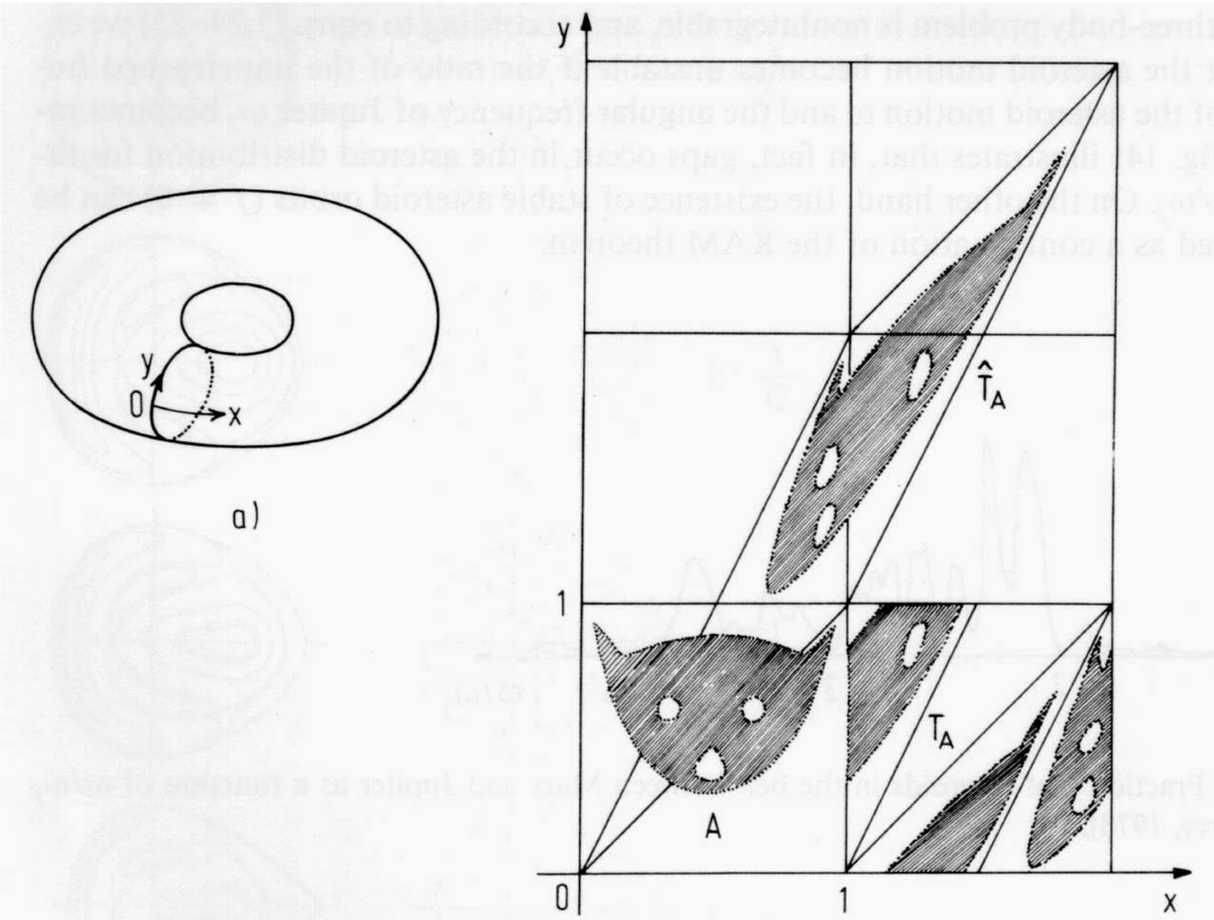
Classical Dynamics [1]: Unimodular Group (UMG)

$$\mathbb{T}^2 \ni \boldsymbol{x}_n \mapsto \boldsymbol{x}_{n+1} := T\,\boldsymbol{x}_n \quad (\mathrm{mod}\ 1) \quad \in \mathbb{T}^2$$

$$T := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad a, b, c, d \in \mathbb{Z} \quad , \quad \det(T) = 1$$

T is a toral automorphism and a generalization of the so called Arnold Cat Map. Depending on $\mathrm{Tr}\,(T)$ we have two kind of dynamics:

- $|\mathrm{Tr}\,(T)| > 2$ Chaotic Systems $\log \lambda = \log \left(\frac{\mathrm{Tr}(T) + \sqrt{\mathrm{Tr}(T)^2 - 4}}{2} \right)$
- $|\mathrm{Tr}\,(T)| \leqslant 2$ Regular Systems



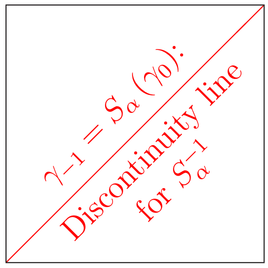
An extension of UMG:

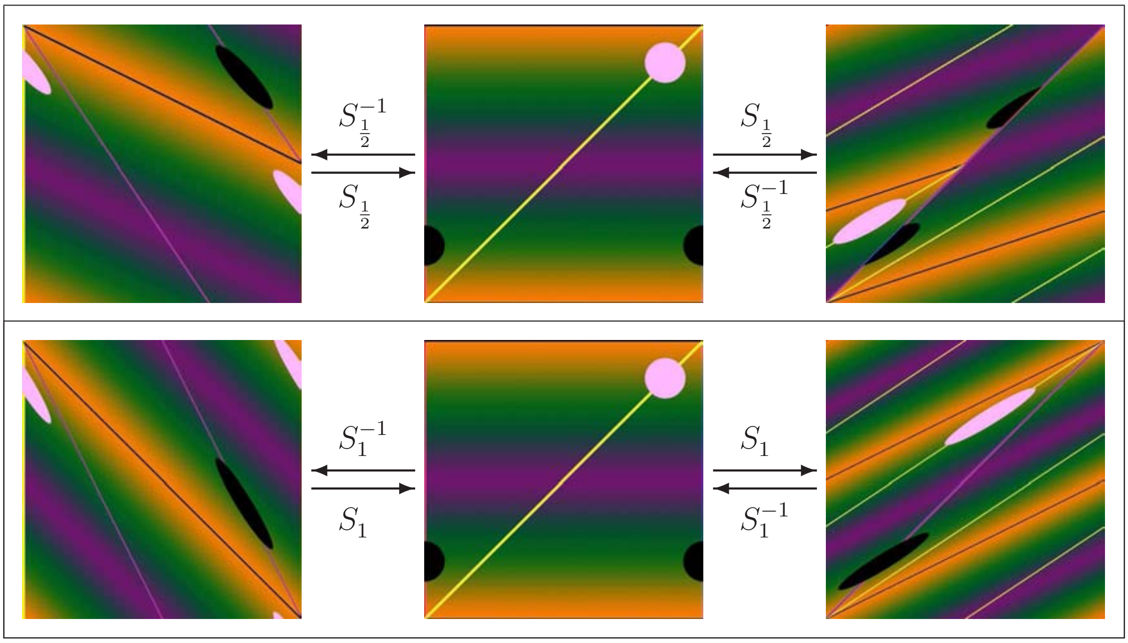
Classical Dynamics [2]: Sawtooth Maps (SM)

$$S_\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1+\alpha & 1 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \langle x_1 \rangle \\ x_2 \end{pmatrix} \pmod 1 , \quad \alpha \in \mathbb{R}$$

Sawtooth Maps are:

- ☺ measure preserving
- ☺ invertible
- ☹ DISCONTINUOUS



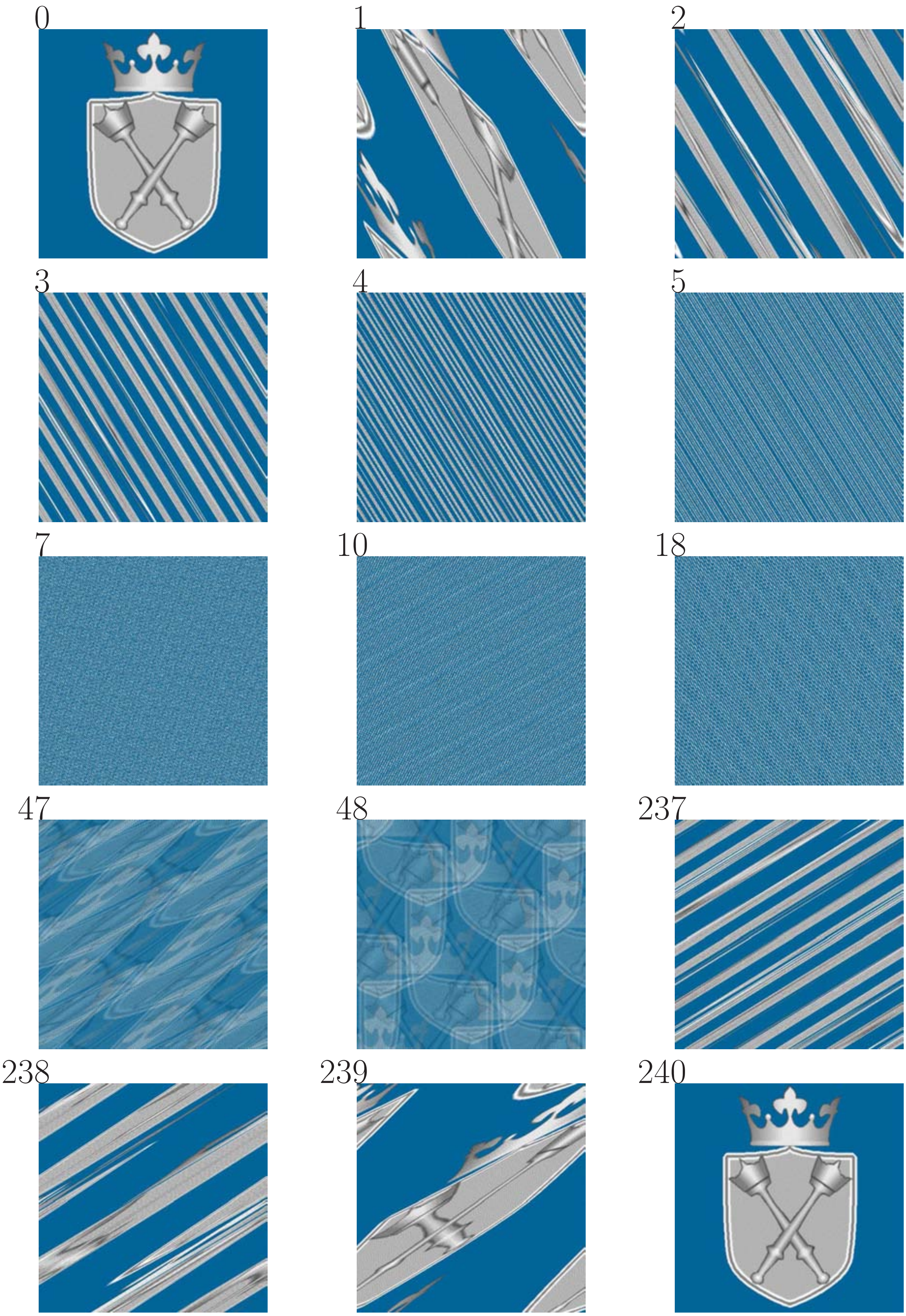


Depending on the value of α we have two kind of dynamics:

- $\alpha \notin [-4, 0]$ Chaotic Systems $\log \lambda = \log \left(\frac{\alpha+2+\sqrt{\alpha(\alpha+4)}}{2} \right)$
- $\alpha \in [-4, 0]$ Regular Systems

By using SM we dispose of a continuum set of Lyapunov Exponent

Chaos in Discrete Systems. . .



. . . but also Periodicity

Algebraic description of Dynamical Systems

1) Classical Dynamical Systems (CDS $(\mathcal{M}_\mu, \omega_\mu, \Theta)$)

$$\begin{cases} \mathcal{M}_\mu : \text{Algebra of observables} \quad \left(\mathcal{C}^0\left(\mathbb{T}^2\right), L^\infty_\mu\left(\mathbb{T}^2\right)\right) \\ \omega_\mu : \text{State over } \mathcal{M}_\mu \quad \left(\omega_\mu\left(f\right) := \int_{\mathbb{T}^2} f\left(\boldsymbol{x}\right) \mathrm{d}^2\boldsymbol{x}\right) \\ \Theta : \text{Discrete group of } \omega_\mu\text{–preserving} \quad \left(\Theta^k\left(f\left(\boldsymbol{x}\right)\right) := f\left(T^k\,\boldsymbol{x}\right)\right) \\ \qquad \text{automorphisms of } \mathcal{M}_\mu \end{cases}$$

2) Quantum Dynamical Systems (QDS $(\mathcal{M}_N, \omega_N, \Theta_N)$)

$$\begin{cases} \mathcal{M}_N : \text{Finite dimensional algebra} \quad (N \times N \text{ Full–matrix algebra}) \\ \omega_N : \text{State over } \mathcal{M}_N \quad (\Theta_N\text{–invariant}) \quad \left(\omega_N\left(m\right) := \frac{1}{N}\,\mathrm{Tr}\left(m\right)\right) \\ \Theta_N : \text{Unitary dynamics on } \mathcal{M}_N \quad \left(\Theta_N\left(m\right) := U\,m\,U^\dagger\right) \end{cases}$$

3) Discrete Dynamical Systems (DDS $(\mathcal{D}_N, \omega_N, \Theta_N)$)

They differ from QDS $(\mathcal{M}_N, \omega_N, \Theta_N)$ only in

- $\mathcal{D}_N$: Finite dimensional algebra ($N^2 \times N^2$ Diagonal–matrix algebra)

QUANTIZATION

To quantize a CDS $(\mathcal{M}_\mu, \omega_\mu, \Theta)$ means to find two linear maps $\mathcal{J}_{N,\infty}$ and $\mathcal{J}_{\infty,N}$ such that:

$$\begin{array}{rcccclcl} \mathcal{J}_{N,\infty} & : & \mathcal{M}_\mu & \longmapsto & \mathcal{M}_N & ; & \mathcal{J}_{N,\infty}\left(f\right) = M_N \\ \mathcal{J}_{\infty,N} & : & \mathcal{M}_N & \longmapsto & \mathcal{M}_\mu & ; & \mathcal{J}_{\infty,N}\left(M_N\right) = f \end{array}$$

Classical limit is: $\mathcal{J}_{\infty,N} \circ \mathcal{J}_{N,\infty} \xrightarrow{N \longrightarrow \infty} \mathbb{1}_{\mathcal{M}_\mu}$

DISCRETIZATION

Analogously, in order to discretize the CDS $(\mathcal{M}_\mu, \omega_\mu, \Theta)$, we must find two linear maps $\mathcal{J}_{N,\infty}$ and $\mathcal{J}_{\infty,N}$ such that:

$$\begin{array}{lllllr} \mathcal{J}_{N,\infty} & : & \mathcal{M}_\mu & \longmapsto & \mathcal{D}_N & ; & \mathcal{J}_{N,\infty}(f) = D_N \\ \mathcal{J}_{\infty,N} & : & \mathcal{D}_N & \longmapsto & \mathcal{M}_\mu & ; & \mathcal{J}_{\infty,N}(D_N) = f \end{array}$$

satisfying the continuous limit: $\mathcal{J}_{\infty,N} \circ \mathcal{J}_{N,\infty} \xrightarrow{N \longrightarrow \infty} \mathbb{1}_{\mathcal{M}_\mu}$

CLASSICAL (CONTINUOUS) LIMIT FOR THE DYNAMICS

Quantum (Discrete) evolution $\xrightarrow[\text{LIMIT}]{\text{CLASSICAL (CONTINUOUS)}}$ Classical (Continuous) evolution

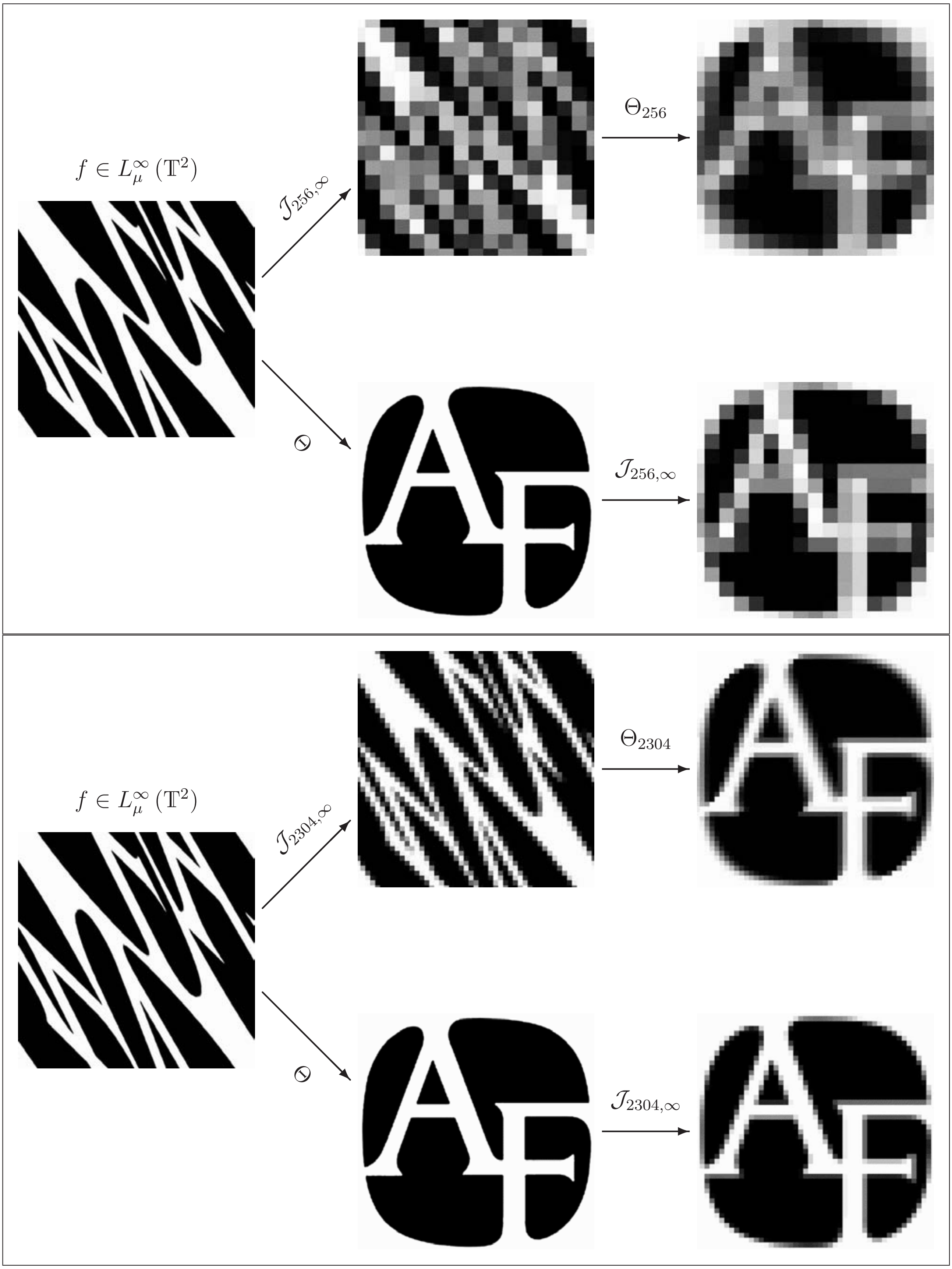
In general it holds true ONLY IF time–evolution does not cross a BREAKING TIME, depending on N!

In particular, depending on the Dynamical System considered, it exists an α such that for any given $f \in L^\infty_\mu(\mathbb{T}^2)$ it holds true

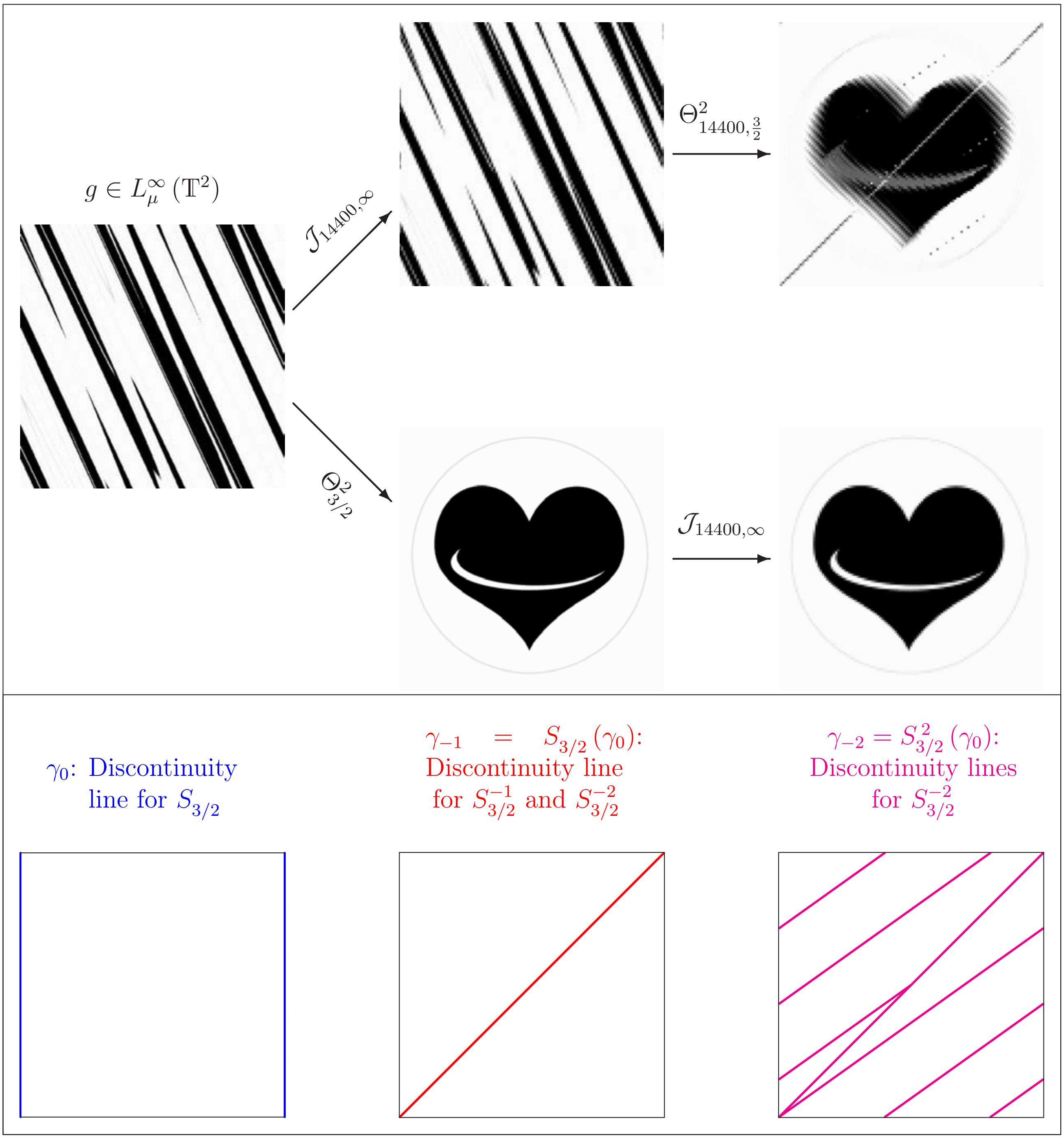
$$\lim_{\substack{k,\ N \to \infty \\ k < \alpha \log N}} \left\| \left(\Theta^k - \mathcal{J}_{\infty,N} \circ \Theta^k_N \circ \mathcal{J}_{N,\infty} \right)(f) \right\|_2 = 0$$

where $\|\cdot\|_2$ is the $L^2_\mu(\mathbb{T}^2)$ norm $\|g\|_2 := \sqrt{\int_{\mathbb{T}^2} |g|^2 \mu(\mathrm{d}\boldsymbol{x})}$

Continuous limit for the UMG: a 16 × 16 and a 48 × 48 lattices



Continuous limit for the SM family : a 48 × 48 lattice



ANTI–WICK QUANTIZATION (DISCRETIZATION)

Using a "well defined" set of Coherent States $\{|C_N(\boldsymbol{x})\rangle \mid \boldsymbol{x} \in \mathcal{X}\}$:

$$\mathcal{J}^{\mathsf{AW}}_{N\infty}(f) := N^2 \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, f(\boldsymbol{x})\, |C_N(\boldsymbol{x})\rangle\langle C_N(\boldsymbol{x})|$$

$$\mathcal{J}^{\mathsf{AW}}_{\infty N}(X)(\boldsymbol{x}) := \langle C_N(\boldsymbol{x}), X\, C_N(\boldsymbol{x})\rangle$$

Properties of $\{|C_N(\boldsymbol{x})\rangle \mid \boldsymbol{x} \in \mathcal{X}\}$

1. Measurability: $\boldsymbol{x} \mapsto |C_N(\boldsymbol{x})\rangle$ is measurable on $\mathcal{X}$;

2. Normalization: $\|C_N(\boldsymbol{x})\|^2 = 1$, $\boldsymbol{x} \in \mathcal{X}$;

3. Overcompleteness: $N^2 \int_{\mathcal{X}} \mu(\mathrm{d}\boldsymbol{x})\, |C_N(\boldsymbol{x})\rangle\langle C_N(\boldsymbol{x})| = \mathbb{1}$;

4′. Localization: given $\varepsilon > 0$ and $d_0 > 0$, there exists $N_0(\epsilon, d_0)$ such that for $N \geq N_0$ and $d(\boldsymbol{x}, \boldsymbol{y}) \geq d_0$ one has

$$N^2 |\langle C_N(\boldsymbol{x}), C_N(\boldsymbol{y})\rangle|^2 \leq \varepsilon.$$

4″. Dynamical localization:
There exists an $\alpha > 0$ such that for all choices of $\varepsilon > 0$ and $d_0 > 0$ there exists an $N_0 \in \mathbb{N}$ with the following property: if $N > N_0$ and $k \leq \alpha \log N$, then $N |\langle C_N(\boldsymbol{x}), U_N^k\, C_N(\boldsymbol{y})\rangle|^2 \leq \varepsilon$ whenever $d(T^k \boldsymbol{x}, \boldsymbol{y}) \geq d_0$.

(U_N^k is the single step unitary evolution operator).

Our “Lattice States” family (LS) $\{|C_N(\boldsymbol{x})\rangle \mid \boldsymbol{x} \in \mathcal{X}\}$

$$\mathbb{T}^2 \ni \boldsymbol{x} \longmapsto \Big|C_N(\boldsymbol{x})\Big\rangle = \Big|\ \lfloor N x_1 + \tfrac{1}{2} \rfloor\, , \lfloor N x_2 + \tfrac{1}{2} \rfloor\ \Big\rangle \in \mathcal{H}_N = \mathbb{C}^{N^2}$$

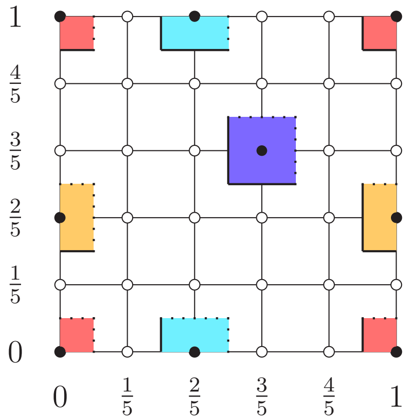


Figure 1: The above picture represents a square lattice (L_5) of spacing $\frac{1}{5}$ by circles and connecting lines. All points in the blue square $I_{\left(\frac{3}{5},\frac{3}{5}\right)} := \left[\frac{5}{10}, \frac{7}{10}\right) \times \left[\frac{5}{10}, \frac{7}{10}\right) \subset \mathbb{T}^2$ are associated with the grid point $\left(\frac{3}{5}, \frac{3}{5}\right)$ (black dot). Thus, for all $\boldsymbol{x} \in I_{\left(\frac{3}{5},\frac{3}{5}\right)}$, it turns out that $|C_{\mathcal{N}}(\boldsymbol{x})\rangle = |\,(3,3)\rangle \in \mathcal{H}_{\mathcal{N}}$.

RESULTS

PROPERTIES: 1 2 3 4′ $\Longrightarrow$ Classical limit

1 2 3 4′ 4″ $\Longrightarrow$ Classical limit of the dynamic

WEYL GROUP

- On compact phase space, we cannot make a finite dimensional quantization with CCR
$$\left[\hat{Q},\hat{P}\right] = i\,\hbar\,\mathbb{1}\ ;$$
- We can find U_N and V_N that behave as $e^{2\pi i\,\hat{P}}$, respectively $e^{-2\pi i\,\hat{Q}}$

$$\boxed{N = \frac{1}{\hbar}}$$

WEYL OPERATORS

$W_N(\boldsymbol{n}) = e^{2\pi i\left(n_1\hat{P}-n_2\hat{Q}\right)} = e^{\frac{i\pi}{N}n_1 n_2}\,V_N^{n_2}U_N^{n_1}$ provide the so–called

WEYL QUANTIZATION:

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \hat{f}_{\boldsymbol{n}}\ e^{2\pi i\,\sigma(\boldsymbol{n},\boldsymbol{x})} \qquad \boxed{\sigma(\boldsymbol{n},\boldsymbol{x}) = n_1x_2 - n_2x_1}$$

can be mapped in $\quad M_f = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \hat{f}_{\boldsymbol{n}}\,W_N(\boldsymbol{n})$

by means of the Weyl quantization operator

$$\mathcal{J}^{\mathsf{W}}_{N,\infty}\ :\ \mathcal{M}_\mu \longmapsto \mathcal{M}_N \qquad ; \qquad \mathcal{J}^{\mathsf{W}}_{N,\infty}(f) = M_f$$

DYNAMICAL EVOLUTION OF THE WEYL OPERATORS:

$$\Theta_N^j\left(W_N(\boldsymbol{n})\right) = W_N\left(T^j\cdot\boldsymbol{n}\right)\ \cdot$$

Such a relation guarantees:

$$\left(\Theta_N^j\circ\mathcal{J}^{\mathsf{W}}_{N,\infty}\right)(f) = \left(\mathcal{J}^{\mathsf{W}}_{N,\infty}\circ\Theta_N^j\right)(f)\ \cdot$$

WEYL DISCRETIZATION

WEYL OPERATORS

$$\mathcal{D}_N \ni W_N(\boldsymbol{n}) := \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} e^{\frac{2\pi i}{N}\,\boldsymbol{n}\boldsymbol{\ell}} \,|\boldsymbol{\ell}\rangle\langle\boldsymbol{\ell}| \ , \qquad \boldsymbol{\ell} = (\ell_1, \ell_2)\,\cdot$$

provide a WEYL DISCRETIZATION:

$$\mathcal{M}_\mu \ni f(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \hat{f}_{\boldsymbol{n}}\, e^{2\pi i\,\boldsymbol{n}\boldsymbol{x}}$$

can be mapped in $\quad \mathcal{D}_N \ni D_f = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \hat{f}_{\boldsymbol{n}}\, W_N(\boldsymbol{n})$

by means of the Weyl Discretization Operator

$$\mathcal{J}^{\mathsf{W}}_{N,\infty} \ : \ \mathcal{M}_\mu \longmapsto \mathcal{D}_N \qquad ; \qquad \mathcal{J}^{\mathsf{W}}_{N,\infty}(f) = D_f$$

Moreover, for $f \in \mathcal{C}^0\left(\mathbb{T}^2\right)$, we have

$$f \longmapsto \mathcal{J}^{\mathsf{W}}_{N,\infty}(f) := \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \hat{f}_{\boldsymbol{n}}\, W_N(\boldsymbol{n}) = \sum_{\boldsymbol{\ell}\in(\mathbb{Z}/N\mathbb{Z})^2} f\left(\frac{\boldsymbol{\ell}}{N}\right) |\boldsymbol{\ell}\rangle\langle\boldsymbol{\ell}| \ \cdot$$

DYNAMICAL EVOLUTION OF THE WEYL OPERATORS:

$$\Theta^j_N\left(W_N(\boldsymbol{n})\right) = W_N\left(\left(T^\dagger\right)^j \cdot \boldsymbol{n}\right) \ \cdot$$

Such a relation guarantees:

$$\left(\Theta^j_N \circ \mathcal{J}^{\mathsf{W}}_{N,\infty}\right)(f) = \left(\mathcal{J}^{\mathsf{W}}_{N,\infty} \circ \Theta^j_N\right)(f) \ \cdot$$

The Alicki Lindblad Fannes Dynamical Entropy

Given a Quantum (or Discrete) Dynamical System $(\mathcal{M}_N, \omega_N, \Theta_N)$ (with $\mathcal{M}_N$ denoting both a full–matrix and a diagonal–matrix algebra), we introduce:

- $\mathcal{Y} := \{y_\ell\}_{\ell=1}^{D}$; $\sum_{\ell=1}^{D} y_\ell^\dagger y_\ell = \mathbb{1}_{\mathcal{M}_0}$ – PARTITION OF UNIT

  $y_\ell \in \mathcal{M}_0 \subseteq \mathcal{M}_N$; $\mathcal{M}_0$ (subalgebra) s.t. $\Theta_N(\mathcal{M}_0) = \mathcal{M}_0$

- EXAMPLE: Partition of 2 elements $(D = 2)$

$$\begin{aligned} \mathcal{M}_N &:: \quad N^2 \times N^2 \text{ diagonal–matrix algebra} \\ \omega_N &:: \quad \omega_N(m) := \frac{1}{N^2} \mathrm{Tr}\,(m) \\ \Theta_N &:: \quad \Theta_N(m) := U\, m\, U^\dagger \\ \mathcal{Y} &:: \quad \Big\{ M_0, M_1 \Big\} \quad (\text{fulfilling } M_0^\dagger M_0 + M_1^\dagger M_1 = \mathbb{1}_N) \end{aligned}$$

- the time-evolving partition of unit: $\Theta_N^k(\mathcal{Y}) := \{\Theta_N^k(y_i)\}_{i=1}^{D}$

- EXAMPLE: with the partition $\Big\{ M_0, M_1 \Big\}$

$$\Theta_N^k(\mathcal{Y}) :: \quad \Big\{ \Theta_N^k(M_0), \Theta_N^k(M_1) \Big\} = \Big\{ U^k M_0\, U^{\dagger k}, U^k M_1\, U^{\dagger k} \Big\}$$

- the refined partition:

$$\mathcal{Y}_{\Theta_N}^{[0,n-1]} = \Big\{ \Theta_N^{n-1}\Big(y_{i_{n-1}}\Big)\ \Theta_N^{n-2}\Big(y_{i_{n-2}}\Big) \cdots \Theta_N(y_{i_1})\, y_{i_0} \Big\}_{\boldsymbol{i} \in \Omega_D^n}$$

- EXAMPLE: with the partition $\left\{M_0, M_1\right\}$ and $n = 3$

$$\mathcal{Y}_{\Theta_N}^{[0,2]} :: \quad \left\{M_{(000)}, M_{(001)}, M_{(010)}, M_{(011)}, M_{(100)}, M_{(101)}, M_{(110)}, M_{(111)}\right\}$$

$$\begin{cases} M_{(000)} := \Theta_N^2(M_0)\ \Theta_N(M_0)\ M_0 = U^2 M_0\, U^{\dagger 2}\ U M_0\, U^\dagger\ M_0 = U^2 M_0\, U^\dagger M_0\, U^\dagger M_0 \\ M_{(001)} := \Theta_N^2(M_1)\ \Theta_N(M_0)\ M_0 = U^2 M_1\, U^{\dagger 2}\ U M_0\, U^\dagger\ M_0 = U^2 M_1\, U^\dagger M_0\, U^\dagger M_0 \\ M_{(010)} := \Theta_N^2(M_0)\ \Theta_N(M_1)\ M_0 = U^2 M_0\, U^{\dagger 2}\ U M_1\, U^\dagger\ M_0 = U^2 M_0\, U^\dagger M_1\, U^\dagger M_0 \\ M_{(011)} := \Theta_N^2(M_1)\ \Theta_N(M_1)\ M_0 = U^2 M_1\, U^{\dagger 2}\ U M_1\, U^\dagger\ M_0 = U^2 M_1\, U^\dagger M_1\, U^\dagger M_0 \\ M_{(100)} := \Theta_N^2(M_0)\ \Theta_N(M_0)\ M_1 = U^2 M_0\, U^{\dagger 2}\ U M_0\, U^\dagger\ M_1 = U^2 M_0\, U^\dagger M_0\, U^\dagger M_1 \\ M_{(101)} := \Theta_N^2(M_1)\ \Theta_N(M_0)\ M_1 = U^2 M_1\, U^{\dagger 2}\ U M_0\, U^\dagger\ M_1 = U^2 M_1\, U^\dagger M_0\, U^\dagger M_1 \\ M_{(110)} := \Theta_N^2(M_0)\ \Theta_N(M_1)\ M_1 = U^2 M_0\, U^{\dagger 2}\ U M_1\, U^\dagger\ M_1 = U^2 M_0\, U^\dagger M_1\, U^\dagger M_1 \\ M_{(111)} := \Theta_N^2(M_1)\ \Theta_N(M_1)\ M_1 = U^2 M_1\, U^{\dagger 2}\ U M_1\, U^\dagger\ M_1 = U^2 M_1\, U^\dagger M_1\, U^\dagger M_1 \end{cases}$$

- the $D^n \times D^n$ density matrices $\rho\left[\mathcal{Y}_{\Theta_N}^{[0,n-1]}\right]$ with elements
$\left[\rho\left[\mathcal{Y}_{\Theta_N}^{[0,n-1]}\right]\right]_{\boldsymbol{i},\boldsymbol{j}} := \omega_N\left(\left[\mathcal{Y}_{\Theta_N}^{[0,n-1]}\right]_{\boldsymbol{j}}^\dagger \left[\mathcal{Y}_{\Theta_N}^{[0,n-1]}\right]_{\boldsymbol{i}}\right)$.

- EXAMPLE: with the partition $\mathcal{Y}_{\Theta_N}^{[0,2]}$

$$\left[\rho\left[\mathcal{Y}_{\Theta_N}^{[0,2]}\right]\right]_{\boldsymbol{i},\boldsymbol{j}} := \omega_N\left(M^\dagger_{(j_0,j_1,j_2)} M_{(i_0,i_1,i_2)}\right) \qquad \text{and so}$$

$$\left[\rho\left[\mathcal{Y}_{\Theta_N}^{[0,2]}\right]\right]_{(010),(100)} := \omega_N\left(M^\dagger_{(100)} M_{(010)}\right)$$

$$:= \frac{1}{N}\,\mathrm{Tr}\left(M_1^\dagger U M_0^\dagger U M_0^\dagger M_0\, U^\dagger M_1\, U^\dagger M_0\right)$$

- the Von Neumann Entropy:

$$H_{\omega_N,\mathcal{M}_0}\left[\mathcal{Y}_{\Theta_N}^{[0,n-1]}\right] = -\operatorname{Tr}\left(\rho\left[\mathcal{Y}_{\Theta_N}^{[0,n-1]}\right]\log\rho\left[\mathcal{Y}_{\Theta_N}^{[0,n-1]}\right]\right) \cdot$$

Then, the ALF–entropy of $(\mathcal{M}_N, \omega_N, \Theta_N)$ is given by:

$$h^{\mathsf{ALF}}_{\omega_N,\mathcal{M}_0}(\Theta_N) := \sup_{\mathcal{Y}\subset\mathcal{M}_0} h^{\mathsf{ALF}}_{\omega_N,\mathcal{M}_0}(\Theta_N,\mathcal{Y}) \,,$$

$$\text{where} \quad h^{\mathsf{ALF}}_{\omega_N,\mathcal{M}_0}(\Theta_N,\mathcal{Y}) := \limsup_n \frac{1}{n} H_{\omega_N}\left[\mathcal{Y}^{[0,n-1]}\right] \cdot$$

PROPOSITION 1

Let $\left(\mathcal{A}_{\mathcal{X}}, \omega_\mu, \Theta\right)$ represent a classical dynamical system. Then,

$$h^{\mathsf{ALF}}_{\omega_\mu,\mathcal{A}_{\mathcal{X}}}(\Theta) = h_\mu(T) \cdot$$

PROPOSITION 2

If $(\mathcal{M}_N, \omega_N, \Theta_N)$ be a quantum (discrete) dynamical system with $\mathcal{M}_N$ finite dimensional, then

$$h^{\mathsf{ALF}}_{\omega,\mathcal{M}_N}(\Theta_N) = 0 \cdot$$

h^{ALF}

- behave as $h_\mu(T)$ for CDS

(so) - test CHAOS as $h_\mu(T)$ do

- reveal NO CHAOS on finite dimensional systems

- All these quantities are computed in the $n \longrightarrow \infty$ limit…
- What about the running Von Neumann entropies?

RESULT (For the ALF entropy)

Finding a correspondence (actually the more natural) between

- a classical partition $\mathcal{E}$ KS
- a partition of unit $\mathcal{Y}$ ALF

and using the dynamical localization condition, we get

$$\lim_{\substack{k,N\to\infty \\ k\leq \alpha \log N}} \frac{1}{k} \left| H_{\omega_N}[\mathcal{Y}_{\Theta_N}^{[0,k-1]}] - S_\mu(\mathcal{E}^{[0,k-1]}) \right| = 0 \cdot$$

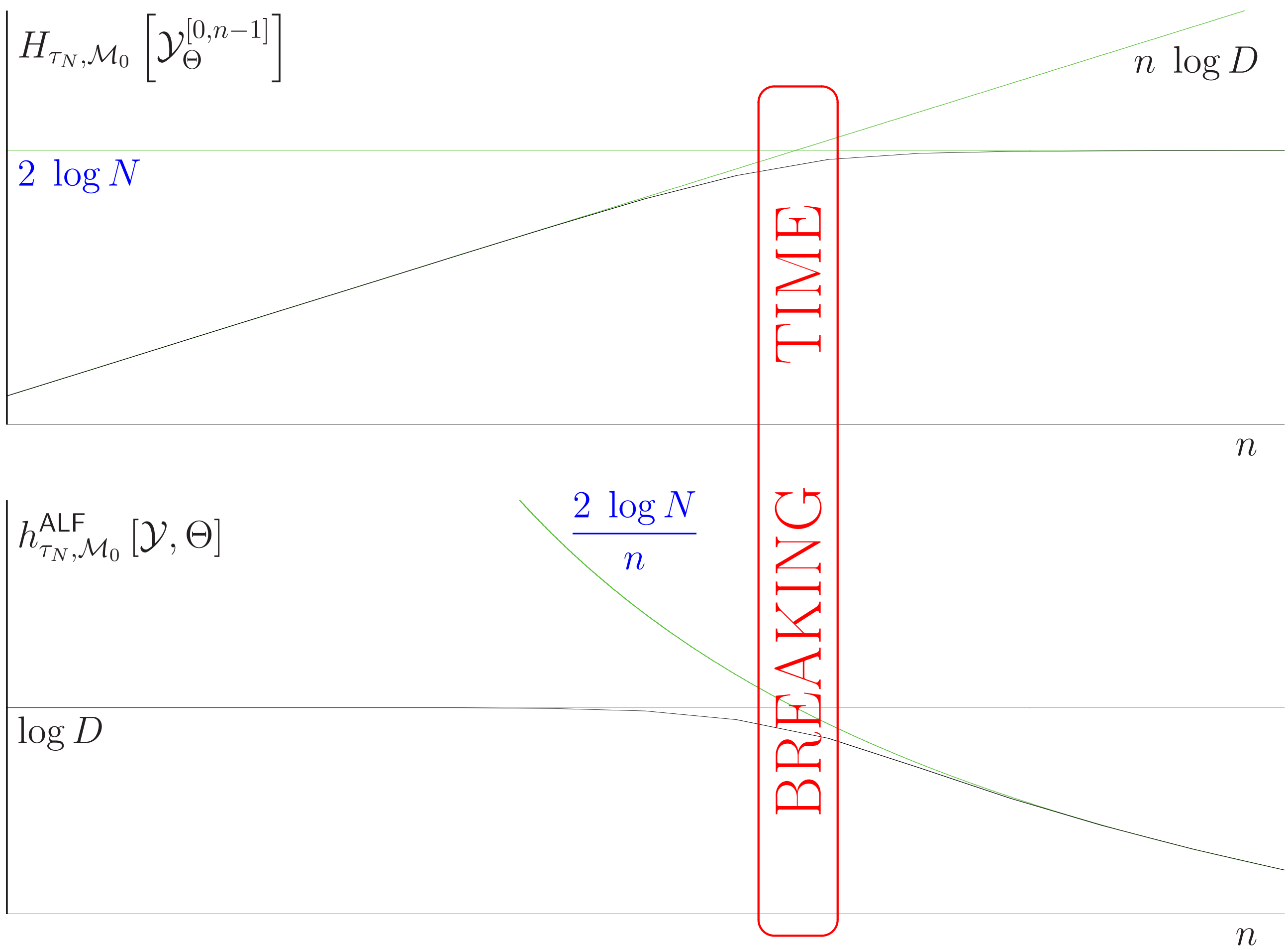
– Remember –

$$\begin{array}{ccccc}
S_\mu(\mathcal{E}^{[0,n-1]}) & \xrightarrow{\frac{1}{n}\lim_{n\longrightarrow\infty}} & h_\mu(T,\mathcal{E}) & \xrightarrow[\mathcal{E}]{\sup} & h_\mu(T) \\
\Big\uparrow \lim\limits_{\substack{k,N\to\infty \\ k\leq\alpha\log N}} \frac{1}{k}\cdots & & & & \\
H_{\omega_N,\mathcal{M}_0}\left[\mathcal{Y}_{\Theta_N}^{[0,n-1]}\right] & \xrightarrow[\frac{1}{n}\limsup_{n\longrightarrow\infty}]{} & h^{\mathsf{ALF}}_{\omega_N,\mathcal{M}_0}(\Theta_N,\mathcal{Y}) & \xrightarrow[\sup_{\mathcal{Y}}]{} & h^{\mathsf{ALF}}_{\omega_N,\mathcal{M}_0}(\Theta_N)
\end{array}$$

$\mathcal{Y}$: partition of D elements that maximize the entropy rate

Time n	1	2	3		m	m'
Size$\left(\rho_\Theta^{[0,n-1]}\right)$	$D \times D$	$D^2 \times D^2$	$D^3 \times D^3$	$\cdots$	$D^m \times D^m$	$D^{m'} \times D^{m'}$
Max. number of eigenvalues of $\rho_\Theta^{[0,n-1]}$ different from 0	D	D^2	D^3	$\cdots$	D^m	N^2
$\left\langle \text{Eigenvalues} \right\rangle$	$\frac{1}{D}$	$\frac{1}{D^2}$	$\frac{1}{D^3}$	$\cdots$	$\frac{1}{D^m}$	$\frac{1}{N^2}$
$H_{\tau_N,\mathcal{M}_0}\left[\mathcal{Y}_\Theta^{[0,n-1]}\right]$	$\log D$	$2\ \log D$	$3\ \log D$	$\cdots$	$m\ \log D$	$2\ \log N$
$h^{\mathsf{ALF}}_{\tau_N,\mathcal{M}_0}[\mathcal{Y},\Theta]$	$\log D$	$\log D$	$\log D$	$\cdots$	$\log D$	$\frac{2\ \log N}{m'}$

BREAKING TIME



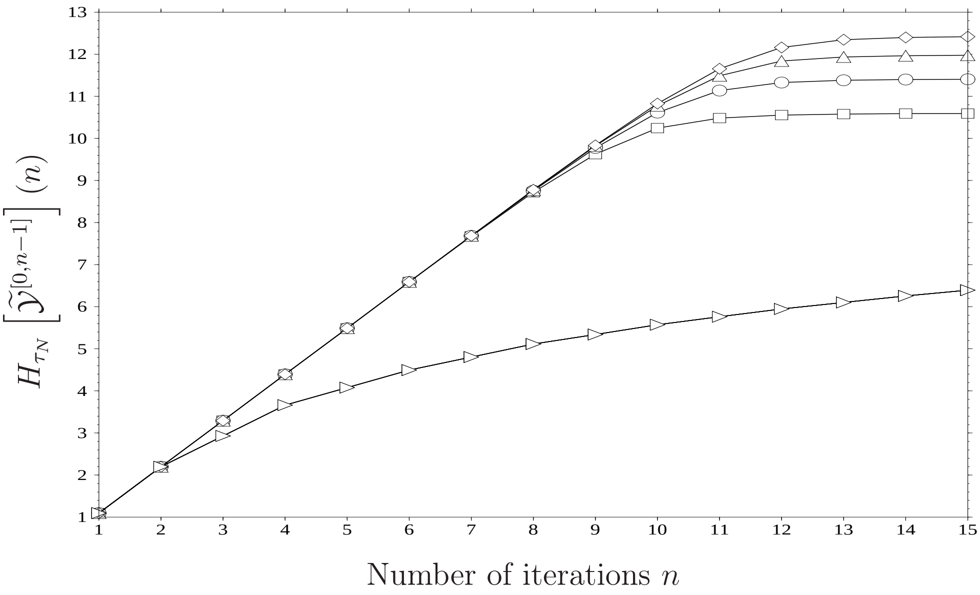


Figure 2: Von Neumann entropy $H_{\tau_N}(n)$ in four hyperbolic ($\alpha = 1$ for $\diamond$, $\triangle$, $\circ$, $\square$) and four elliptic ($\alpha = -2$ for $\triangleright$) cases, for three randomly distributed $\boldsymbol{r}_i$ in Λ. Values for N are: $\diamond = 500$, $\triangle = 400$, $\circ = 300$ and $\square = 200$, whereas the curve labeled by $\triangleright$ represents four elliptic systems with $N \in \{200, 300, 400, 500\}$.

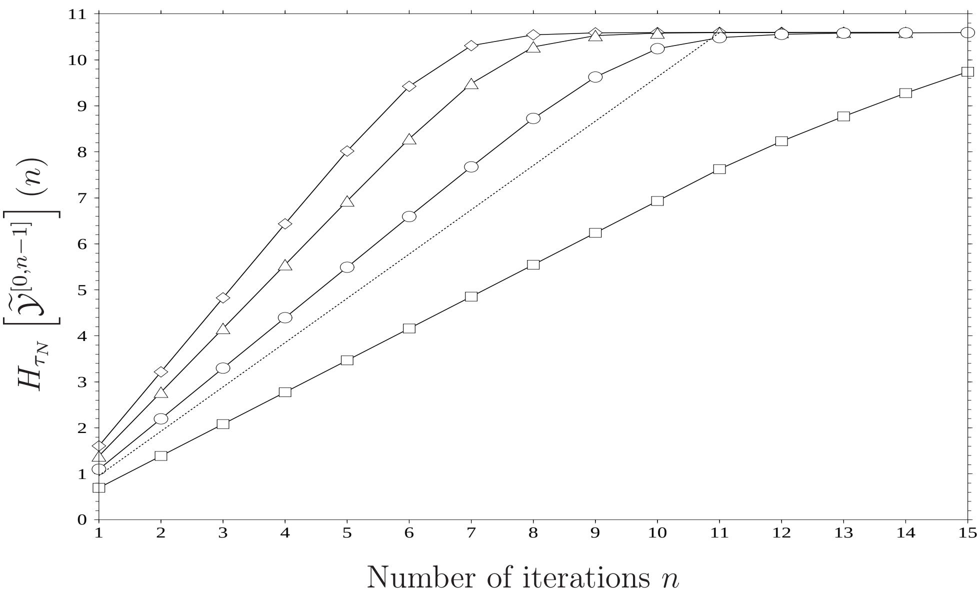


Figure 3: Von Neumann entropy $H_{\tau_N}(n)$ in four hyperbolic ($\alpha = 1$) cases, for D randomly distributed $\boldsymbol{r}_i$ in Λ, with $N = 200$. Value for D are: $\diamond = 5$, $\triangle = 4$, $\circ = 3$ and $\square = 2$. The dotted line represents $H_{\tau_N}(n) = \log\lambda \cdot n$ where $\log\lambda = 0.962\ldots$ is the Lyapounov exponent at$\alpha = 1$.

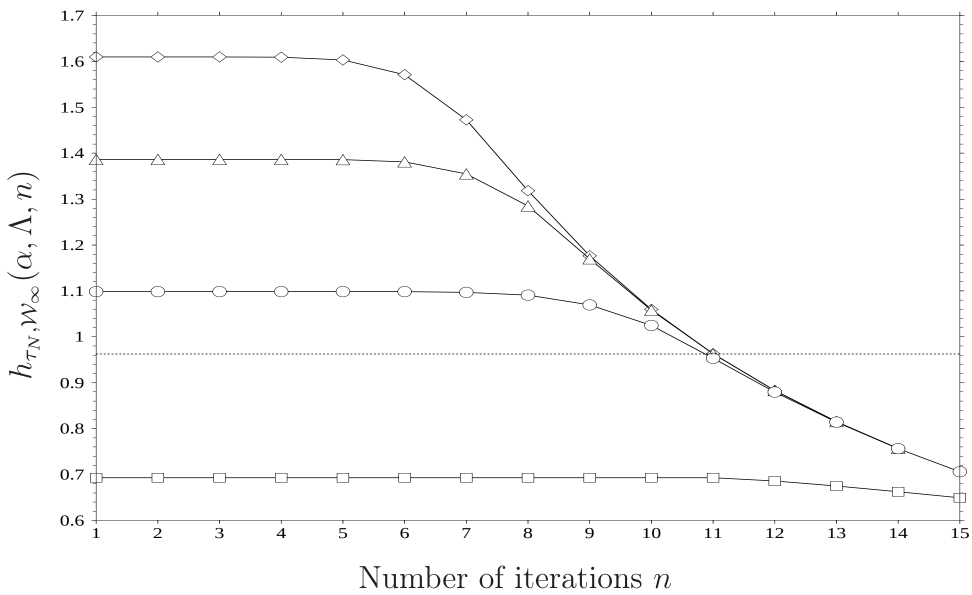


Figure 4: Entropy production $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n)$ in four hyperbolic ($\alpha = 1$) cases, for D randomly distributed $\boldsymbol{r}_i$ in Λ, with $N = 200$. Values for D are: $\diamond = 5$, $\triangle = 4$, $\circ = 3$ and $\square = 2$. The dotted line corresponds to the Lyapounov exponent $\log\lambda = 0.962\ldots$ at $\alpha = 1$.

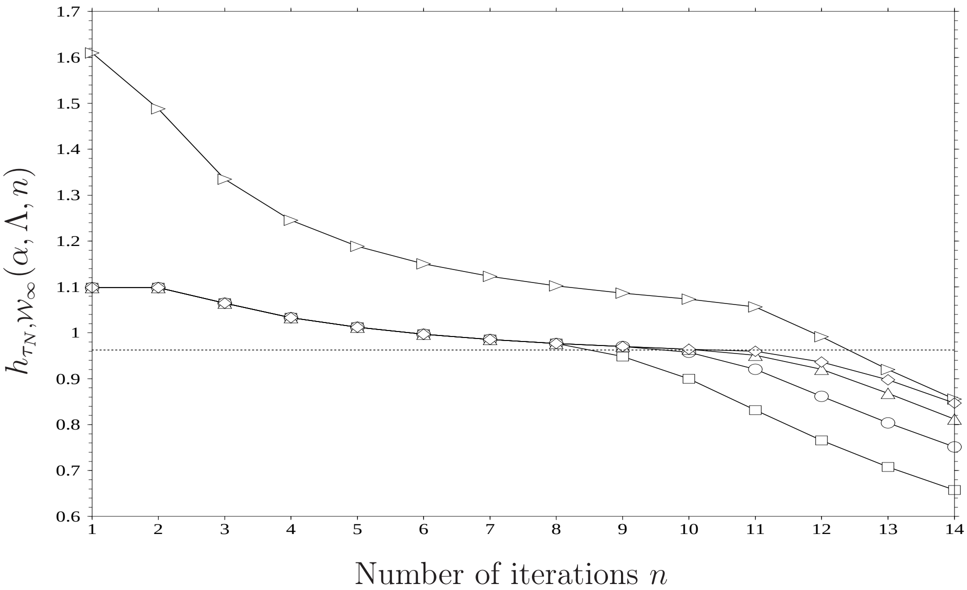


Figure 5: Entropy production $h_{\tau_N,\mathcal{W}_\infty}(\alpha,\Lambda,n)$ in five hyperbolic ($\alpha = 1$) cases, for D nearest neighboring points $\boldsymbol{r}_i$ in Λ. Values for (N, D) are: $\triangleright = (200, 5)$, $\diamond = (500, 3)$, $\triangle = (400, 3)$, $\circ = (300, 3)$ and $\square = (200, 3)$. The dotted line corresponds to the Lyapounov exponent $\log\lambda = 0.962\ldots$ at $\alpha = 1$ and represents the natural asymptote for all these curves in absence of breaking–time.

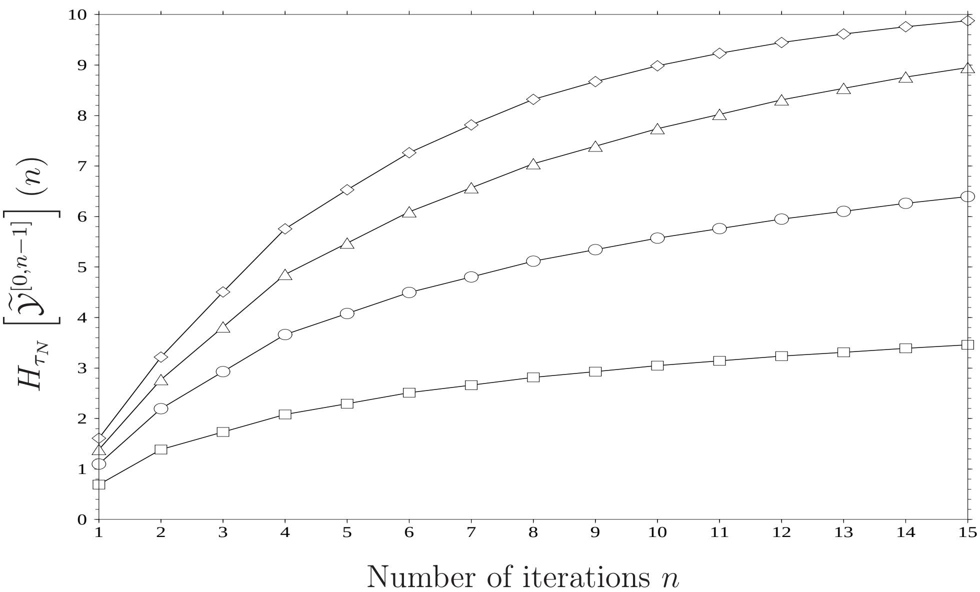


Figure 6: Von Neumann entropy $H_{\tau_N}(n)$ in four elliptic ($\alpha = -2$) cases, for D randomly distributed $\boldsymbol{r}_i$ in Λ, with $N = 200$. Value for D are: $\diamond = 5$, $\triangle = 4$, $\circ = 3$ and $\square = 2$.

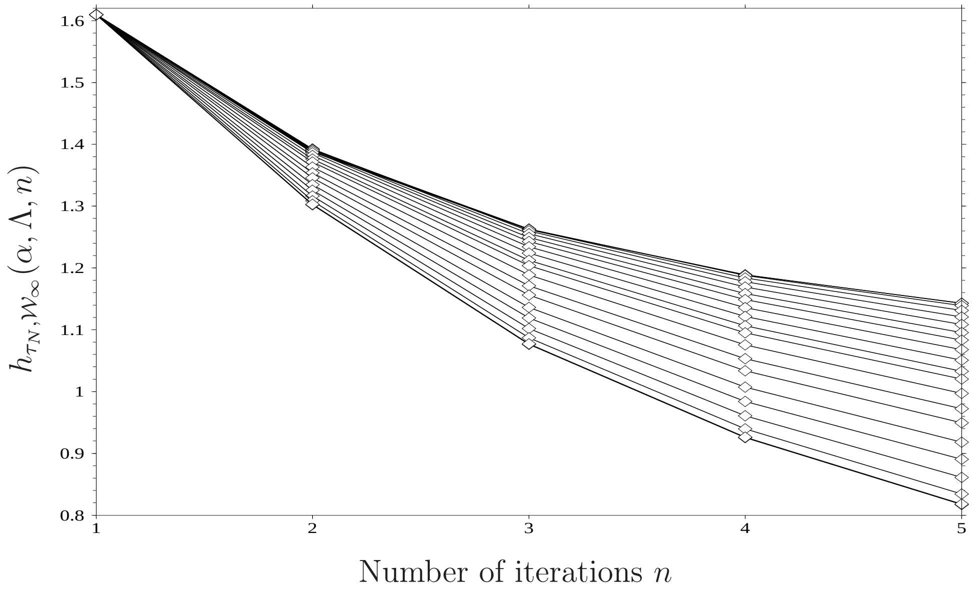


Figure 7: Entropy production $h_{\tau_N,\mathcal{W}_\infty}(\alpha, \Lambda, n)$ for 21 hyperbolic Sawtooth maps, relative to a for a cluster of 5 nearest neighborings points $\boldsymbol{r}_i$ in Λ, with $N = 38$. The parameter α decreases from $\alpha = 1.00$ (corresponding to the upper curve) to $\alpha = 0.00$ (lower curve) through 21 equispaced steps.

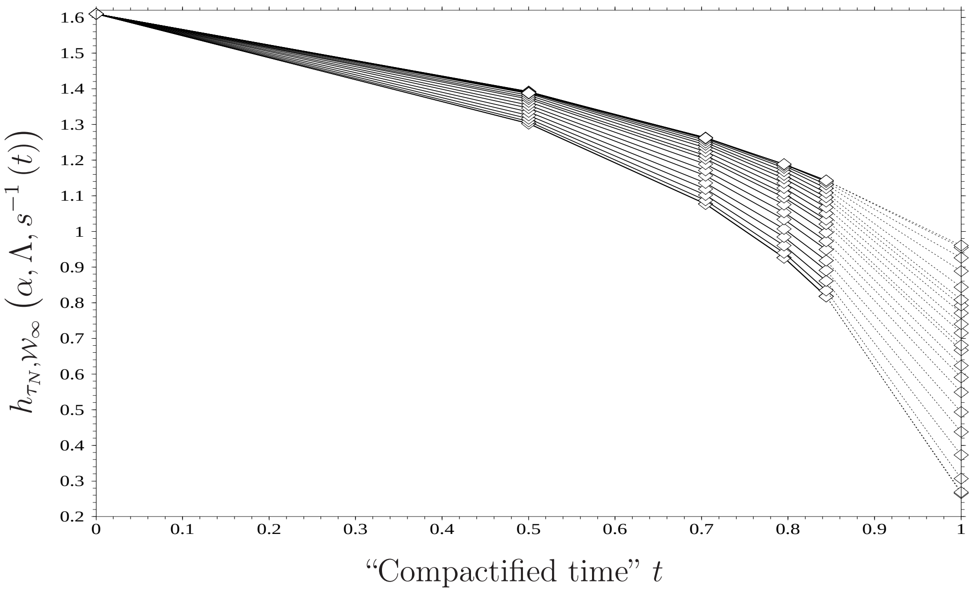


Figure 8: The solid lines correspond to $(s_n\,,\, h_{\tau_N,\mathcal{W}_\infty}(\alpha, \Lambda, n))$, with $n \in \{1, 2, 3, 4, 5\}$, for the values of α considered in figure 7. Every α–curve is continued as a dotted line up to $(1, l^5_\alpha)$, where l^5_α is the Lyapounov exponent extracted from the curve by fitting all the five points via a Lagrange polynomial $\mathcal{P}^m(t)$.

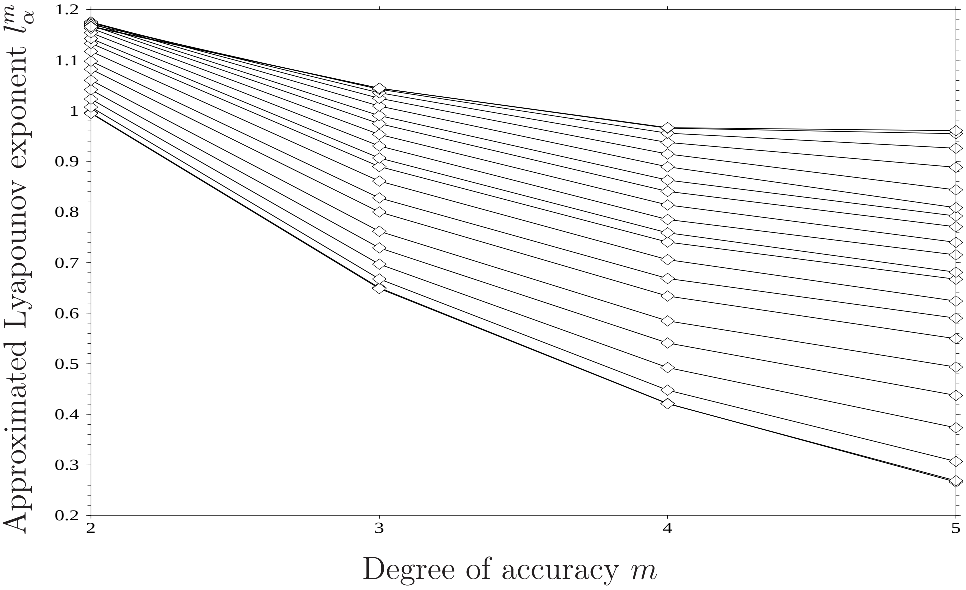


Figure 9: Four estimated Lyapounov exponents l_α^m plotted vs. their degree of accuracy m for the values of α considered in figures 7 and 8.

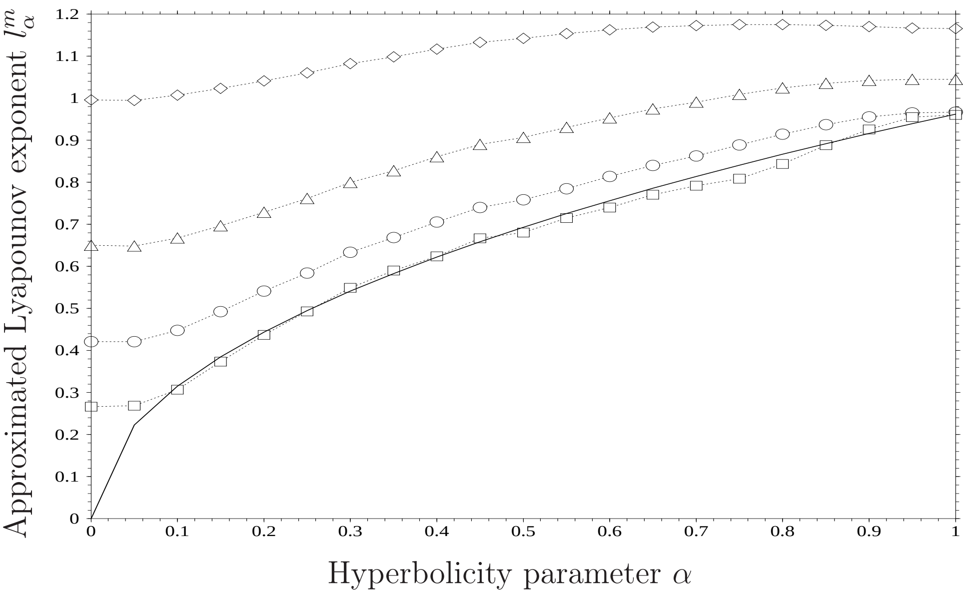


Figure 10: Plots of the four estimated of Lyapounov exponents l_α^m of figure 9 vs. the considered values of α. The polynomial degree m is as follows: $\diamond = 2$, $\triangle = 3$, $\circ = 4$ and $\square = 5$. The solid line corresponds to the theoretical Lyapounov exponent $\log\lambda_\alpha = \log\left(\alpha + 2 + \sqrt{\alpha\,(\alpha+4)}\,\right) - \log 2$.

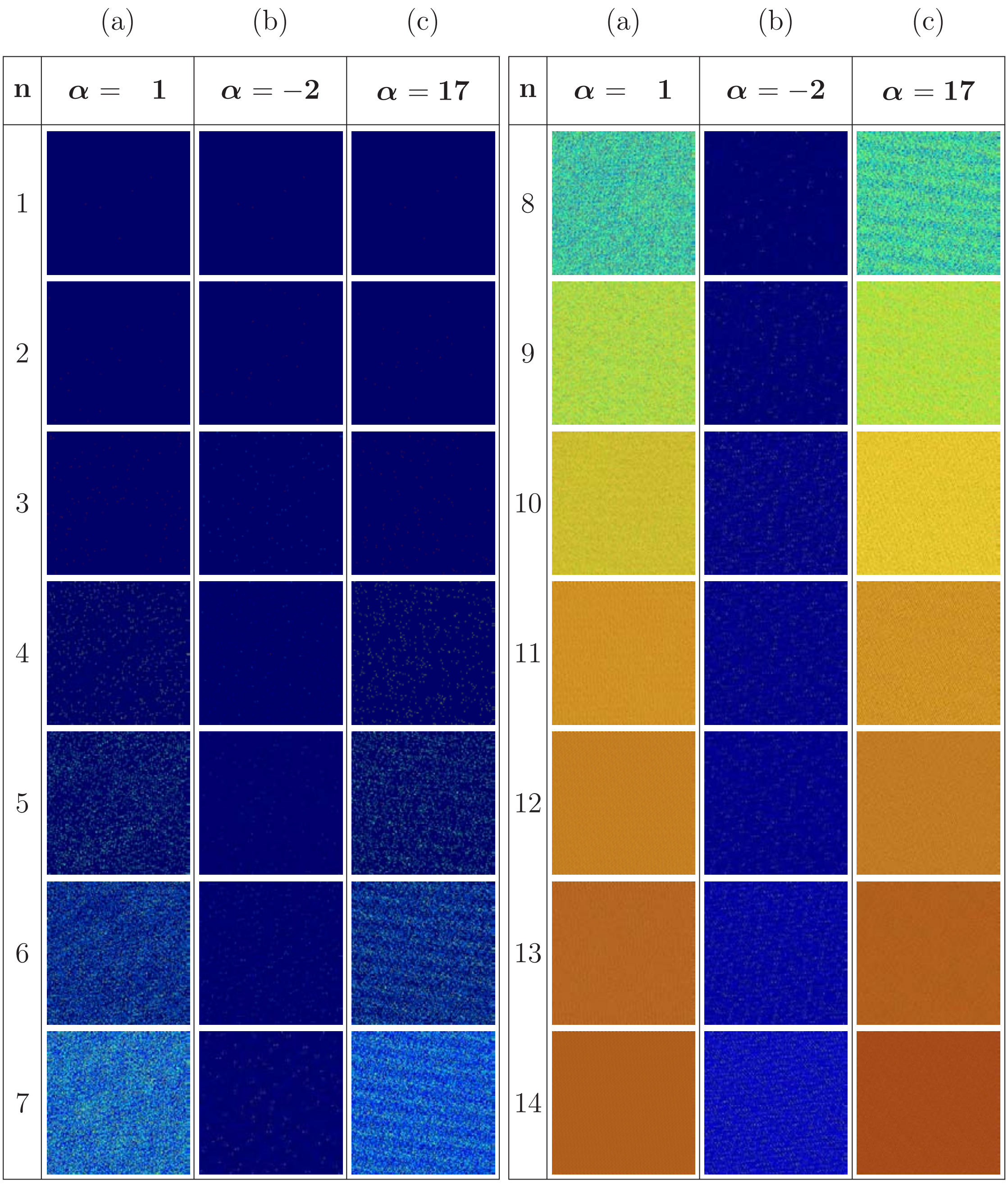


Figure 11: Temperature–like plots showing the frequencies $\nu^{(n),N}_{\Lambda,\alpha}$ in two hyperbolic regimes (columns a and c) and an elliptic one (col. b), for five randomly distributed $\boldsymbol{r}_i$ in Λ with $N = 200$. Pale–blue corresponds to $\nu^{(n),N}_{\Lambda,\alpha} = 0$. In the hyperbolic cases, $\nu^{(n),N}_{\Lambda,\alpha}$ tends to equidistribute on $(\mathbb{Z}/N\mathbb{Z})^2$ with increasing n and becomes constant when the breaking–time is reached.

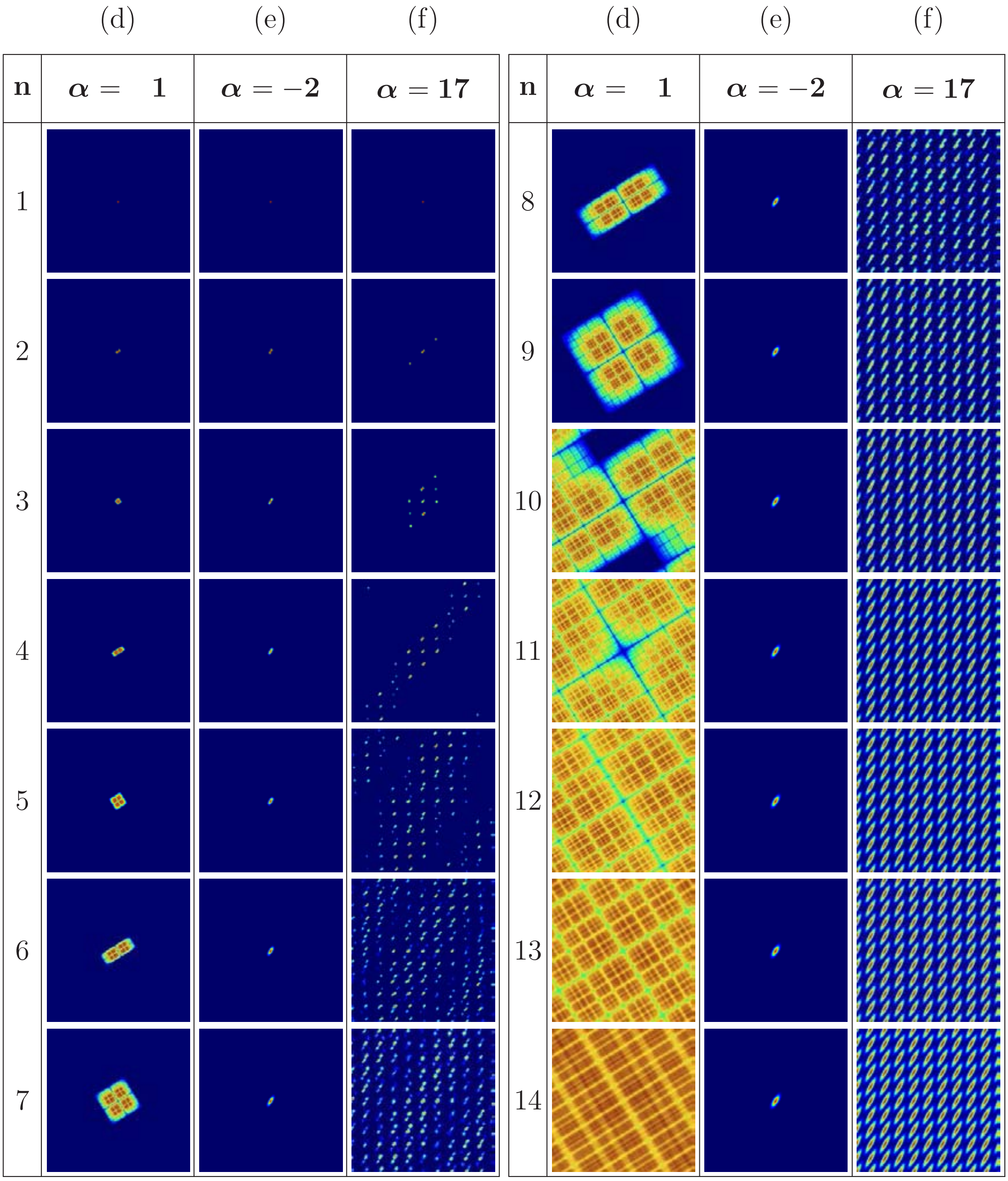
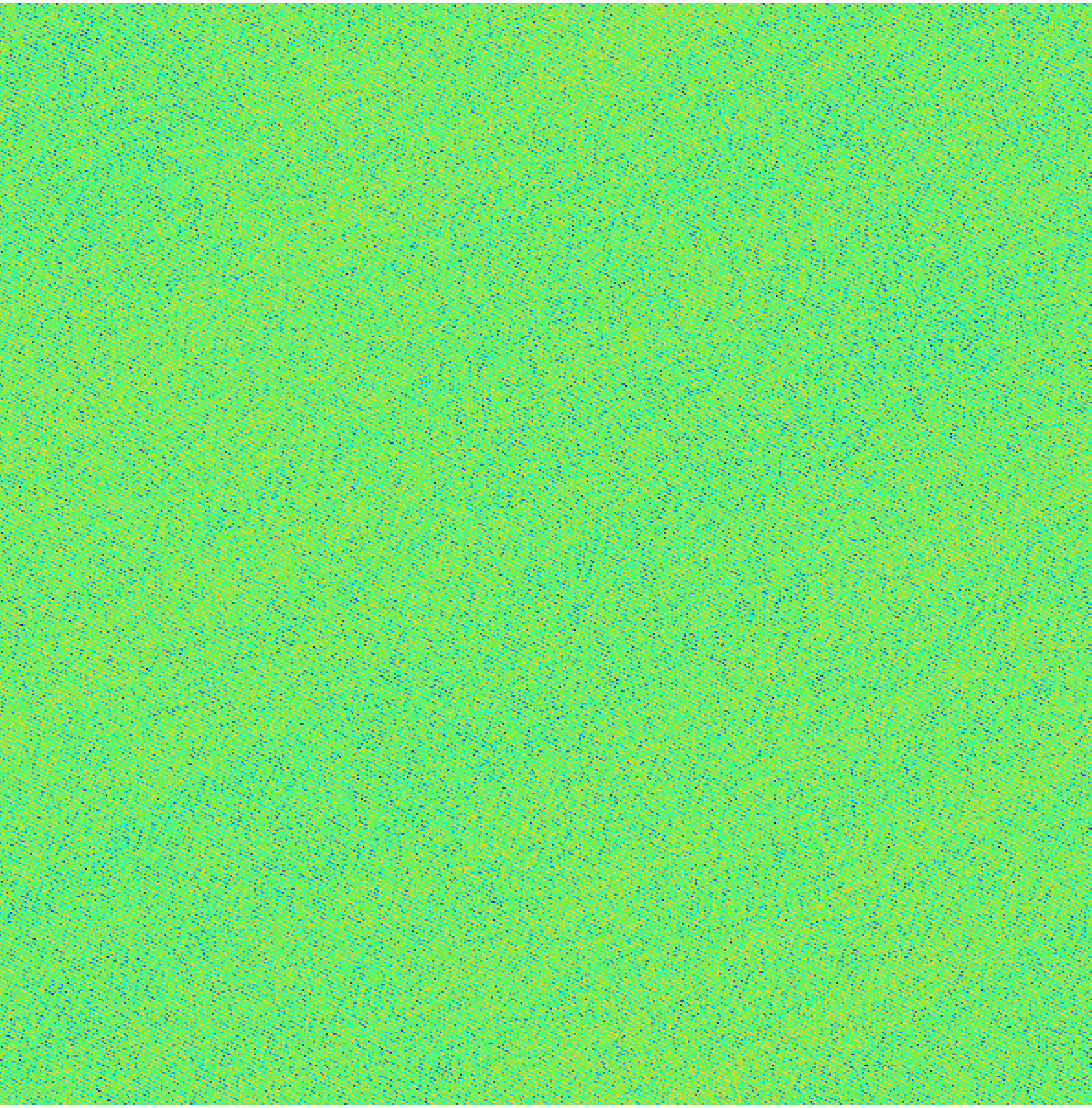
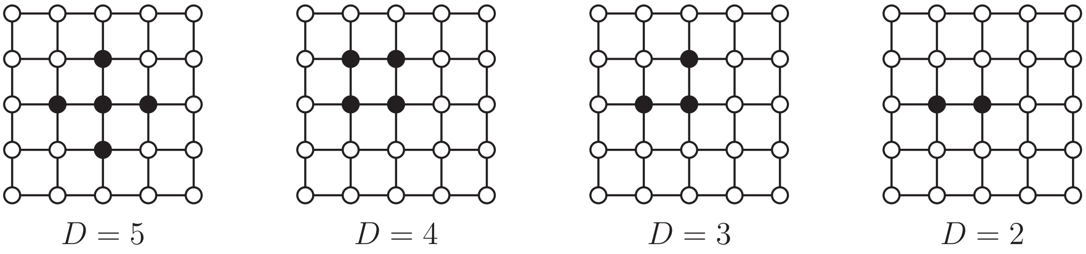
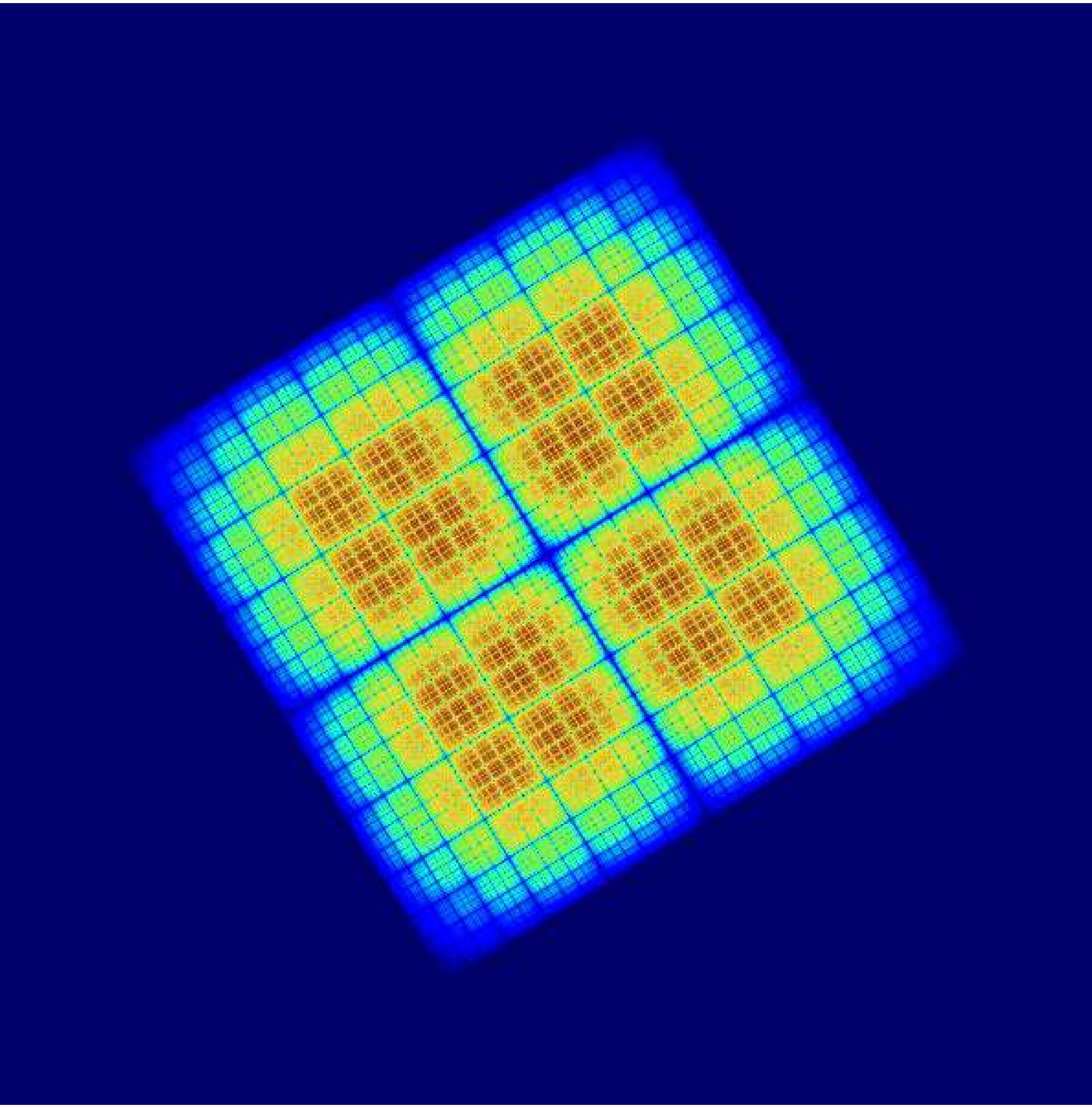
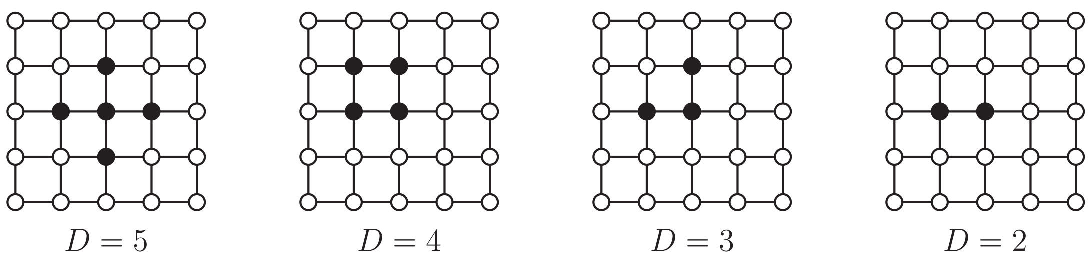


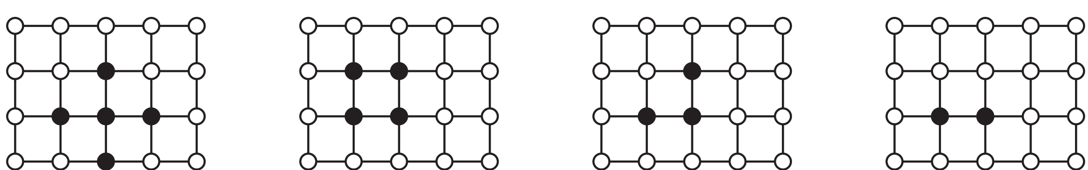
Figure 12: Temperature–like plots showing $\nu_{\Lambda,\alpha}^{(n),N}$ in two hyperbolic (columns d and f) and one elliptic (col. e) regime, for five nearest neighboring $\boldsymbol{r}_i$ in Λ ($N = 200$). Pale–blue corresponds to $\nu_{\Lambda,\alpha}^{(n),N} = 0$. When the system is chaotic, the frequencies tend to equidistribute on $(\mathbb{Z}/N\mathbb{Z})^2$ with increasing n and to approach, when the breaking–time is reached, the constant value $\frac{1}{N^2}$. Col. (f) shows how the dynamics can be confined on a sublattice by a particular combination (α, N, Λ) with a corresponding entropy decrease.




$D = 5$
$D = 4$
$D = 3$
$D = 2$




$D = 5$
$D = 4$
$D = 3$
$D = 2$


$D = 5$
$D = 4$
$D = 3$
$D = 2$

Given a Quantum Dynamical System $(\mathcal{M}_N, \omega_N, \Theta_N)$ we introduce:

$\mathcal{Y} := \{y_\ell\}_{\ell=1}^{D}$; $\sum_{\ell=1}^{D} y_\ell^* y_\ell = \mathbb{1}_{\mathcal{M}_0}$ – PARTITION OF UNIT (PU)

WAVE PACKET REDUCTION POSTULATE $\Longrightarrow$ $\rho \xrightarrow{\text{MEASURE}} \mathcal{I}_{\mathcal{Y}}(\rho) := \sum_j y_j \, \rho \, y_j^*$

- The map $\mathcal{I}_{\mathcal{Y}}$ is called an instrument;
- it describe the change in the state ρ caused by the measure;
- $\omega \left[y_j \, \rho \, y_j^* \right]$ is the probability that the measure select the i^{th} value.

—— The CS Instrument ——

$$\mathcal{E} := \{E_\ell\}_{\ell=1}^{D} \; ; \quad \begin{cases} \bigcup_{\ell=1}^{D} E_\ell = \mathcal{X} \\ E_\ell \cap E_k = \emptyset \end{cases} \quad \text{– CLASSICAL PARTITION (CP)}$$

With the family of CS $\{|C_N(\boldsymbol{x})\rangle \mid \boldsymbol{x} \in \mathcal{X}\}$ and $\boxed{P_{\boldsymbol{x}} := |C_N(\boldsymbol{x})\rangle\langle C_N(\boldsymbol{x})|}$:

- the map $\mathcal{I}(E_\ell)(\rho) := \mathcal{N} \int_{E_\ell} P_{\boldsymbol{x}} \, \rho \, P_{\boldsymbol{x}} \, \mu(\mathrm{d}\boldsymbol{x})$ is called a CS–instrument;
- it describe the change in the state ρ caused by the E_ℓ–dependent measurement process;
- $\omega \left[\mathcal{I}(E_\ell)(\rho) \right]$ is the probability that the measure gives values in E_ℓ, when the pre–measurement state is ρ.

—— Time–stroboscopic CS measurement ——

$$\mathcal{P}_{\boldsymbol{i}}^{\mathsf{CS}} = \mathcal{P}_{i_0,i_1,\ldots,i_{n-1}}^{\mathsf{CS}} := \omega \left[\mathcal{I}\left(E_{i_{n-1}}\right) \circ \Theta \circ \mathcal{I}\left(E_{i_{n-2}}\right) \circ \Theta \circ \cdots \circ \mathcal{I}\left(E_{i_1}\right) \circ \Theta \circ \mathcal{I}\left(E_{i_0}\right)(\rho) \right]$$

is the probability that several measure, taken stroboscopically at times $t_0 = 0$, $t_1 = 1$, … , $t_{n-1} = n-1$, give values in $E_{i_0}, E_{i_1}, \ldots, E_{i_{n-1}}$.

CS Quantum Entropies

With the probabilities $\mu_{\boldsymbol{i}}$ we can compute the SHANNON ENTROPY

$$S(U,\mathcal{I},\mathcal{E},\rho,n) := -\sum_{\boldsymbol{i}\in\Omega_D^n} \mathcal{P}_{\boldsymbol{i}}^{\mathsf{CS}} \log \mathcal{P}_{\boldsymbol{i}}^{\mathsf{CS}} \quad ;$$

its production per time step, is defined as CS quantum entropy

$$H(U,\mathcal{I},\mathcal{E},\rho) := \lim_{n\to\infty} \frac{1}{n}\, S(U,\mathcal{I},\mathcal{E},\rho,n)$$

and it is decomposable in two part: the Measurement CS Quantum Entropy

$$H_{\mathrm{meas}}(\mathcal{I},\mathcal{E},\rho) := H(\mathbb{1}_{\mathcal{N}},\mathcal{I},\mathcal{E},\rho) \quad ,$$

and the remaining part, which is supposed to incorporate the dynamics

$$H_{\mathrm{dyn}}(U,\mathcal{I},\mathcal{E},\rho) := H(U,\mathcal{I},\mathcal{E},\rho) - H_{\mathrm{meas}}(\mathcal{I},\mathcal{E},\rho) \quad .$$

PROPOSITION 1

Consider the Classical Dynamical System $(\mathcal{X},\mu,T)$ endowed with a classical partition $\mathcal{E}$. Then it is possible to define the automorphism U and the classical instrument $\mathcal{I}$ in such a way that

$$H(U,\mathcal{I},\mathcal{E},\rho) = h_\mu(T,\mathcal{E})$$

holds true.

PROPOSITION 2

For finite dimensional systems

$$H(U,\mathcal{I},\mathcal{E},\rho) = 0$$

RESULT (For the CS Quantum Entropy)

If we assume that dynamical localization condition holds, and we take for ρ the tracial state $\frac{1}{N}\,\mathbb{1}_{\mathcal{N}}$, we find an α such that it holds true

$$\lim_{\substack{n,N\to\infty \\ n<\alpha\log N}} \frac{1}{n}\left| S(U,\mathcal{I},\mathcal{E},\rho,n) - S_\mu(\mathcal{E}_{[0,n-1]})\right| = 0 \quad .$$

Moreover, this effect is purely related to the dynamic component of the entropy, indeed it exists an α' such that

$$\lim_{\substack{n,N\to\infty \\ n<\alpha'\log N}} \frac{1}{n}\, S(\mathbb{1}_{\mathcal{N}},\mathcal{I},\mathcal{E},\rho,n) = 0 \quad .$$

– Remember –

$$\begin{array}{ccccc}
S_\mu(\mathcal{E}^{[0,n-1]}) & \xrightarrow{\frac{1}{n}\lim_{n\longrightarrow\infty}} & h_\mu(T,\mathcal{E}) & \xrightarrow[\mathcal{E}]{\sup} & h_\mu(T) \\
\Big\uparrow \lim_{\substack{k,N\to\infty \\ k\leq\alpha\log N}} \frac{1}{k}\cdots & & & & \\
S(U,\mathcal{I},\mathcal{E},\rho,n) & \xrightarrow[\frac{1}{n}\limsup_{n\longrightarrow\infty}]{} & H(U,\mathcal{I},\mathcal{E},\rho) & \longrightarrow & \begin{cases} H_{\mathrm{meas}}(\mathcal{I},\mathcal{E},\rho) \\ H_{\mathrm{dyn}}(U,\mathcal{I},\mathcal{E},\rho) \end{cases}
\end{array}$$

CONCLUSION

- We used QDE to find footprint of CHAOS in quantum (or discrete) systems, obtained from classical continuous one.
- We found that the correspondence between Classical and Quantum Dynamics lasts much less than the Heisenberg time Breaking Time BT.
- The BT scales logarithmically in the dimension of the Hilbert space, moreover it is inversely proportional to the Lyapounov exponent.
- For the Quantum Cat Maps we exactly determined BT$= \frac{1}{2}\frac{\log N}{\log \lambda}$
- We showed how Quantum Dynamical Entropies can be profitably used in a Classical Discretized context.